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Second-Order Displacement Functions for Three-Dimensional Discontinuous Deformation Analysis (3-D DDA)

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Abstract. *The development of second-order displacement functions for a Three-Dimensional Discontinuous Deformation Analysis (3-D DDA) is made by incorporating the complete second order terms. Formulations of stiffness and force matrices in second-order due to elastic stress, initial stress, point load, body force, inertia force and fixed point are derived. Two illustrative examples of 3-D beams subjected to various loads are used to validate the new formulations and code. By contrast, the results calculated for the same model by use of the original first-order 3-D DDA are far from the theoretical solutions.*

Keywords: *Numerical method; Three-Dimensional Discontinuous Deformation Analysis (3-D DDA); Second-order displacement functions; Rock mechanics.*

INTRODUCTION

Numerical methods applicable to rock mechanics can be placed in two main categories: (1) Continuum approaches such as Finite Element Method (FEM) and Boundary Element Method (BEM) in which the effects of discontinuities are equivalently included in a rock mass; (2) Discontinuum approaches such as Distinct Element Method (DEM) and Discontinuous Deformation Analysis (DDA) in which a rock mass is regarded as an assemblage of discrete blocks. Although rock mass discontinuities can be modeled in a discrete manner with FEM and BEM using special joint elements, the description of discontinuities is usually difficult and there are often restrictions on the degree of deformation permitted [1]. Furthermore, the number of discontinuities that can be handled is limited. On the other hand, the Discrete Element Method (DEM) is generally tailored for problems in which there are many material discontinuities placing special emphasis on how the contacts are handled. It also allows for large deformations along discontinuities and can reproduce block movements (translation and rotation) quite well [2].

The Discontinuous Deformation Analysis (DDA) method is a recently developed technique that is a member of the family of DEM methods. It is a pioneering method used to analyze the mechanical behavior of discrete blocks. In contrast, DDA as a complete block kinematics (a key component in dealing with interacting discrete blocks) guarantees the system equilibrium at any time and is a real-time analysis. Both static and dynamic analyses can be conducted with the DDA method [3].

The method has the following major characteristics [3]:

1. The principle of minimum total potential energy is used to calculate an approximate solution similar to FEM;
2. Dynamic and static problems can be solved by applying the same formulations;
3. Any constitutive law can be incorporated;
4. Any contact criterion (i.e., Mohr-Coulomb criterion), boundary condition (i.e., constraint displacement) and loading condition (i.e., initial stress, inertia force, volume force, etc.) can be modeled.

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Original DDA formulation utilizes first order displacement functions to describe the block movement and deformation. Therefore, stress or strain is assumed constant through the block and the capability of block deformation is limited. This may yield unreasonable results when the block deformation is large and the geometry of the block is irregular. In 2-D, to overcome these limitations, some approaches have been attempted. An approach to resolve this problem was to glue small blocks together to form a larger block using artificial joints [4-6] and sub-blocks [7-9]. Some researchers added finite element meshes in the blocks, so that stress variations within the blocks could be accounted for [10-12]. An alternative approach is to include more polynomial terms in the displacement function. Chern et al. [13] and Koo et al. [14] added the second-order to DDA. Ma et al. [15] and Koo and Chern [16] implemented the third-order displacement function in the 2-D DDA method. Hsiung [17] developed a more general formulation of the 2-D DDA. The high-order displacement functions made possible the accurate modeling of complicated stress and strain fields in blocks.

In 3-D, there are some works but they use a linear displacement function.

In this paper, the 3-D DDA with second-order displacement functions is derived. The details for the second-order 3-D DDA are given for program coding and used to calculate three 3-D beams deformation under various forces and an example of discontinuous problem.

FUNDAMENTAL OF 3-D DDA

In the DDA method, the formulation of blocks is very similar to the definition of a finite element mesh. A problem of the finite element type is solved in which all elements of physically isolated blocks are bounded by pre-existing discontinuities. When the blocks are in contact, Coulomb's law is applied to the contact interfaces and the simultaneous equilibrium equations are formed and solved at each loading or time increment. The large displacements are the accumulation of incremental displacements and deformation at each time step. Within each time step, the incremental displacements of all points are small and, hence, the displacements can be reasonably represented by the first order approximation [18].

