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Improving Penalty Functions for Structural Optimization

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Abstract. *New penalty functions, which have better convergence properties, as compared to the commonly used exterior and interior penalty functions, have been proposed in this paper. The convergence behavior and accuracy of ordinary penalty functions depend on the selection of appropriate penalty parameters. The optimization of ordinary penalty functions is accomplished after several rounds of optimization where, at each round a different but fixed value of penalty parameter is used. While some useful hints and rules for the selection of suitable penalty parameter values have been provided by different authors, the objective of this paper has been to improve this procedure by including the penalty parameter in the design vector, so that it can be modified during the optimization, automatically, in order to improve the convergence characteristics. This can also help accomplish optimization in only one round, which is of considerable importance when it is desired to solve a constrained problem by using genetic algorithms. The proof of convergence to the optimum solution of the proposed functions is also included in the paper. Ten-bar and three-bar truss examples are used for illustration through which the convergence of ordinary and new functions are evaluated and compared. The results show that the new penalty functions can outperform the ordinary functions, especially in combination with genetic algorithms.*

Keywords: *Optimization; Minimum weight design; Trusses; Genetic algorithms; Structural design.*

INTRODUCTION

The general form of a minimization problem with inequality and equality constraints is as follows:

$$\text{find} \quad \mathbf{x}, \tag{1a}$$

$$\text{Minimize} \quad f(\mathbf{x}), \tag{1b}$$

subject to:

$$g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \cdots, m, \tag{1c}$$

and:

$$h_j(\mathbf{x}) = 0, \quad j = 1, 2, \cdots, p, \tag{1d}$$

where $\mathbf{x}$ = n-dimensional vector of design variables, $f(\mathbf{x})$ = objective function to be minimized, and g and h = the inequality and equality constraints, respectively; hence there are m inequality and p equality constraints.

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However, in the majority of structural optimization problems, the equality constraints constitute a part of or the whole equilibrium equation and, so, are excluded from the above formulation and hence are treated implicitly.

Solving constrained optimization problems is generally cumbersome. While there are methods such as the feasible directions method [1,2] that solves the constrained problems directly, the main strategy behind the penalty methods is to change the form of the above constrained optimization problem to an unconstrained form so that unconstrained optimization techniques can be applied. To this end, a global objective function, ψ, called the penalty function, is defined, which includes both the main objective function and all the constraints. ψ is so defined that, through its optimization by unconstrained optimization techniques, both the constraints are satisfied and also the objective function is minimized.

Penalty function methods are the most popular constraint handling methods among users [3]. Two main branches of penalty method have been proposed in the literature: 1) Exterior, 2) Interior, which is also called the barrier method. The general formulation of these methods is as follows.

Exterior Penalty Method

The widely used form of ψ in the exterior penalty method is:

$$\psi = \psi_e(\mathbf{x}, r) = f(\mathbf{x}) + r_e \left(\sum < g_i(\mathbf{x}) >^2 + \sum h_j(\mathbf{x})^2 \right), \quad r_e \rightarrow \infty, \tag{2a}$$

where:

$$< g_i(\mathbf{x}) >= \max(0, g_i(\mathbf{x})), \quad i = 1, 2, \cdots, m, \tag{2b}$$

and r_e is a parameter, which is modified at the beginning of each round of optimization. Each optimization round is defined here as a complete optimization of $\psi_e(\mathbf{x}, r)$ for a fixed value of r_e until the convergence is achieved. The optimum point, $\mathbf{x}^*$, at the end of each round serves as the starting point, $\mathbf{x}_1$, of the next round of optimization with a larger r_e. Figure 1 shows the general optimization procedure of the exterior penalty method for problems with inequality and equality constraints [2].

The selection of appropriate r_e values is vital for faster convergence and more precision [4]. In some cases, the user might specify the value of r_e at the end of each optimization round. This technique is very interactive and time consuming and is not generally preferred. Another dominant technique is to define a function that automatically determines the r_e value at the beginning of a new round of optimization. Denoting the optimization steps by k, then $k = 1$ at the onset of optimization and $r_{e,k} = r_{e,1}$. The selection of an appropriate $r_{e,1}$ plays a key role in the convergence behavior of the method. Haftka and Gurdal [4] have proposed to use:

$$r_{e,1} = \frac{16 f(\mathbf{x}_1)}{n}, \tag{2c}$$

where $\mathbf{x}_1$ is the initial value of $\mathbf{x}$. They have also suggested that r_e be updated according to:

$$r_{e,i+1} = c r_{e,i}, \tag{2d}$$

where for most structural problems a value of:

$$c = 5, \tag{2e}$$

has been found satisfactory.

The third alternative to determine appropriate r_e is to use the so called intelligent and adaptive techniques such as fuzzy logic and neural networks [5,6].

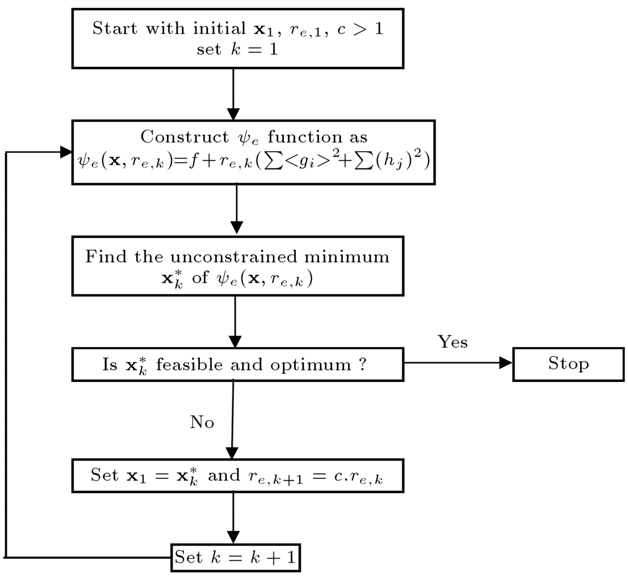


Figure 1. Flow chart for exterior penalty function method.

Interior Penalty Method

The original form of the interior penalty function,ψ_{in}, is as follows [1]:

$$\psi_{in}(\mathbf{x}, r) = f(\mathbf{x}) - r_{in} \left(\sum \frac{1}{g_i(\mathbf{x})} + \sum \frac{1}{h_j(\mathbf{x})} \right), \quad r_{in} \rightarrow 0, \tag{3a}$$

where r_{in} reduces from a high value to 0 gradually.

Rao [2] has proposed the following function for selecting r_{in} at the start of the optimization procedure:

$$r_1 = (0.1 \sim 1) \times \frac{f(\mathbf{x}_1)}{-\sum \frac{1}{g(\mathbf{x}_1)}}, \tag{3b}$$

where $\mathbf{x}_1$ is the initial point in the feasible region. The optimization procedure is similar to the exterior penalty function method except that r_{in} reduces to 0 gradually. Figure 2 summarizes the optimization procedure of the interior penalty method. Here, the reduction follows [2]:

$$r_{in_{i+1}} = c \times r_{in_i}, \tag{3c}$$

where c is a coefficient less than 1.

Hence, ordinary penalty functions generally require the value of some coefficients to be specified at the beginning of optimization. However, these coefficients usually have no clear physical meaning. Consequently, it is very difficult to select appropriate values for these coefficients even through experience [7].

