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Research Note

Fundamental Frequency of Tapered Plates Using Modified Modes

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Abstract. *The fundamental frequency of a rectangular orthotropic plate having an arbitrary thickness variation is computed by using the method of Modified Vibrational Mode (MVM) shapes. The change of thickness within a plate is characterized by introducing a tapering index. It is shown that the vibrational mode shapes of a tapered plate is in fact a linear combination of various mode shapes of intact plates. This phenomenon is used to estimate the vibrational mode shapes of stepped plates. In turn, these mode shapes are incorporated to evaluate their fundamental frequency. Many numerical analyses are carried out to represent the accuracy and robustness of the proposed method by comparing the results to the works presented by other researchers. The major advantage of the present method over the existing ones is its simplicity for handling the problem of force vibration of tapered plates.*

Keywords: *Fundamental frequency; Stepped plate; Dynamic equation of motion; Modified Vibrational Mode shape (MVM)*

INTRODUCTION

Tapered plates are being increasingly used in modern engineering structures. The increasing use is due to the distributed flexural stiffness that helps to reduce the weight of structural elements and improve the utilization of the material. The flexural stiffness, vibrational and buckling capacities of these plates may be significantly increased by appropriate tapering. Furthermore, provision of openings in the perforated plates can greatly enhance the applicability and access for inspection, services and maintenance of these members in civil, mechanical, aeronautical and marine structures. Among all advantages, provision of openings in these members will result in changes in the vibrational capacity of plates that should be taken into account. On the other hand, thickness non-uniformity within a plate leads to changes in the dynamic characteristics such as vibration responses, natural frequencies and mode shapes. Therefore, stiffness changes within a structure can be used to look for the influence of higher vibrational modes to determine the more accurate modes of such tapered plates.

For the estimation of fundamental frequency of intact plates many theories are proposed in the textbooks by Meirovitch, Timoshenko and Gere, and Reddy [1-3]. Many research have been directed towards the study of the vibration of isotropic plates having a linear thickness variation. Gumeniuk presented a finite difference approach to compute the fundamental frequency of simply supported plates [4]. Chopra and Durvasula presented the vibration of simply supported skew plates having a linear variation in thickness in one direction [5]. They have made approximate analysis by using the Lagrange's equations and employing the double Fourier sine series in oblique coordinates to represent the deflected surface. Natural frequencies are obtained for rhombic plates for several ranges of thickness variation and skew angle. The nodal patterns plotted for a few typical configurations show interesting metamorphoses with variation in thickness and skew angle. Chopra investigated the free vibration of stepped plates as a composition of uniform domains, and the thickness was allowed to vary from domain to domain [6]. In his study the overall eigenvalue problem was formulated by assuming the boundary conditions and continuity conditions at

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the location of abrupt change of thickness. Ng and Araar employed the Galerkin method to investigate the vibration of clamped rectangular plates [7]. Kukreti et al. have employed the differential quadratures method to compute the fundamental frequency of rectangular plates [8]. They have considered simply supported, clamped and mixed boundary conditions in their study. Gutierrez and Laura used the method of differential quadratures to study vibration of rectangular and circular plates [9,10]. To study the vibration of rectangular plates with elastic point constraints, Liew and Lam used two-dimensional Gram-Schmidt orthogonal polynomials [11]. Kitipornchai et al. incorporated the Lagrange multiplier method to study the vibration of point-supported Mindlin plates [12].

A numerical finite element solution was developed by Rossi et al. to study the transverse vibration of rectangular cantilevered plates having a discontinuous thickness variation [13]. Barton has employed the method of eigensensitivity analysis to determine a quadratic expression in order to compute the fundamental frequency of a rectangular, isotropic plate with a linear thickness variation in one direction [14]. Various support conditions were analyzed in their study, including simply supported and clamped boundary conditions. They have compared their results with those of provided by Kukreti [8].

Cheung and Zhou studied the problem of the free vibrations of tapered rectangular plates [15]. A wide range of non-uniform rectangular plates in one or two directions are considered. The thickness of the plate is continuously varying and proportional to the power function, $x^s y^t$. Kang used 3-D Ritz analysis procedure to determine extensive and accurate frequency data for thick, tapered, circular and annular plates [16]. The analysis uses the 3-D equations of the theory of elasticity in their general forms for isotropic materials. Cheung and Zhou investigated the free vibrations of rectangular Mindlin plates with variable thickness in one or two directions [17]. They have represented the thickness variation of the plate by a continuous power function of the rectangular coordinates. They have developed two sets of new admissible functions, respectively, to approximate the flexural displacement and the angle of rotation due to bending of the plate. Cheung and Zhou have also obtained the eigen-frequency equation by using the Rayleigh-Ritz method. In this study the complete solutions of displacement and angle of rotation due to bending for a tapered Timoshenko beam (a strip taken from the tapered Mindlin plate in some direction) under a Taylor series of static load have been derived, which are used as the admissible functions of the rectangular Mindlin plates with taper thickness in one or two directions.