Block Deformation and Displacement

The motion of an arbitrary point in a 3-D block can be divided into translations, rotations, normal strains and shear strains as shown in Figure 1.

Hence, the unknown degree of freedom consists of 12 unknowns, or 3 of each of these terms, to describe the motion of a 3-D block. In addition, the block displacements function is equivalent to the complete first-order displacement approximation; constant strains and stresses are assumed within each block. The complete first-order displacement function has the following form:

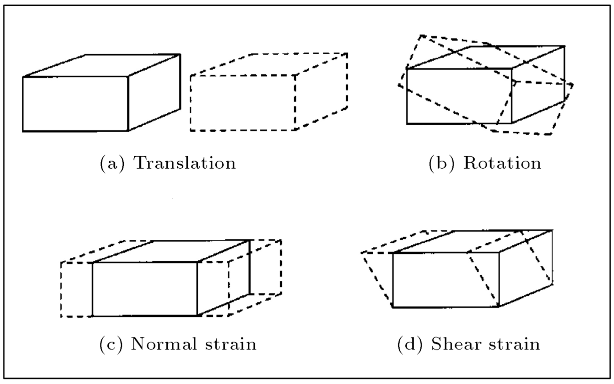

Figure 1. Three-dimensional displacements of block [18].

$$\begin{cases} u = a_0 + a_1 x + a_2 y + a_3 z \\ \nu = b_0 + b_1 x + b_2 y + b_3 z \\ w = c_0 + c_1 x + c_2 y + c_3 z \end{cases} \tag{1}$$

where, u, ν and w are the displacements of a point within the block in the X, Y, and Z directions; x, y, and z are the coordinates of a point within the block; $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3$ and c_0, c_1, c_2, c_3 are unknown parameters.

Assume that the coordinates x_c, y_c, z_c are the center of gravity of a block, and u_0, ν_0, w_0 are its displacements; substituting them into Equation 1 results in:

$$\begin{cases} u_c = a_0 + a_1 x_c + a_2 y_c + a_3 z_c \\ \nu_c = b_0 + b_1 x_c + b_2 y_c + b_3 z_c \\ w_c = c_0 + c_1 x_c + c_2 y_c + c_3 z_c \end{cases} \tag{2}$$

Subtracting Equation 2 from Equation 1 gives:

$$\begin{cases} u = u_c + a_1(x - x_c) + a_2(y - y_c) + a_3(z - z_c) \\ \nu = \nu_c + b_1(x - x_c) + b_2(y - y_c) + b_3(z - z_c) \\ w = w_c + c_1(x - x_c) + c_2(y - y_c) + c_3(z - z_c) \end{cases} \tag{3}$$

The rotations of a block can be expressed as:

$$\begin{cases} r_x = \frac{1}{2}(\frac{\partial w}{\partial y} - \frac{\partial \nu}{\partial z}) = \frac{1}{2}(c_2 - b_3) \\ r_y = \frac{1}{2}(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}) = \frac{1}{2}(a_3 - c_1) \\ r_z = \frac{1}{2}(\frac{\partial \nu}{\partial x} - \frac{\partial u}{\partial y}) = \frac{1}{2}(b_1 - a_2) \end{cases} \tag{4}$$

The normal strains are given by:

$$\begin{cases} \varepsilon_x = \frac{\partial u}{\partial x} = a_1 \\ \varepsilon_y = \frac{\partial \nu}{\partial y} = b_2 \\ \varepsilon_z = \frac{\partial w}{\partial z} = c_3 \end{cases} \tag{5}$$

and the shear strains are given by:

$$\begin{cases} \frac{1}{2}\gamma_{yz} = \frac{1}{2}(\frac{\partial w}{\partial y} + \frac{\partial \nu}{\partial z}) = \frac{1}{2}(c_2 + b_3) \\ \frac{1}{2}\gamma_{zx} = \frac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) = \frac{1}{2}(a_3 + c_1) \\ \frac{1}{2}\gamma_{xy} = \frac{1}{2}(\frac{\partial \nu}{\partial x} + \frac{\partial u}{\partial y}) = \frac{1}{2}(b_1 + a_2) \end{cases} \tag{6}$$