Besides the classical nonlinear optimization methods, smart computational techniques, especially Genetic Algorithms, have been applied to find the minimum of the response surface, ψ. To this end, some

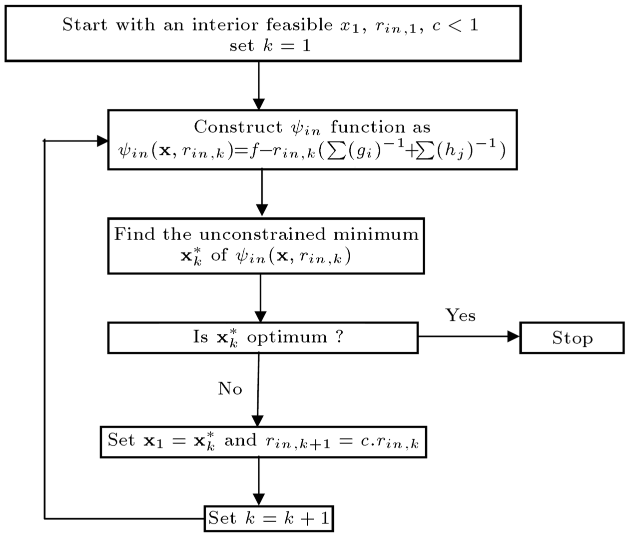


Figure 2. Flow chart for interior penalty function method.

special penalty functions such as the Death Penalty, Static and Dynamic Penalty and Maximum-violation Penalty methods have also been introduced and discussed in the literature [8-11].

In this paper, a new technique has been proposed for the determination of suitable r_e and r_{in} values, where they have been treated as design variables and modified automatically through the optimization procedure. To this end, new penalty functions have been proposed for the interior and exterior penalty approaches. The new functions have been used in the optimization of the 10 and 3 bar truss benchmark problems [4] using the steepest descent method. Also, to assess the efficiency of the functions in combination with GA methods, the problems have been solved using GA too.

NEW PENALTY FUNCTIONS

Two new penalty functions are introduced with their proof of convergence: 1) New exterior penalty function and 2) New interior penalty function.

New Exterior Penalty Function

The function is as follows:

$$\psi'_e(\mathbf{x}, r'_e) = f(\mathbf{x}) + r'_e \Big(\sum < g_i(\mathbf{x}) >^2 + \sum h_j(\mathbf{x})^2 \Big) + u(r'_e), \tag{4a}$$

where:

$$< g_i(\mathbf{x}) >= \max(0, g_i(\mathbf{x})), \quad i = 1, 2, \cdots, m, \tag{4b}$$

and r'_e and $u(r'_e)$ have the following properties:

$$r'_e > 0, \tag{4c}$$

$$u(r'_e) \geq 0, \quad \forall r'_e \geq 0, \tag{4d}$$

$$\lim_{r'_e \to \infty} u(r'_e) = 0. \tag{4e}$$

Also, $u(r'_e)$ is monotonically decreasing with r'_e, that is:

$$\text{if} \quad r'_{e1} \leq r'_{e2} \quad \text{then} \quad u(r'_{e1}) \geq u(r'_{e2}). \tag{4f}$$

Hence, it is expected that the penalty function is minimized gradually with the advancement of the optimization, when r'_e increases towards very high values, forcing the constraints to be satisfied and $u(r'_e)$reduce to 0. Any function satisfying the above criteria could be used for $u(r'_e)$. In this paper, the following simple function has been found useful:

$$u(r'_e) = \frac{\alpha_e}{r'^2_e}, \tag{5}$$

where α_e is a constant, the designer should specify. The general rule is that a larger α_e should be selected for more accuracy.

A suitable set of convergence criteria is:

$$\|\mathbf{x}_k - \mathbf{x}_{k-1}\| \leq \varepsilon. \tag{6}$$

For accuracy, where $\|\boldsymbol{\nu}\|$ is the length of vector $\boldsymbol{\nu}$ and ε is a small positive number specified by the designer.

Proof of Convergence

Theorem

The minimization of the new ψ' function provides the optimal solution, whether the answer to the constrained optimization problem is a point, $\mathbf{x}^*$, inside the feasible region or on the feasible surface, and it never provides an answer outside the feasible region.

Proof

Assuming that f, g_i, $i = 1, 2, \cdots, m$ and h_j, $j = 1, 2, \cdots, p$ are continuous, that an optimum solution exists for the given problem and, also, that the optimum of ψ' is point x' inside the feasible region, then:

$$\sum < g_i(\mathbf{x}') >^2 + \sum h_j(\mathbf{x}')^2 = 0, \tag{7a}$$

$$\Rightarrow \psi'(\mathbf{x}', r'_e) = f(\mathbf{x}') + 0 + \frac{\alpha_e}{r'^2_e}. \tag{7b}$$

But f and $\frac{\alpha_e}{r'^2_e}$ are independent, so, ψ' is minimized when both these functions are minimized independently. The minimum of $(\frac{\alpha_e}{r'^2_e})$ is zero when $r'^*_e \to +\infty$

and the minimum of f inside the feasible region is obtained for $\mathbf{x} = \mathbf{x}^*$, hence $\mathbf{x}' = \mathbf{x}^*$.

Now, assume that $\mathbf{x}'$, the optimal answer to ψ', is outside the feasible region. Then:

$$\psi'(\mathbf{x}', r'_e) = f(\mathbf{x}') + r'_e \left(\sum < g_i(\mathbf{x}') >^2 + \sum h_j(\mathbf{x}')^2 \right) + \frac{\alpha_e}{r'^2_e}, \tag{8a}$$

ψ' is a function of r'_e. Its optimum is obtained by putting its first derivative equal to zero:

$$\frac{\partial \psi'}{\partial r'_e} = 0 \Rightarrow \sum < g_i(\mathbf{x}') >^2 + \sum h_j(\mathbf{x}')^2 - \frac{2\alpha_e}{r'^3_e} = 0. \tag{8b}$$

Then, solving for r'_e:

$$r'^*_e = \left(\frac{2\alpha_e}{\sum < g_i(\mathbf{x}') >^2 + \sum h_j(\mathbf{x}')^2} \right)^{\frac{1}{3}}, \tag{8c}$$

substituting r'_e in Equation 8a,

$$\Rightarrow \psi'^*(\mathbf{x}', r'_e) = f(\mathbf{x}') + \left(\frac{2\alpha_e}{\sum < g_i(\mathbf{x}') >^2 + \sum h_j(\mathbf{x}')^2} \right)^{\frac{1}{3}} \left(\sum < g_i(\mathbf{x}') >^2 + \sum h_j(\mathbf{x}')^2 \right) + \frac{\alpha_e}{\left(\frac{2\alpha_e}{\sum < g_i(\mathbf{x}') >^2 + \sum h_j(\mathbf{x}')^2} \right)^{\frac{2}{3}}}, \tag{8d}$$

$$\Rightarrow \psi'^*(\mathbf{x}', r'_e) = f(\mathbf{x}') + \left[(2\alpha_e)^{\frac{1}{3}} + \left(\frac{\alpha_e}{4} \right)^{\frac{1}{3}} \right] \left(\sum < g_i(\mathbf{x}') >^2 + \sum h_j(\mathbf{x}')^2 \right)^{\frac{2}{3}}. \tag{8e}$$

If α_e is assigned a sufficiently large value, so that the second term in Equation 8e is significantly larger than $f(\mathbf{x}')$, then the answer $\mathbf{x}'$ tends to be a point which is firstly on the constraints surface so that:

$$\sum < g_i(\mathbf{x}') >^2 + \sum h_j(\mathbf{x}')^2 = 0, \tag{8f}$$

and, secondly, for which $f(\mathbf{x}')$ is minimized; hence, $\mathbf{x}'$ is the solution to the original constrained optimization problem of the set of Equations 1.