Xiang and Wei presented an analytical approach for studying the buckling and vibration behavior of rectangular Mindlin plates with multiple steps [18]. The Levy solution method is employed in connection with the domain decomposition technique that is used to cater for the step variation in the plates. They have investigated the influence of the step length ratios, step thickness ratios and the number of steps on the buckling and vibration behavior of square and rectangular Mindlin plates.

In this article, the method of modified vibrational modes is used to compute the fundamental frequency of stepped rectangular orthotropic plates. The main advantage of this method is that it can be applied to determine the fundamental frequency of tapered plates that non-uniformity within the thickness of plate can be introduced with arbitrary function including polynomial, trigonometric etc. The procedure may be readily extended to obtain modal properties of plates having various boundary conditions where their corresponding vibrational mode shapes are known.

THEORY AND FORMULATIONS

General

Figure 1 shows an orthotropic elastic stepped rectangular plate of length, a, width, b, mass density per unit volume, ρ, and the intact Young's modulus, E. The plate is formed by two subogines named O_1 and O_2 where in the middle zone, the plate has a thickness of h_1, and the rest of the plate has thickness of h_0. By assuming the expression $h_1 < h_0$, it is considered that the plate has a damaged part of lengths, λa, and

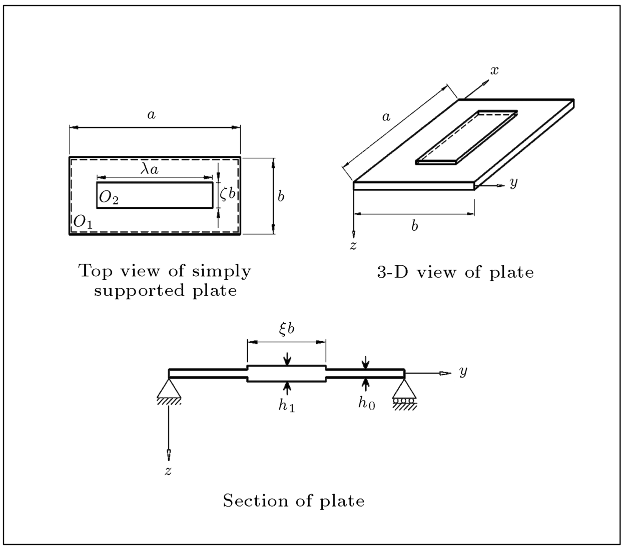

Figure 1. Coordinate and geometry of abruptly varying thickness plate.

width, ζb, located at $x_i < x < x_{i+1}$ and $y_i < y < y_{i+1}$ where i refers to the ith damaged or stepped segment. The basic procedures in the vibrational analysis of such stepped plates involve the following parts:

I. The dynamic equation of motion is used to obtain the Modified Vibrational Mode (MVM) shapes of tapered plates.

II. The Modified Vibrational Mode shapes (MVM) are invoked to evaluate the fundamental frequency of tapered plates.

The equation of motion governing the linear bending of an orthotropic elastic plate is given by the following equation:

$$D_{11}\frac{\partial^4 w_o}{\partial x^4} + 2(D_{12}+2D_{66})\frac{\partial^4 w_o}{\partial x^2.\partial y^2} + D_{22}\frac{\partial^4 w_o}{\partial y^4} + kw_o = -\left(\frac{\partial^2 M_{xx}^T}{\partial x^2} + 2\frac{\partial^2 M_{xy}^T}{\partial x \partial y} + \frac{\partial^2 M_{yy}^T}{\partial y^2}\right) + q(x,y) - I_0.\frac{\partial^2 w_o}{\partial t^2} + I_2\left(\frac{\partial^4 w_o}{\partial t^2.\partial x^2} + \frac{\partial^4 w_o}{\partial t^2.\partial y^2}\right), \quad (1)$$

where k is the elastic foundation modulus (force per unit surface area), w_o denotes the out-of-plane deflection measured from the pre-deformed state, M_{xx}^T and M_{yy}^T are the normal thermal bending moments, M_{xy}^T is the twisting thermal moment, and q is the intensity of the distributed transverse load.

In Equation 1, the D factors are bending stiffness coefficients and are defined as follows:

$$D_{11} = \frac{Eh^3}{12(1-\nu^2)}, \qquad D_{22} = \frac{E_2}{E_1}D_{11},$$

$$D_{12} = \nu D_{22}, \qquad D_{66} = \frac{G_{12}h^3}{12}.$$

Also I_0 and I_2 are mass moments of inertia and defined as:

$$I_0 = \rho \times h,$$

$$I_2 = \frac{\rho \times h^3}{12},$$

where ρ is the material density in the undeformed body and h is the plate's thickness.