Hence, the parameters in Equation 1 can be computed as:

$$\begin{cases} a_1 = \varepsilon_x, \quad b_2 = \varepsilon_y, \quad c_3 = \varepsilon_z \\ c_2 = \frac{1}{2}\gamma_{yz} + r_x, \quad b_3 = \frac{1}{2}\gamma_{yz} - r_x, \quad a_3 = \frac{1}{2}\gamma_{zx} + r_y \\ c_1 = \frac{1}{2}\gamma_{zx} - r_y, \quad b_1 = \frac{1}{2}\gamma_{yz} + r_z, \quad a_2 = \frac{1}{2}\gamma_{xy} - r_z \end{cases} \tag{7}$$

Equation 3 can be written as:

$$\begin{bmatrix} u(x,y,z) \\ \nu(x,y,z) \\ w(x,y,z) \end{bmatrix} = [T(x,y,z)]\{D\}, \tag{8}$$

where:

$$[T_i] =$$

$$\begin{bmatrix} 1 & 0 & 0 & -y' & 0 & z' \\ 0 & 1 & 0 & x' & -z' & 0 \\ 0 & 1 & 1 & 0 & y' & -x' \end{bmatrix}$$

$$\begin{matrix} x' & 0 & 0 & y'/2 & 0 & z'/2 \\ 0 & y' & 0 & x'/2 & z'/2 & 0 \\ 0 & 0 & z' & 0 & y'/2 & x'/2 \end{matrix}\Bigg],$$

and:

$$x' = x - x_0, \quad y' = y - y_0, \quad z' = z - z_0,$$

$$[D_i] = (u_0, \nu_0, w_0, r_x, r_y, r_z, \varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})^T.$$

Using this equation, the displacements of arbitrary points in a block can be calculated with $\{D\}$, indicating a vector of unknowns or variables representing the displacements and deformations of a block.

Simultaneous Equations

Since 3-D DDA conforms to the principle minimum total potential energy, the total potential energy is the summation of all potential energy sources for each block such as those contributed by a) the elastic deformation of the blocks; b) the initial stresses; c) the point load on a block; d) the inertia forces; e) the constraint displacement points. The fixed point is the point where the prescribed constraint displacement equals zero in DDA.

For a system with N blocks, the total potential energy can be expressed in matrix form as follows:

$$\pi = \frac{1}{2}\left[\{D_1\}^T \quad \{D_2\}^T \quad \{D_3\}^T \quad \cdots \quad \{D_N\}^T\right]$$

$$\begin{bmatrix} [K_{11}] & [K_{12}] & [K_{13}] & \cdots & [K_{1N}] \\ [K_{21}] & [K_{22}] & [K_{23}] & \cdots & [K_{2N}] \\ [K_{31}] & [K_{32}] & [K_{33}] & \cdots & [K_{3N}] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ [K_{N1}] & [K_{N2}] & [K_{N3}] & \cdots & [K_{NN}] \end{bmatrix} \begin{bmatrix} \{D_1\} \\ \{D_2\} \\ \{D_3\} \\ \vdots \\ \{D_N\} \end{bmatrix}$$

$$+ \left[\{D_1\}^T \quad \{D_2\}^T \quad \{D_3\}^T \quad \cdots \quad \{D_N\}^T\right]$$

$$\begin{bmatrix} \{F_1\} \\ \{F_2\} \\ \{F_3\} \\ \vdots \\ \{F_N\} \end{bmatrix} + C, \tag{9}$$

where $\{D_i\}$ represent the displacement variables and $\{F_i\}$ indicates the loading and moments caused by the external forces and stress acting on block i. The stiffness submatrices $[K_{ii}]$ depend on the material properties of block i with $[K_{ij}]_{i \neq j}$, being defined by the contacts between blocks i and j; and C is the energy produced by the friction force. There are 12 displacement variables for each block. As a result, $\{D_i\}$ and $\{F_i\}$ are 12×1 matrices and $[K_{ij}]$ is a 12×12 matrix.