New Interior Penalty Function

The new penalty function is defined as:

$$\psi'_{in}(\mathbf{x}, r_{in}) = f(\mathbf{x}) + r_{in} \left(\sum \frac{1}{g_i(\mathbf{x})} + . \sum \frac{1}{h_j(\mathbf{x})} \right) + u(r_{in}),$$
$$r_{in} \to 0, \tag{9a}$$

where the following function is proposed for $u(r_{in})$:

$$u(r_{in}) = \alpha_{in} r^2_{in}, \tag{9b}$$

where more accuracy is obtained by taking smaller values of α_{in}. The proof of convergence is similar to that of the new exterior penalty method.

EXAMPLE PROBLEMS

The results of application of the old and new exterior and interior penalty functions in conjunction with the steepest descent method are reported in this section.

Ten-bar Truss

The 10 bar truss benchmark problem, as shown in Figure 3a, which has been previously studied by other authors [4], has been optimized using both old and new penalty functions, and the results are compared.

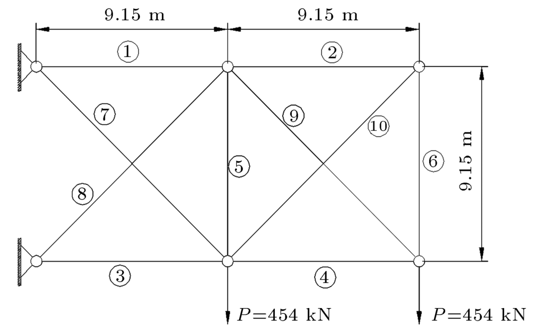
It has been desired to minimize the weight of the truss (W) as the objective function by changing $\mathbf{x}$ = the vector of cross sectional areas of the members, subject to stress constraints and minimum cross sectional area of 6.452×10^{-5} m^2 (0.1 in^2) for all the members. The maximum allowable stress has been the same in compression and tension equal to 172375 kPa, for all the members except bar 9 for which the allowable stress has been 517125 kPa [4]. The density and elastic modulus of the material have been assumed 2768 kg/m^3 and 6.895×10^7 kPa, respectively. The optimization problem can be formulated as:

Minimize:

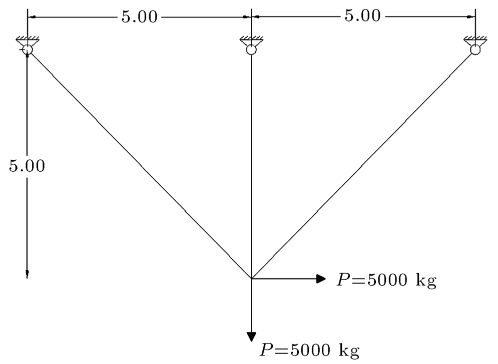
$$W(\mathbf{x}) = \text{weight} = \text{sum over weight of 10 bars}, \tag{10a}$$

subject to:

$$\sigma_i(\mathbf{x}) - 172375 \leq 0, \quad i = 1, 2, \cdots, 8, 10,$$



(a) 10-bar truss



(b) 3-bar truss

Figure 3. Truss problems.

$$-\sigma_i(\mathbf{x}) - 172375 \leq 0, \quad i = 1, 2, \cdots, 8, 10,$$

$$\sigma_9(\mathbf{x}) - 517125 \leq 0,$$

$$-\sigma_9(\mathbf{x}) - 517125 \leq 0,$$

$$-x_i + 6.452 \times 10^{-5} \leq 0, \quad i = 1, 2, ..., 10. \tag{10b}$$

Using the allowable stress design method, the optimum weight for this truss has been obtained as 680 gN [4], where $g = 9.81$ m/s^2 is the gravitational acceleration. The cross sectional areas have been:

$$\mathbf{x} = [50.97 \quad 0.65 \quad 52.26 \quad 25.16 \quad 0.65 \quad 0.65$$
$$37.42 \quad 35.55 \quad 23.74 \quad 0.90] \times 10^{-4} \text{ m}^2. \tag{11}$$

In the following sections, the old and new exterior and interior penalty functions for this example problem have been optimized by the steepest descent method. Next, in order to demonstrate the advantages of the new functions when combined with GA, the old and new exterior penalty functions have been solved by GA and a comparison has been made.

Optimization by Old and New Exterior Penalty Functions

The old and new exterior penalty functions have been defined as:

$$\psi(\mathbf{x}, r_e) = 2768 \times \left(9.15 \times \sum\nolimits_1^6 x_i + 9.15\sqrt{2}\sum\nolimits_7^{10} x_i\right)$$
$$+ r_e \times \left(\sum\nolimits_1^8 < \frac{|\sigma_i| - 172375}{1723.75} >^2\right.$$
$$+ < \frac{|\sigma_9| - 517125}{5171.25} >^2 + < \frac{|\sigma_{10}| - 172375}{1723.75} >^2$$
$$\left. + \sum\nolimits_1^{10} < \frac{x_i - 6.452}{0.06452} >^2\right), \tag{12a}$$

and:

$$\psi'(\mathbf{x}, r'_e) = 2768 \times \left(9.15 \times \sum\nolimits_1^6 x_i + 9.15\sqrt{2}\sum\nolimits_7^{10} x_i\right)$$
$$+ r'_e \times \left(\sum\nolimits_1^8 < \frac{|\sigma_i| - 172375}{1723.75} >^2\right.$$
$$+ < \frac{|\sigma_9| - 517125}{5171.25} >^2$$
$$+ < \frac{|\sigma_{10}| - 172375}{1723.75} >^2$$
$$\left. + \sum\nolimits_1^{10} < \frac{x_i - 6.452}{0.06452} >^2\right) + \frac{\alpha_e}{r'^2_e}, \tag{12b}$$

respectively. For both cases, minimization by the steepest descent method has been utilized. The gradient of the exterior penalty function has been calculated numerically by calculating the change in ψ because of a small change Δ in the ith design parameter. Taking $\Delta = 0.001$ in^2, the slope of ψ with respect to the ith parameter can be calculated as follows:

$$\frac{\partial\psi}{\partial x_i} = \frac{\psi(x_1, \cdots, x_{i-1}, x_i + 0.001, x_{i+1}, \cdots, x_n) - \psi(x)}{0.001}. \tag{12c}$$

Also, r_e has been increased after each complete round of optimization according to Equations 2d and 2e:

$$r_{e_{i+1}} = 5 r_{e_i}, \tag{12d}$$

where i represents the ith round of optimization.