In the absence of thermal loadings and when no elastic foundation is present, Equation 1 reduces to:

$$D_{11}\frac{\partial^4 w_0}{\partial x^4} + 2(D_{12}+2D_{66}).\frac{\partial^4 w_0}{\partial x^2.\partial y^2} + D_{22}.\frac{\partial^4 w_0}{\partial y^4} + I_0.\frac{\partial^2 w_0}{\partial t^2} - I_2\left(\frac{\partial^4 w_0}{\partial t^2.\partial x^2} + \frac{\partial^4 w_0}{\partial t^2.\partial y^2}\right) = q(x,y). \quad (2)$$

In the case of natural vibrations, the solution to Equation 2 is assumed to be periodic as shown in Equation 3:

$$w_0(x,y,t) = w(x,y)e^{i\Omega t}, \quad (3)$$

where $i = \sqrt{-1}$, and Ω is the frequency of natural vibration associated with the mode shape. Substituting Equation 3 into Equation 2 and setting $q = 0$ for no lateral loading, it yields:

$$\left\{D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} - \Omega^2\left[I_0\Omega - I_2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\right]\right\} e^{i\Omega t} = 0. \quad (4a)$$

Equation 4a must hold for any time. So it can be rewritten:

$$\left\{D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} - \Omega^2\left[I_0\Omega - I_2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\right]\right\} = 0. \quad (4b)$$

In case of a rectangular plate with all sides simply supported, the following boundary condition can be applied:

$$w = 0, \qquad w,xx = 0,$$

$$x = 0, a, \qquad w = 0,$$

$$w,yy = 0, \qquad y = 0, b. \quad (5)$$

For a uniform thickness plate, the following Navier solution can be considered:

$$w_{mn}(x,y) = \Lambda \sin\frac{m\pi x}{a}.\sin\frac{n\pi y}{b},$$

$$m,n = 1,2,3,\cdots. \quad (6)$$

Solving Equation 6 for the natural frequency of a uniform section rectangular orthotropic plate and neglecting the rotary inertia I_2, the following expression yield:

$$\Omega_{mn}^2 = \frac{\pi^4}{\rho h b^4}\left[D_{11}.m^4\left(\frac{b}{a}\right)^4 + 2\left(D_{12}+2D_{66}\right)m^2n^2\left(\frac{b}{a}\right)^2 + D_{22}n^4\right]. \quad (7a)$$

For isotropic rectangular plates, this expression can be represented by:

$$\Omega_{mn} = \frac{\pi^2}{b^2}\sqrt{\frac{D}{\rho h}}\left(m^2\frac{b^2}{a^2} + n^2\right). \tag{7b}$$

This way, the fundamental frequency of a simply supported uniform section plate is given by taking $m = n = 1$:

$$\Omega_{11} = \frac{\pi^2}{b^2}\sqrt{\frac{D}{\rho h}}\left[\left(\frac{b}{a}\right)^2 + 1\right]. \tag{7c}$$

Modified Vibrational Mode Shapes

By introducing $I_2 = 0$ in dynamic equation of motion (Equation 2) and differentiating from this equation with respect to x, y and neglecting the change in the mass distribution, the following equation can be obtained:

$$D_{11}\frac{\partial^4 dw_\alpha}{\partial x^4} + dD_{11}\frac{\partial^4 w_\alpha}{\partial x^4} + 2(D_{12} + 2D_{66}).\frac{\partial^4 dw_\alpha}{\partial x^2.\partial y^2}$$
$$+ 2(dD_{12} + 2dD_{66}).\frac{\partial^4 w_\alpha}{\partial x^2.\partial y^2} + D_{22}.\frac{\partial^4 dw_\alpha}{\partial y^4}$$
$$+ dD_{22}.\frac{\partial^4 w_\alpha}{\partial y^4} + \rho h d\ddot{w}_\alpha = 0, \tag{8}$$

where ρh is the mass per unit surface area of plate. It is assumed that the changes in the αth mode shape of tapered plate can be written as a linear combination of M natural modes of the corresponding uniform thickness plate:

$$dw_\alpha = \sum_\beta \alpha_{\alpha\beta} w_\beta,$$
$$\alpha, \beta = 1, 2, 3, \cdots, M. \tag{9}$$

Substituting Equation 9 in the dynamic equation of motion for the αth mode shape yields:

$$\sum\left[D_{11}\alpha_{\alpha\beta}\frac{\partial^4 w_\beta}{\partial x^4} + dD_{11}\frac{\partial^4 w_\alpha}{\partial x^4}\right.$$
$$\left.+2(dD_{12} + 2dD_{66}).\frac{\partial^4 w_\alpha}{\partial x^2.\partial y^2} + dD_{22}.\frac{\partial^4 w_\alpha}{\partial y^4}\right]$$
$$+ D_{22}\sum \alpha_{\alpha\beta}.\frac{\partial^4 w_\beta}{\partial y^4} + 2(D_{12} + 2D_{66})$$
$$.\sum \alpha_{\alpha\beta}.\frac{\partial^4 w_\beta}{\partial x^2 \partial y^2} - \rho h d\Omega_\alpha^2 w_\alpha$$
$$- \rho h \Omega_\alpha^2 w_\alpha \sum \alpha_{\alpha\beta}.w_\beta = 0. \tag{10}$$