The result of minimization is shown in Figures 4a and 5a, where the weight of the truss and r_e are plotted as functions of N_s= number of steps of minimization by the steepest descent method. As can be seen, after

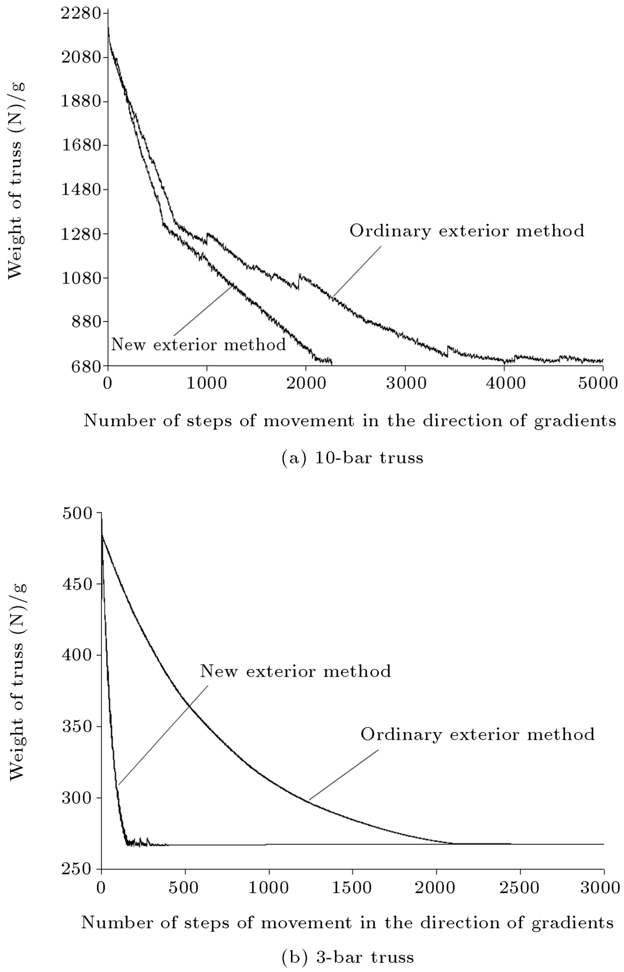


Figure 4. Convergence to final solution using old and new exterior penalty functions.

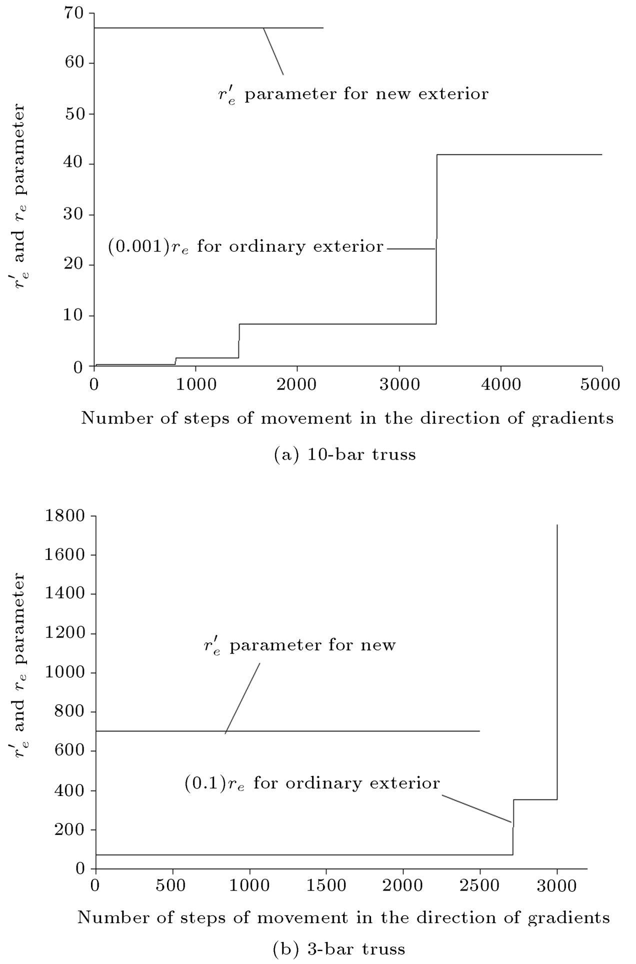


Figure 5. History of change in response surface factor for the ordinary and new exterior penalty functions.

6 rounds of changing r_e, which are comprised of a total number of 5000 steps of movement in the steepest descent directions (N_s=5000), the minimum weight, design vector and r_e have been:

$$W_e^* = 697.36\text{g}N, \tag{12e}$$

$$\mathbf{x}_e^* = [50.32 \quad 2.45 \quad 54.00 \quad 24.32 \quad 3.94 \quad 1.74$$
$$39.29 \quad 34.19 \quad 25.81 \quad 2.84] \times 10^{-4}\ \text{m}^2, \tag{12f}$$

$$r_e = 209820. \tag{12g}$$

After $N_s = 4000$ steps, the increase in r_e has not provided much improvement and the reduction in weight after 3700 steps has been just 2.5%.

To solve the problem, by using the new exterior penalty function, the gradient of the new penalty function has also been calculated from Equation 12c by ψ' replacing ψ. Also, r'_e has been considered as an additional variable. The result of using the new penalty function (Relations 4a to 4f) has been shown in Figures 4a and 5a. In this case,

$$\alpha_e = 1000, \tag{13a}$$

has been used in Equation 5 and r'_e has been changed as a design variable, in order to help minimize the weight. The convergence of the solution has been much smoother and monotonic in the case of the new function. After 2260 steps, the weight, design vector and r'_e have been:

$$W_e'^* = 681.1\ \text{gN}, \tag{13b}$$

$$\mathbf{x}_e'^* = [48.84 \quad 2.19 \quad 53.81 \quad 23.23 \quad 1.16 \quad 2.26$$
$$39.55 \quad 32.71 \quad 22.00 \quad 3.10] \times 10^{-4}\ \text{m}^2, \tag{13c}$$

$$r'_e = 67.03. \tag{13d}$$

The new function has been more successful than the old exterior penalty function. Referring to Figure 4, it is noteworthy that the curve of the new penalty function is always below the curve of the ordinary penalty function. Hence, the result has always been better, regardless of the number of steps.

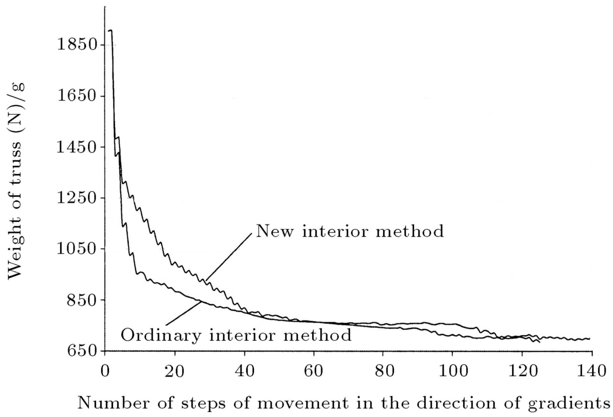
While r_e has increased monotonically and stepwise, r'_e has changed in order to help minimization more effectively. However, in this problem, the change in r'_e has been slight and it only has changed 1.7%.

To study how dependent on α_e the result could be, the problem was solved by selecting different values for α_e, where in Figure 6a the value of W'^*_e has been plotted against the number of steps for α_e= 1, 100 and 10000. The result has not been sensitive to the value of α_e.

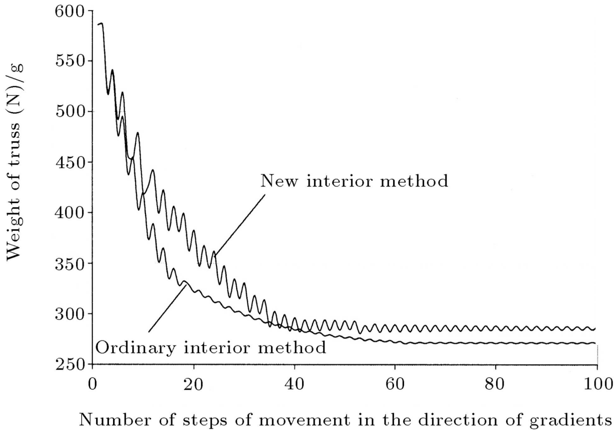
Optimization by Old and New Interior Penalty Functions

The same procedure as for the exterior and new exterior penalty functions has been repeated for the interior and new interior penalty functions and the results have been shown in Figure 7a. Again, the new interior penalty function has proven to be more successful than the old interior penalty function. r_{in} has been reduced monotonically and stepwise, however, r'_{in} has been optimally changed to help the minimization procedure. It is noteworthy that r'_{in} has changed mainly monotonically in this example, though this observation might not be generalized to other cases. The optimum solutions have been W^*_{in} = 694 gN after about 145 steps for the interior and W'^*_{in} = 685.4 gN after about 125 steps for the new interior penalty functions.