Multiplying both side of Equation 10 by w_γ (displacement of plate related to this mode) where $\gamma \neq \alpha$, and integrating for the whole area of plate yields:

$$\sum \alpha_{\alpha\beta}\left\{\int_\Gamma \left[D_{11}\frac{\partial^4 w_\alpha}{\partial x^4} + 2(D_{12} + 2D_{66}).\frac{\partial^4 w_\alpha}{\partial x^2.\partial y^2}\right.\right.$$
$$\left.\left.+D_{22}.\frac{\partial^4 w_\alpha}{\partial y^4}\right] w_\gamma.dx.dy\right\} + \int_\Gamma \left[dD_{11}\frac{\partial^4 w_\alpha}{\partial x^4}\right.$$
$$+ 2(dD_{12} + 2dD_{66}).\frac{\partial^4 w_\alpha}{\partial x^2.\partial y^2}$$
$$\left.+dD_{22}.\frac{\partial^4 dw_\alpha}{\partial y^4}\right] w_\gamma dx.dy$$
$$- \rho h d\Omega_\alpha^2 \int_\Gamma w_\alpha w_\gamma dx.dy$$
$$+ \rho h \Omega_\alpha^2 \sum \alpha_{\alpha\beta} \int_\Gamma w_\beta w_\gamma dxdy = 0, \tag{11}$$

where Γ is the surface area of plate, i.e. $\Gamma = a \times b$. The orthogonal property implies that:

$$\int_\Gamma \rho h.w_\alpha.w_\beta.dx.dy = \delta_{\alpha\beta},$$
$$\alpha, \beta = 1, 2, 3, \cdots, M, \tag{12}$$

$$\int_\Gamma w_\alpha \left[D_{11}\frac{\partial^4 w_\beta}{\partial x^4} + 2(D_{12} + D_{66})\frac{\partial^4 w_\beta}{\partial x^2 \partial y^2}\right.$$
$$\left.+D_{22}\frac{\partial^4 w_\beta}{\partial y^4}\right] dx.dy = \Omega_\alpha^2 \delta_{\alpha\beta},$$
$$\alpha, \beta = 1, 2, 3, \cdots, M, \tag{13}$$

where Ω_α is the natural frequencies of the uniform thickness plate, M indicates the number of normal modes superposed in the analysis, and $\delta_{\alpha\beta}$ is the Kronecker symbol defined as follows:

$$\delta_{\alpha\beta} = \begin{cases} 1 & \text{for} \quad \alpha = \beta \\ 0 & \text{for} \quad \alpha \neq \beta \end{cases}$$

For brevity, the contracted subscripts for the mode numbers will be consistently used in the following manner, i.e. α is used for various compounds of m and n, and β is used for various compounds of r and s where m, n, r and s range for various mode shapes.

According to the orthogonal property in Equation 13, in order to avoid dissipation of the first term in Equation 11, β and γ should be equal, $\gamma = \beta$. On the other hand, since $\gamma \neq \alpha$, that term becomes zero, thus the first term in Equation 11 will always become zero, therefore:

$$\alpha_{\alpha\gamma}\Omega_\gamma^2 + \int_\Gamma \left[dD_{11}\frac{\partial^4 w_\alpha}{\partial x^4} + 2(dD_{12} + 2dD_{66}) .\frac{\partial^4 w_\alpha}{\partial x^2.\partial y^2} + dD_{22}.\frac{\partial^4 w_\alpha}{\partial y^4}\right] w_\gamma dx.dy - \Omega_\alpha^2.\alpha_{\alpha\gamma} = 0. \tag{14}$$

From the above equation, the effect of βth mode shape on changes in the αth mode shape can be derived as follows:

$$\alpha_{\alpha\beta} = \frac{1}{\Omega_\alpha^2 - \Omega_\beta^2}.\int_\Gamma \left[dD_{11}\frac{\partial^4 w_\alpha}{\partial x^4} + 2(dD_{12} + 2dD_{66}) .\frac{\partial^4 w_\alpha}{\partial x^2.\partial y^2} + dD_{22}.\frac{\partial^4 dw_\alpha}{\partial y^4}\right] .w_\beta dx.dy. \tag{15}$$

Differentiating the orthogonality property of mode shapes with respect to mass would yield to:

$$\int_0^a\int_0^b d(\rho h).w_\alpha^2.dydx + 2\int_0^a\int_0^b (\rho h).w_\alpha dw_\alpha.dydx = 0. \tag{16}$$