Also, Figure 8a shows the change in r_{in} and r'_{in} through the procedure. The effect of α_{in} on the result has been investigated too as depicted in Figure 9a. Convergence has not been achieved if a large α_{in} (for this example α > 20 has been considered large) has been selected. Using smaller α_{in}, it is possible to get a better answer with more accuracy, though it requires more time to converge.

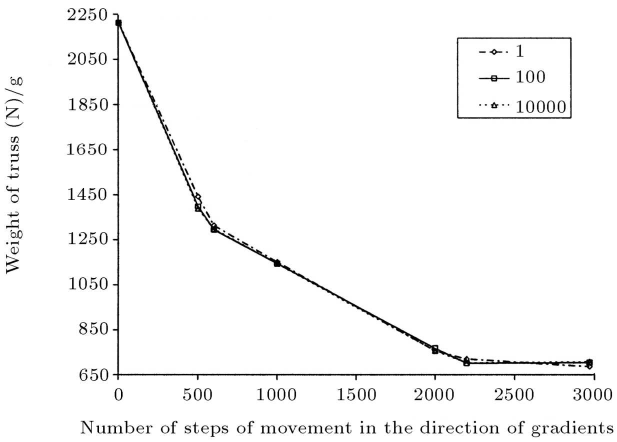


(a) 10-bar truss

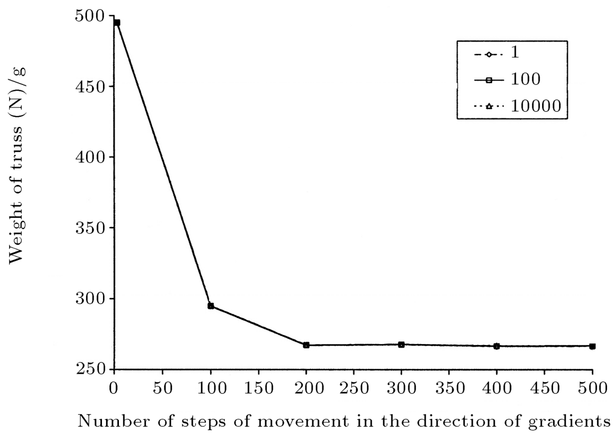


(b) 3-bar truss

Figure 7. Comparison between old and new interior methods.



(a) 10-bar truss



(b) 3-bar truss

Figure 6. Effect of different α_e values on result of new exterior penalty function.

Three-bar Truss

The 3 bar truss benchmark problem shown in Figure 3b has been optimized using both the old and new penalty functions and the results are compared. This problem has been included only to show that the method could be easily applied to other truss problems too.

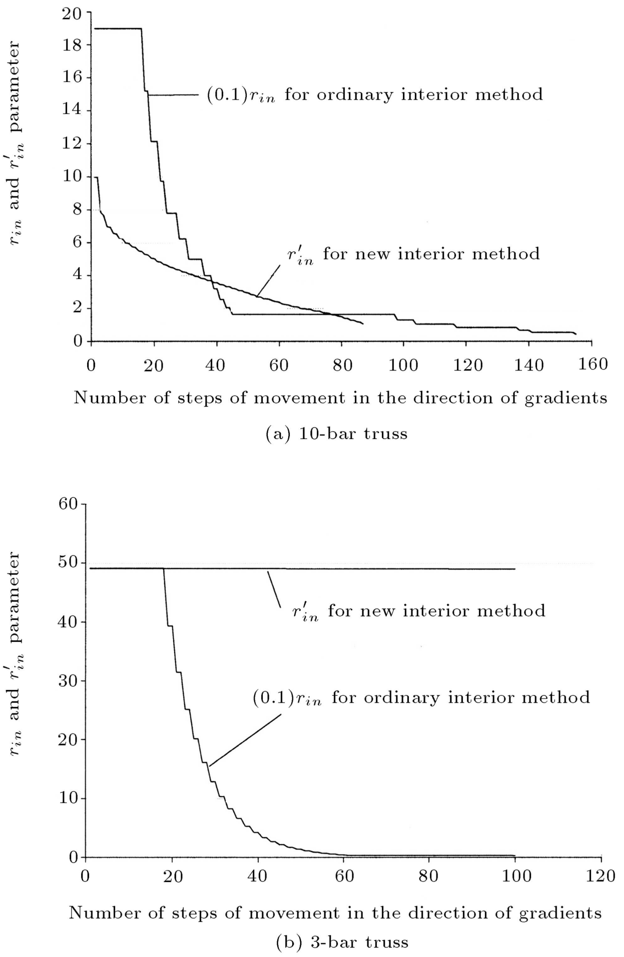


Figure 8. History of change in response surface factor for the ordinary and new interior penalty functions.

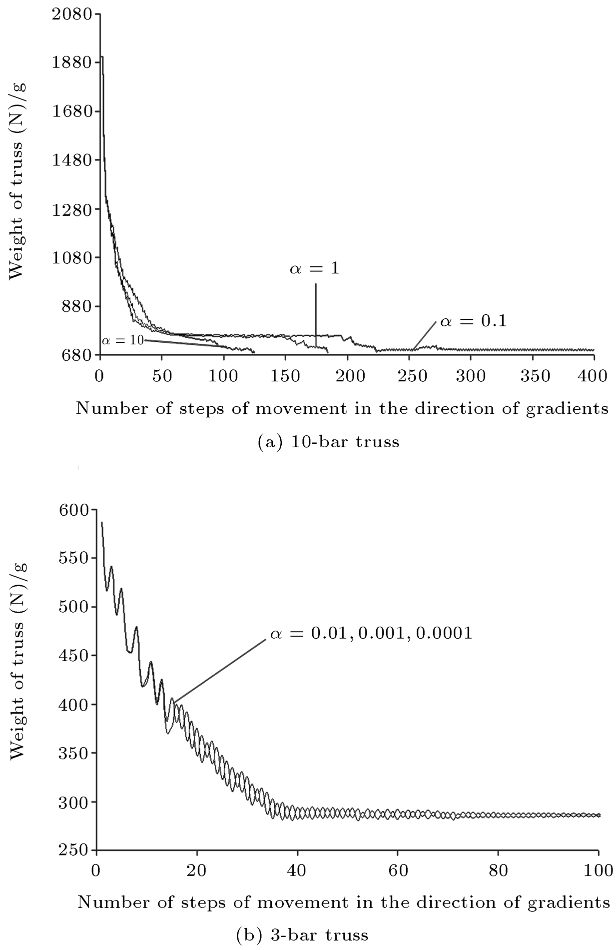


Figure 9. Effect of different α_{in} values on convergence in new interior method.