It is assumed that the changes in the mass distribution are negligible in comparison with the significant changes in the stiffness of the structure. Therefore, the first term in the left side of Equation 16 can be eliminated. Substitute Equation 9 into the second part of above equation gives:

$$2\int_0^a\int_0^b \rho h.w_\alpha.\sum_\beta \alpha_{\alpha\beta}.w_\beta dydx = 0. \tag{17}$$

By using orthogonal property (Equation 12), and applying the Kronecker properties:

$$2.\alpha_{\alpha\alpha}\int_0^a\int_0^b \rho h.w_\alpha^2.dxdy = 0. \tag{18}$$

It can be shown that:

$$\int_0^a\int_0^b \rho h.w_\alpha^2.dydx = 1.$$

Thus:

$$\alpha_{\alpha\alpha} = 0. \tag{19}$$

Generally, vibrational mode shape of tapered plates is given by:

$$(w_d)_\alpha = w_\alpha + dw_\alpha, \tag{20}$$

where $(w_d)_\alpha$ indicates the αth mode shape of tapered plates, w_α refers to various mode shapes of uniform thickness plates, and dw_α represents changes in the αth mode shape. Substituting dw_α from Equation 9 into Equation 20 may yield:

$$(w_d)_\alpha = w_\alpha + \sum_\beta \alpha_{\alpha\beta} w_\beta, \quad \alpha, \beta = 1, 2, 3, \cdots, M. \tag{21}$$

w_α and w_β are the natural modes satisfying the eigenvalue problem for the intact plate, $\alpha_{\alpha\beta}$ are the effect of βth mode shapes of intact plate on the αth mode shape of tapered plate, and is defined as:

$$\alpha_{\alpha\beta} = \begin{cases} \frac{1}{\Omega_\alpha^2 - \Omega_\beta^2}.\int_A \left[dD_{11}\frac{\partial^4 w_\alpha}{\partial x^4} + 2(dD_{12} + 2dD_{66}).\frac{\partial^4 w_\alpha}{\partial x^2.\partial y^2} + dD_{22}.\frac{\partial^4 w_\alpha}{\partial y^4}\right] .w_\beta dx.dy & \text{for} \quad \alpha \neq \beta \\ 0 & \text{for} \quad \alpha = \beta \end{cases} \tag{22}$$

D_{ij} are the bending stiffness of plate. Ω_α and Ω_β are the natural frequencies of uniform thickness plates. In order to demonstrate various domains of integration, an example is presented in this section. Zero thicknesses for zones 1, 2 and 3 in the plate shown in Figure 2 imply that the plate has a uniform thickness, and non-zero values mean that the plate has abrupt changes in the thickness. dD_{ij} in an orthotropic plate or dD in an isotropic plate is the tapering function, which characterizes the state of non-uniformity in various parts and involves changes in the primary stiffness of plate. dD_{ij} is generally a function of variables x and y, and can be represented in various forms, such as continuous, piecewise, single-valued, and may take positive values for stiffened or negative values for damaged plates.

The main advantage of this method is that the tapering function can be applied to any form of damaged plates, having various boundary conditions. That is, of course, if the mode shapes of intact plates are known. For an elastic isotropic plate Equation 22, becomes:

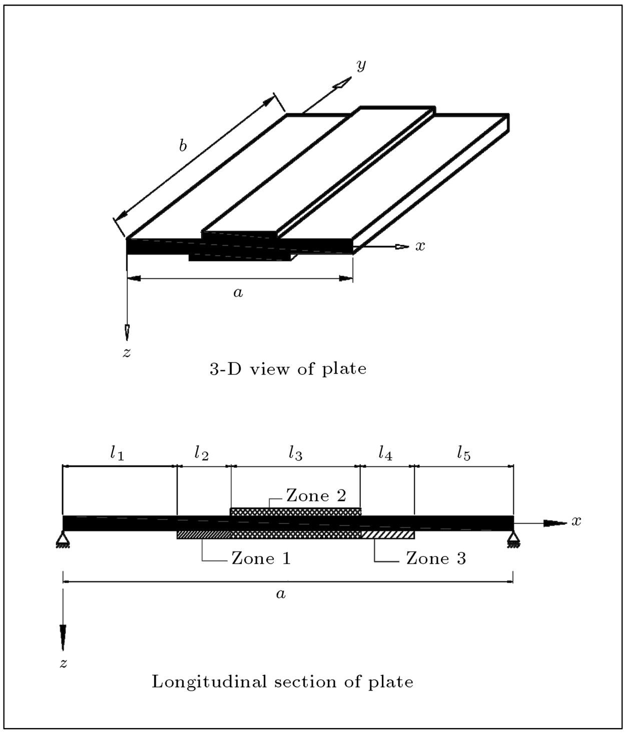

Figure 2. 3-D view and coordinate of stepped plates.

$$\alpha_{\alpha\beta} = \begin{cases} \frac{1}{\Omega_\alpha^2 - \Omega_\beta^2} \cdot \int\limits_\Gamma \left[dD \left(\frac{\partial^4 w_\alpha}{\partial x^4} + 2(1+\nu) \frac{\partial^4 w_\alpha}{\partial x^2 . \partial y^2} + \frac{\partial^4 dw_\alpha}{\partial y^4} \right) . \right] . w_\beta dx.dy & \alpha \neq \beta \\ 0 & \alpha = \beta \end{cases} \tag{23}$$

The domain of the integral in Equation 23 varies for the tapered region of plate (see Figure 2) as follow:

Zone (1):

$$l_1 < x < l_1 + l_2, \qquad 0 < y < b,$$

$$h = h_1.$$

Zone (2):

$$l_1 + l_2 < x < l_1 + l_2 + l_3, \qquad 0 < y < b,$$

$$h = 2h_2.$$

Zone (3):

$$l_1 + l_2 + l_3 < x < l_1 + l_2 + l_3 + l_4, \qquad 0 < y < b,$$

$$h = h_1.$$

Vibrational modes of a taper plate can be obtained by superposing M natural modes of corresponding intact plate and applying Equation 23 into Equation 21. Based on these modified vibrational mode shapes, their corresponding frequency can be obtained. The least value among all frequency is the fundamental frequency of tapered plate. The major advantage of the present method over the existing ones is its simplicity for handling the problem of force vibration of tapered plates.

NUMERICAL RESULTS

General

Equation 21 provides a general closed-form equation which is used to evaluate the vibrational mode shape of tapered plate. The vibrational mode shape of the plate in the intact state and the modification coefficient $\alpha_{\alpha\beta}$ from Equation 23 are then incorporated to predict the vibrational mode shape and the corresponding frequency of tapered isotropic plates. However, in order to identify the lowest frequency, higher order values of m, n, r and s may be required to be taken into account. This is because during the evaluation process of w_α, there is no prior guarantee that the first mode shape, w_{11}, always induces the least frequency, although this will often be the case. Therefore, higher modes should also be taken into account.

The procedure based on the aforementioned formulation is programmed on a desktop computer and numerical results are presented for different models.

Fundamental Frequency of Simply Supported Tapered Plates

To demonstrate the versatility and to validate the method, several vibrational analysis of plates with variable thicknesses are carried out and compared with existing results. The examples are selected for comparison purpose only. Generally speaking, this method is applicable to all plates whose mode shapes can be expressed by analytical functions, either trigonometric or polynomial.

The deflection surface of a uniform thickness plate, having simply supported edges, can be represented by the following double sinusoidal series:

$$w_{ij} = A. \sin\left(\frac{i\pi x}{a}\right). \sin\left(\frac{j\pi y}{b}\right),$$

$$i, j = 1, 2, 3, \cdots, M. \tag{24}$$

Normalized mode shapes are used in the formulation process, thus using orthogonal property:

$$\int\limits_A m.w_i.w_j.dxdy = \delta_{ij}. \tag{25}$$

is the Kroniker symbol and i and j are the contracted subscripts of the mode numbers. Substituting Equation 38 into 39 will lead to the following equation:

$$m.Q^2.\int_\Gamma \left(\sin\left(\frac{i\pi x}{a}\right).\sin\left(\frac{j\pi y}{b}\right)\right)^2 dxdy = 1,$$

$$i,j = 1,2,3,\cdots,M. \qquad (26)$$

Solving Equation 40 for various normalized mode shapes will result in the following expression:

$$Q = \sqrt{\frac{2}{m.a.b}}. \qquad (27)$$

By substituting Equation 41 into Equation 38 yields to the normalized mode shape of uniform thickness plates:

$$w_{ij} = \sqrt{\frac{2}{m.a.b}}.\sin\left(\frac{i\pi x}{a}\right)\sin\left(\frac{j\pi y}{b}\right),$$

$$i,j = 1,2,3,\cdots,M. \qquad (28)$$

Substituting the natural frequency of simply supported intact plates, Equation 7b, and the normalized mode shape from Equation 42 into Equation 22 and finally using Equation 21 the Modified Vibrational Mode shape (MVM) can be achieved. Introducing this MVM into the governing differential equation of plates (Equation 4b), the corresponding natural frequency can be obtained.