Again, the objective function has been the weight of the truss to be minimized, subject to stress constraints and a minimum cross sectional area of 10^{-4} m^2 for all the members. The maximum allowable stress has been the same under compression and tension equal to 144000 kPa = 1440 kg/cm^2 for all the members. The density and elastic modulus of the material have been assumed to be 7800 kg/m^3 and 2.04×10^8 kPa, respectively, which are those of steel. The optimization problem can be formulated as:

Minimize:

$$W(\mathbf{x}) = \text{weight} = \text{sum over weight of 10 bars}, \tag{14a}$$

subject to:

$$\sigma_i(\mathbf{x}) - 144000 \leq 0, \quad i = 1, 2, 3,$$

$$-\sigma_i(\mathbf{x}) - 144000 \leq 0, \quad i = 1, 2, 3,$$

$$-x_i + 10^{-4} \leq 0, \quad i = 1, 2, 3. \tag{14b}$$

Using the allowable stress design method, the optimum weight for this truss has been obtained as 267 N. The cross sectional areas have been:

$$\mathbf{x} = [4.9 \quad 0.04 \quad 0.02] \times 10^{-4}\text{m}^2. \tag{15}$$

Optimization by Old and New Exterior Penalty Functions

The result of minimization is shown in Figures 4b and 5b, where the weight of the truss and r_e are plotted as functions of N_s= number of steps of minimization by the steepest descent method. As can be seen, after 3 rounds of changing r_e, which are comprised of a total number of 2500 steps of movement in the steepest descent directions (N_s = 2500), the minimum weight, design vector and r_e have been:

$$W_e^* = 267.6 \text{ N}, \tag{16a}$$

$$\mathbf{x}_e^* = [4.899 \quad 0.04 \quad 0.017] \times 10^{-4}\text{m}^2, \tag{16b}$$

$$r_e = 17000. \tag{16c}$$

The results and comparison of the old and new exterior penalty functions can be seen in Figures 4b and 5b. In this case,

$$\alpha_e = 1000. \tag{17a}$$

The convergence has been much faster in the case of the new function. After 500 steps, the weight, design vector and r'_e have been:

$$W'^{*}_e = 267.3\text{N}, \tag{17b}$$

$$\mathbf{x}^*_e = [4.90 \quad 0.037 \quad 0.012] \times 10^{-4}\text{m}^2, \tag{17c}$$

$$r'_e = 703. \tag{17d}$$

The new function has been more successful than the old exterior penalty function. Again, as shown in Figure 4b, the curve of the new penalty function has always been below the curve of the ordinary penalty function, hence, the result has always been better, regardless of the number of steps.

The effect of α_e can be seen in Figure 6b, where the value of W'^{*}_e has been plotted against the number of steps for $\alpha_e = 1$, 1000 and 100000. The plots are not distinguishable and, hence, the result has not been sensitive to the value of α_e.

Optimization by Old and New Interior Penalty Functions

The results of application of the new and old ordinary penalty functions have been as shown in Figure 7b. The optimum solutions have been $W^*_{in} = 271$ N after about 100 steps for the ordinary interior and $W'^{*}_{in} = 287$ N after about 80 steps for the new interior penalty functions where the new interior penalty method has not been as successful as the ordinary interior penalty method.

Also, Figure 8b shows the change in r_{in} and r'_{in} through the procedure. The effect of α_{in} on the result has been investigated too as depicted in Figure 9b, showing that convergence has not be achieved for large values of α_{in} (in this example $\alpha > 10$ has been considered large).

SOLUTION BY GENETIC ALGORITHMS

One category of optimization method, which could enjoy the benefits of using the new penalty functions, is the Genetic Algorithm (GA). Noticing most engineering problems are of constrained type, it is generally cumbersome to solve such problems by GA using the old penalty functions in which a new r value (r_e for exterior and r_{in} for interior penalty function) should be introduced for every new round of optimization. Using the new functions though, it is possible to include r (r'_e or r'_{in} for exterior and interior penalty respectively) as an additional gene in the chromosomes of variables, so that r is also optimized during the optimization.

There are many references to the application of GA to different design optimization problems, such as [12-17]. In this paper, the method explained in [18] has been adopted and used to solve the truss problems of Figure 3. Figure 10 depicts the flowchart of the GA approach in structural optimization. [12]. The genetic operation shown in Figure 10a is described in more detail in Figure 10b. Also, the flowchart of continuous GA was shown in Figure 11 [18].

Here, the GA has been used to minimize both the old and new exterior penalty functions of the example problems.

GA with the Old Exterior Penalty Function

The 10 and 3 bar trusses have been optimized by the old exterior penalty functions in conjunction with the GA.

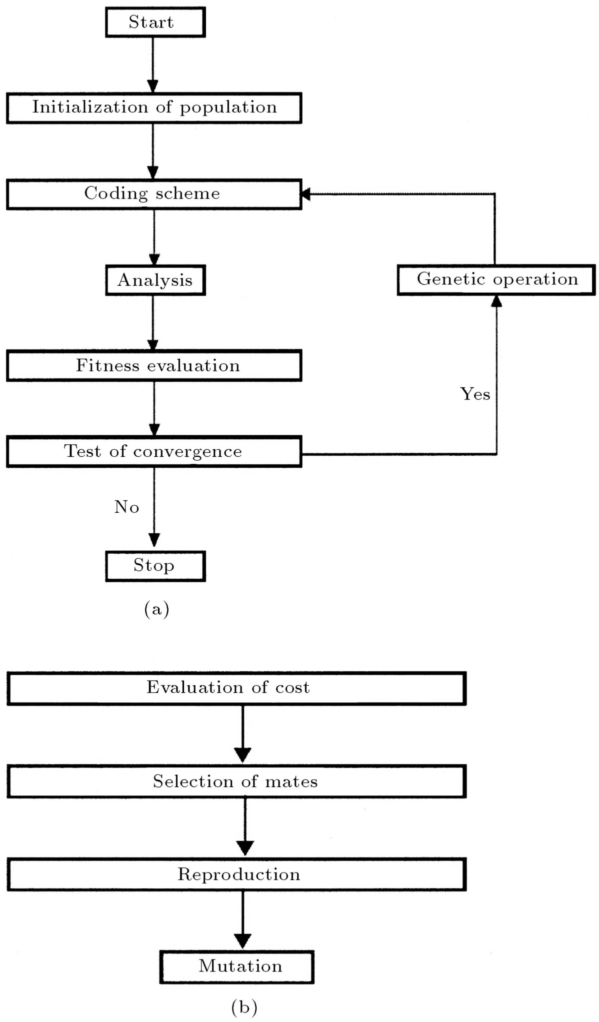


Figure 10. (a) Flowchart of GA approach; (b) Details of genetic operation box.

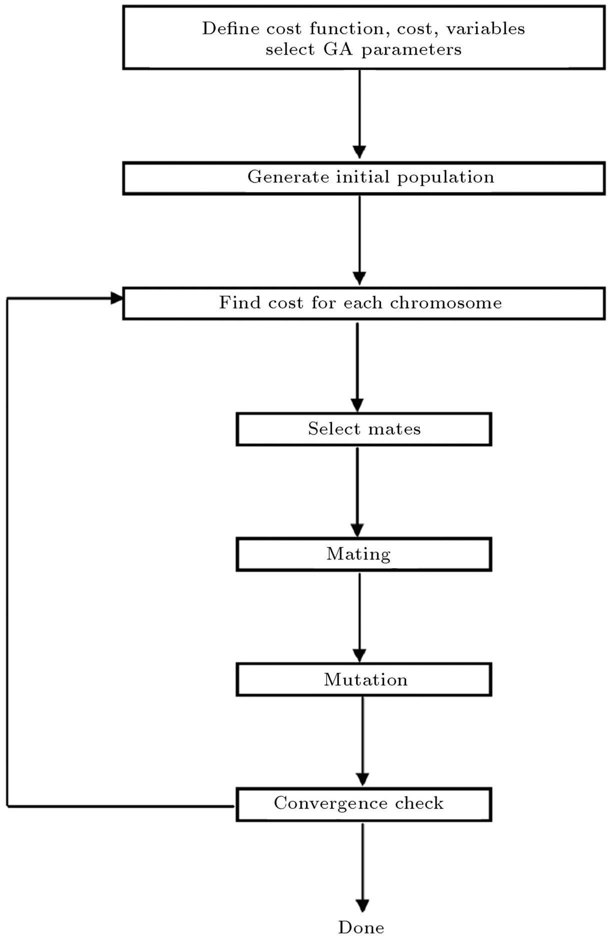


Figure 11. Flowchart of a continuous GA.

For the 10-bar truss the chromosome has been defined as:

$$\text{choromosome} = [x_1, x_2, ..., x_{10}], \tag{18a}$$

where x_i = area of the ith bar, which is considered as the ith gene of the chromosome. The selected parameters for use in Equations 2a and 2b have been:

$$c = 5, \tag{18b}$$

$$r_1 = 1. \tag{18c}$$

The size of the population, which has been considered fixed throughout the optimization, has been $N_{pop} = 48$. The roulette wheel strategy has been used to select the pairs for mating where at each generation N_{pairs} of pairs of chromosomes have been drawn from the pool. Denoting a pair of parents by:

$$\text{parent}_1 = [x_{m,1}x_{m,2}x_{m,3}...x_{m,l}...x_{m,Npar}], \tag{18d}$$

$$\text{parent}_2 = [x_{f,1}x_{f,2}x_{f,3}...x_{f,l}...x_{f,Npar}], \tag{18e}$$

where the subscripts m and f stand for male and female, one of the genes is selected randomly. Assuming the lth gene is selected, the first $(l-1)$ genes in each parent are kept unchanged, while the remaining genes are combined to form two newborn chromosomes. Denoting the two newborns by $n1$ and $n2$, they can be shown as:

$$n1 = [x_{n1,1}x_{n1,2}x_{n1,3}...x_{n1,l}...x_{n1,Npar}], \tag{18f}$$

$$n2 = [x_{n2,1}x_{n2,2}x_{n2,3}...x_{n2,l}...x_{n2,Npar}], \tag{18g}$$

where their genes have the following values:

$$x_{n1,i} = x_{m,i} \qquad i < l, \tag{18h}$$

$$x_{n2,i} = x_{f,i} \qquad i < l, \tag{18i}$$

$$x_{n1,i} = x_{m,i} - \beta[x_{m,i} - x_{f,i}] \qquad i > l, \tag{18j}$$

$$x_{n2,i} = x_{f,i} - \beta[x_{f,i} - x_{m,i}] \qquad i > l, \tag{18k}$$

where β is also a randomly generated real number in the interval [0.0, 1.0].

In the 10-bar truss example, a mutation rate of 10% has been applied to the population. Also, a mutation rate of 1% has been applied to the 3-bar truss optimization.

Figures 12a and 12b show the changes in the value of the exterior penalty function and the corresponding weight, as functions of the generation number for the 10 and 3 bar trusses, respectively. Also, Figures 13a and 13b plot the change in r_e against the generation number.

GA with the New Exterior Penalty Function

In the case of the 10 bar truss, a chromosome has been defined as:

$$\text{choromosome} = [x_1, x_2, ..., x_{10}, r'], \tag{19}$$

where r'_e has been included as an additional gene in the chromosome. It has one more gene (r'_e) than in the ordinary exterior method. The results of optimization are reported in Figures 12a and 13a where the values of the penalty function, weight and r'_e are shown against the generation number. It should be noticed that different scales have been used for r_e and r'_e to show both on the same figure. It is noteworthy that r'_e has been much smaller than r_e. For the 10-bar truss, the best results have been W^*_e = 759.8 gN after 170 generations, using the ordinary and W'^*_e = 684.9 gN after 46 generations, using the new exterior penalty function, which shows a considerable reduction of about 10% in the weight. Also, for the 3-bar truss, the best results have been W^*_e = 280.3 N after 140 generations, using the ordinary, and W'^*_e = 277.8 N after 140 generations, using the new exterior penalty functions where the result for the new penalty function

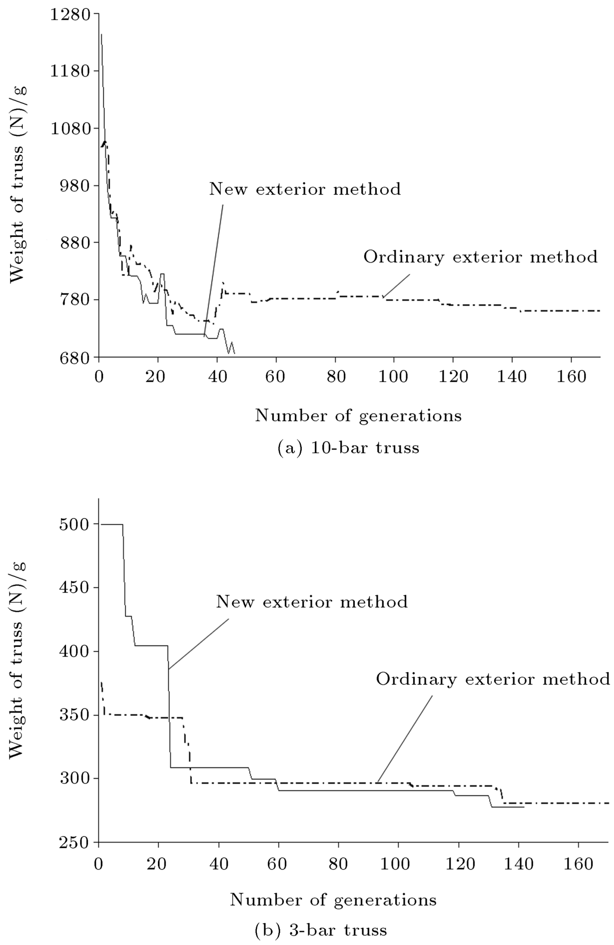


Figure 12. Convergence using ordinary and new exterior penalty functions in GA.

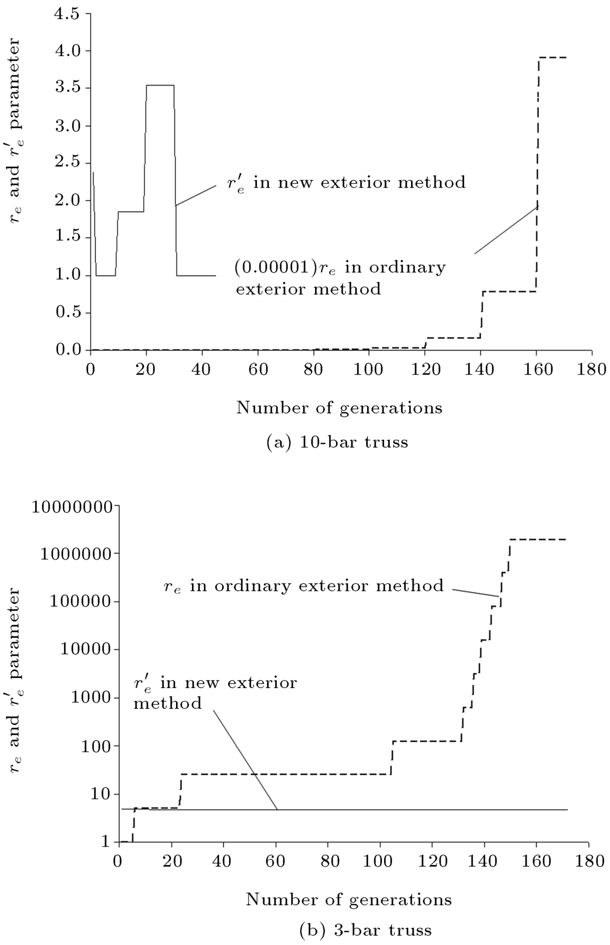


Figure 13. History of change in response surface factor for the ordinary and new exterior penalty functions in GA.