Normalized Frequency of a Simply-Supported Plate

Figure 3 shows a uni-directional stepped plate. The plate is analyzed by the present method (MVM), and the results are compared to those obtained by Barton [14]. Table 1 illustrates results for the simply supported plate on all sides. In this table, columns 1 and 2 provide the values used for the aspect ratio ψ_s and taper ratio λ. The aspect ratios for each plate considered were 0.5, 1 and 2, and the taper parameter λ, varied from 0 to 1. Also the value used for Poisson ratio is $\nu = 0.3$. Columns 3 and 4 provide the normalized fundamental frequency as follow:

$$\zeta = \Omega a^2\sqrt{\frac{\rho h}{D}}. \qquad (29)$$

As shown in Table 1, the results computed by the present method (MVM) are compared to those obtained by the Differential Quadrature Method (DQM) and the approximate closed-form expression, respectively. It means that the result obtained by MVM is close to DQM and approximate closed-form expression methods. As shown the results obtained by the present method are in very good agreement with those obtained by the other method.

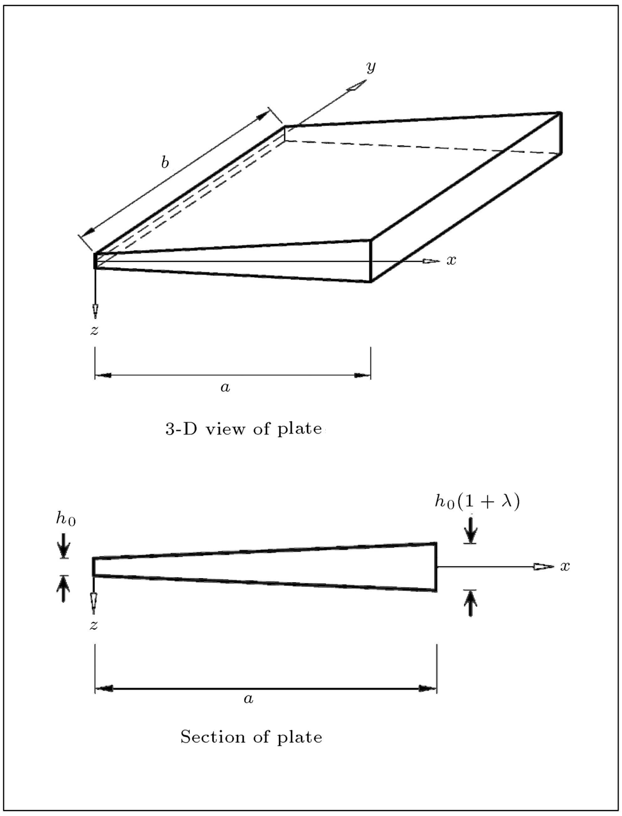

Figure 3. Three-dimensional view of abruptly varying thickness plate.

Normalized Frequency of a S-C-S-C Plate

Table 2 presents the results for a plate with mixed simply supported and clamped boundary conditions. This plate has the simply supported boundary in the tapered direction. The vibrational mode shape of a uniform thickness plate can be expressed as:

$$w_{ij} = A.\sin\left(\frac{i\pi x}{a}\right).\left[\sin\left(\frac{j\pi y}{b}\right)\right]^2,$$

$$m,n = 1,2,3,\cdots,M. \qquad (30)$$

Using the aforementioned procedure used for simply supported stepped column will result in Table 2, which implies the agreement between present method (MVM) and those obtained by Barton [14]. Table 2 demonstrates that the error of MVM method in comparison with two others is less than 1%.

Normalized Frequency of a Clamped Plate

Finally, Table 3 contains the results for the clamped plate on all sides. The aspect ratios for each plate considered were 0.5, 1 and 2, and the taper parameter λ varied from 0 to 1. Also the value used for

Table 1. Comparison of the present analysis results (MVM) with the other results on the normalized fundamental frequency of a simply supported stepped plate (see Figure 3).

Specimen Dimension		ζ		
ψ_s	λ	**DQM [14]**	**Quadratic [14]**	**Present (MVM)**
0.5	0.1	12.9518	12.9482	12.9732
0.5	0.2	13.5539	13.549	13.5834
0.5	0.3	14.1473	14.1412	14.1563
0.5	0.4	14.7332	14.7264	14.7552
0.5	0.5	15.3123	15.306	15.3442
0.5	0.6	15.885	15.8812	15.8912
0.5	0.7	16.451	16.4529	16.4658
0.5	0.8	17.0101	17.022	17.1252
1.0	0.1	20.7296	20.7206	20.7725
1.0	0.2	21.7025	21.6915	21.7652
1.0	0.3	22.6669	22.6541	22.6859
1.0	0.4	23.6239	23.6105	23.7152
1.0	0.5	24.574	24.5624	24.6125
1.0	0.6	25.5177	25.5113	25.5525
1.0	0.7	26.4547	26.4584	26.4785
1.0	0.8	27.3845	27.4048	27.4051
2.0	0.1	51.8244	51.7834	51.3150
2.0	0.2	54.2056	54.1611	54.2315
2.0	0.3	56.5370	56.4916	56.5515
2.0	0.4	58.8237	58.7847	58.8601
2.0	0.5	61.0699	61.0489	61.0814
2.0	0.6	63.2796	63.2915	63.2955
2.0	0.7	65.4562	65.5182	65.4568
2.0	0.8	67.6021	67.7339	67.6541

Poisson ratio is $\nu = 0.3$. This table provides some information on the accuracy and convergence of the present solution with those obtained by Barton [14]. As seen, the results obtained by present method (MVM) is in good agreement with those obtained by other researchers.