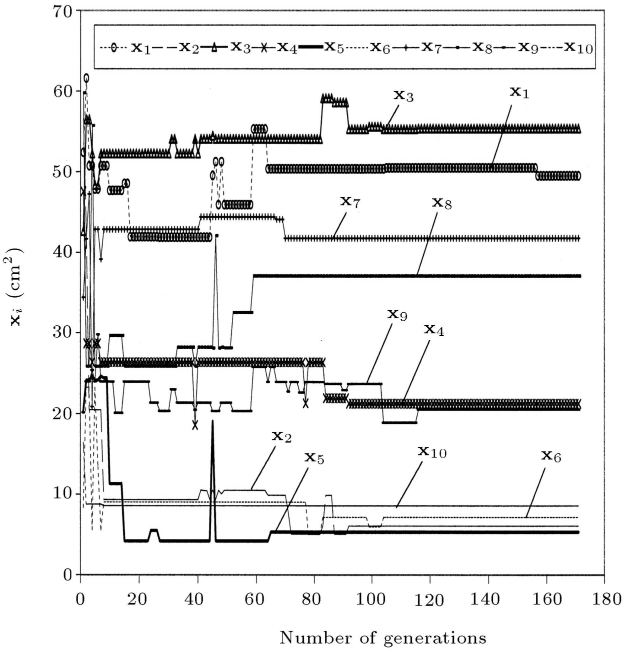
has been slightly better. Figures 12b and 13b show the history of changes in the weight and response factor. Also, changes in x_i values through generation by ordinary exterior and new exterior methods are reported in Figures 14 and 15.

CONCLUSIONS

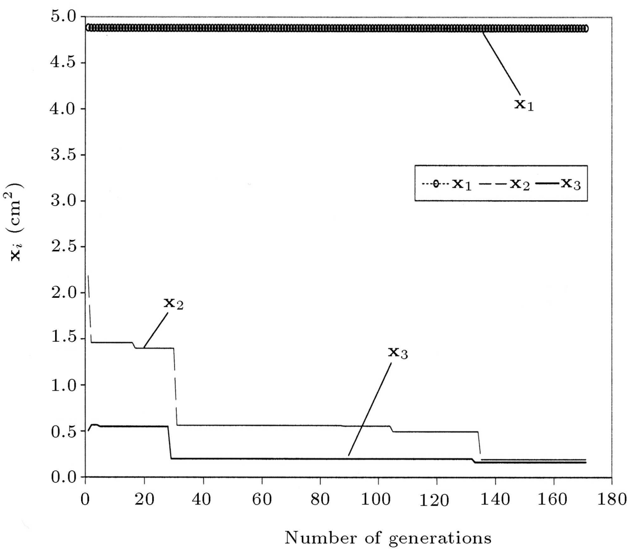
In this paper, the penalty parameter in the ordinary exterior and interior penalty functions has been considered as a design variable and optimized together with the main variables to achieve the best result. This technique has both improved the convergence speed and the final result. Testing on the 10-bar benchmark problem, the ordinary exterior penalty function converged to $W_e^* = 697.4$ gN after about 5000 steps while the new penalty function converged to $W_e'^* = 681.1$ gN in less than 2300 steps, proving that the convergence has been more than twice as fast and the answer has been about 2.3% better. Using the ordinary and new interior penalty functions, the results have been $W_{in}^* = 694$ gN after about 145 steps for the interior, and $W_{in}'^* = 685.4$ gN, after about 125 steps for the new interior penalty functions. Again, the convergence has been 1.25 times faster and the final answer has been 1.3% better.

One area that might enjoy the benefit of using the new penalty functions is Genetic Algorithms (GA). The 10-bar truss problem has been solved by GA using the ordinary and new exterior penalty functions. The best results have been $W_e^* = 759.8$ gN, after 170 generations using the ordinary and, $W_e'^* = 684.9$ gN, after 46 generations using the new exterior penalty functions. The convergence has been 4 times faster and the result has been 11% better using the new exterior penalty function.

To show that the obtained answers have not been limited to the 10-bar truss, a 3-bar truss has also been optimized by the same methods. The results have shown that, while the new exterior penalty function has been obviously successful, the optimum weight obtained from the new interior method has even been slightly higher. Noticing that the new interior penalty has been able to provide slightly better results in the case of the 10-bar truss, it is expected that the new interior penalty method would be less

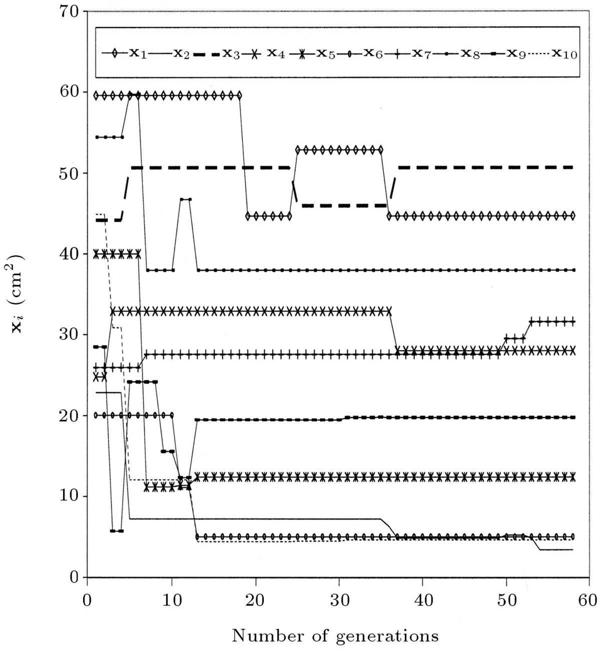


(a) 10-bar truss

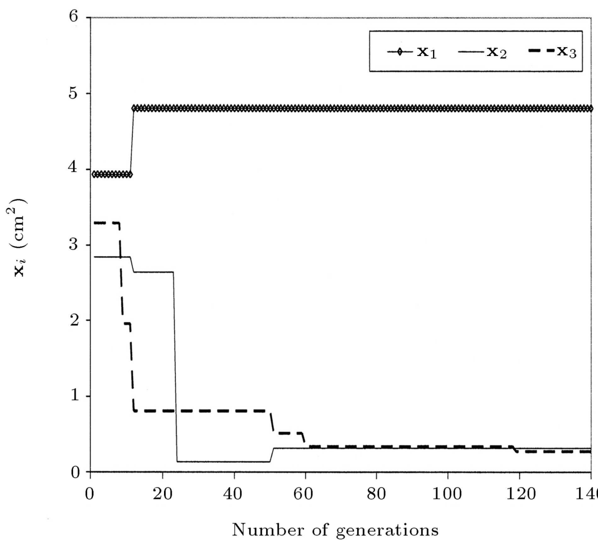


(b) 3-bar truss

Figure 14. Change in x_i values through generations by ordinary exterior method.



(a) 10-bar truss



(b) 3-bar truss

Figure 15. Change in x_i values through generations by new exterior method.

effective, as compared to the new exterior penalty function. However, both the new methods of interior and exterior penalty functions have proven to be helpful methods to be used together with the GA. More research on the subject is in progress by the authors.

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