CONCLUSIONS

Fundamental frequency of simply supported tapered plates in one or two directions, having arbitrary boundary condition, can be achieved by the use of modified vibrational mode shapes. The proposed method can also be used to predict the frequency of tapered plates having any polynomial variation in the thickness. The method is proved to be very efficient, if the mode shapes of corresponding intact plates are known. Comparisons of several analytical and numerical results for various types of tapered plates to those presented by other researchers validate the proposed method.

NOMENCLATURE

a	length of plate
b	width of plate
D	flexural rigidity of isotropic plate
D_{ij}	bending stiffness of orthotropic plate
dD	tapering indices, variations in the primary stiffness in various regions of plate
E	Young's modulus of elasticity

Table 2. Comparison of the present analysis results (MVM) with the other results on the fundamental frequency of a plate clamped and simply supported on parallel sides (see Figure 3).

Specimen Dimension		ζ		
ψ_s	λ	**DQM [14]**	**Quadratic [14]**	**Present (MVM)**
0.5	0.0	23.8157	23.8113	23.8204
0.5	0.2	26.0570	26.1427	26.0654
0.5	0.4	28.0829	28.3959	28.0914
0.5	0.6	29.9778	30.5831	29.9850
0.5	0.8	31.8006	32.7166	31.8125
0.5	1.0	33.5891	34.8036	33.6120
1.0	0.0	28.9515	28.9572	28.9650
1.0	0.2	31.8030	31.7956	31.8450
1.0	0.4	34.5873	34.5371	34.5940
1.0	0.6	37.3319	37.2012	37.328
1.0	0.8	40.0568	39.8010	40.0615
1.0	1.0	42.7752	42.3453	42.7458
2.0	0.0	54.7551	54.8235	54.7665
2.0	0.2	60.1385	60.1797	60.1521
2.0	0.4	65.3754	65.3223	65.3950
2.0	0.6	70.5170	70.2943	70.5325
2.0	0.8	75.6006	75.1258	75.6115
2.0	1.0	80.6513	79.8387	80.6662

Table 3. Comparison of the present analysis results (MVM) with the other results on the fundamental frequency of a plate clamped on all sides (see Figure 3).

Specimen Dimension		ζ		
ψ_s	λ	**DQM [14]**	**Quadratic [14]**	**Present (MVM)**
0.5	0.0	24.5877	24.5789	24.6105
0.5	0.2	26.9971	26.9708	27.0543
0.5	0.4	29.3233	29.2602	29.3354
0.5	0.6	31.5836	31.4856	31.6105
0.5	0.8	33.7888	33.6749	33.7952
0.5	1.0	35.9465	35.846	35.9650
1.0	0.0	36.0056	35.9812	36.0154
1.0	0.2	39.5485	39.5396	39.5518
1.0	0.4	42.9408	43.0339	43.0154
1.0	0.6	46.2504	46.4923	46.3356
1.0	0.8	49.4776	49.9357	49.4845
1.0	1.0	52.6374	53.3772	52.6425
2.0	0.0	98.3475	98.3155	98.3612
2.0	0.2	107.8149	108.0055	107.9518
2.0	0.4	116.6358	117.4573	117.0115
2.0	0.6	124.9588	126.7766	125.1541
2.0	0.8	132.8875	136.0392	133.1548
2.0	1.0	140.5029	145.2929	141.2855

$f(x,t)$	external force
G	shear modulus
h, h_0, h_1	plate thicknesses
$I(x)$	second moment of inertia
I_0, I_2	mass moments of inertia
k	foundation modulus (force per unit surface area)
M	mass moments of inertia
M_{xx}^T, M_{yy}^T	thermal bending
M_{xy}^T	twisting thermal moments
O_1, O_2	zones of stepped plate
q	intensity of the distributed transverse load
t	time
$w(x,y)$	flexural deflection of plate
$(w_d)_\alpha$	the αth modified mode shape of taper plate
w_α, w_β	various mode shapes of intact plates
α	contracted subscripts for various compound of m, n
$\alpha_{\alpha\beta}$	the effect of βth mode shape on the αth mode shape
β	contracted subscripts for various compound of r, s
ν	Poisson's ratio
ξb	width of stepped part
λa	length of stepped part
$\delta_{\alpha\beta}$	Kronecker's symbol
ρ	material density, mass per unit volume
Ω_m	natural frequencies of the intact plate

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