By minimizing the total energy, the simultaneous equations can be expressed in matrix form as follows:

$$\begin{bmatrix} [K_{11}] & [K_{12}] & [K_{13}] & \cdots & [K_{1N}] \\ [K_{21}] & [K_{22}] & [K_{23}] & \cdots & [K_{2N}] \\ [K_{31}] & [K_{32}] & [K_{33}] & \cdots & [K_{3N}] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ [K_{N1}] & [K_{N2}] & [K_{N3}] & \cdots & [K_{NN}] \end{bmatrix}$$

$$\begin{bmatrix} \{D_1\} \\ \{D_2\} \\ \{D_3\} \\ \vdots \\ \{D_N\} \end{bmatrix} = \begin{bmatrix} \{F_1\} \\ \{F_2\} \\ \{F_3\} \\ \vdots \\ \{F_N\} \end{bmatrix}. \tag{10}$$

For only one block, the equilibrium equations for each time step are derived by minimizing the total potential energy, π, in each variable. For block i the following equations:

$$\frac{\partial \pi}{\partial u} = 0, \qquad \frac{\partial \pi}{\partial \nu} = 0, \qquad \frac{\partial \pi}{\partial w} = 0, \tag{11}$$

represent the equilibrium of all loads and contact forces acting on block i along X, Y and Z directions, respectively. The following equations:

$$\frac{\partial\pi}{\partial r_x}=0,\quad \frac{\partial\pi}{\partial r_y}=0,\quad \frac{\partial\pi}{\partial r_z}=0, \tag{12}$$

represent the moment equilibrium of all loads and contact forces acting on block i. The following equations:

$$\begin{cases} \frac{\partial\pi}{\partial\varepsilon_x}=0, & \frac{\partial\pi}{\partial\varepsilon_y}=0, \quad \frac{\partial\pi}{\partial\varepsilon_z}=0,\\ \frac{\partial\pi}{\partial\gamma_{yz}}=0, & \frac{\partial\pi}{\partial\gamma_{zx}}=0, \quad \frac{\partial\pi}{\partial\gamma_{xy}}=0, \end{cases} \tag{13}$$

represent the equilibrium of all external forces and stresses on block i.

The differentiations:

$$\frac{\partial^2\pi}{\partial d_{ri}\partial d_{sj}},\qquad r,s=1,2,\cdots,12, \tag{14}$$

form a 12×12 submatrix, which is submatrix $[K_{ij}]$ in global Equation 10. The differentiations:

$$-\frac{\partial\pi(0)}{\partial d_{ri}},\qquad r,s=1,2,\cdots,12, \tag{15}$$

are the free terms of the equilibrium equations derived by minimizing the total energy, π. Therefore, all terms of Equation 15 form a 12×1 submatrix, which is the submatrix $\{F_i\}$ in Equation 10.

APPROXIMATION OF SECOND-ORDER DISPLACEMENTS IN 3-D DDA

The complete second-order displacement functions have the following form:

$$\begin{cases} u(x,y,z)= & u_1+u_2x+u_3y+u_4z+\\ & u_5xy+u_6yz+u_7xz+u_8x^2+\\ & u_9y^2+u_{10}z^2,\\ \nu(x,y,z)= & \nu_1+\nu_2x+\nu_3y+\nu_4z+\\ & \nu_5xy+\nu_6yz+\nu_7xz+\nu_8x^2+\\ & \nu_9y^2+\nu_{10}z^2,\\ w(x,y,z)= & w_1+w_2x+w_3y+w_4z+\\ & w_5xy+w_6yz+w_7xz+w_8x^2+\\ & w_9y^2+w_{10}z^2. \end{cases} \tag{16}$$

where $u(x,y,z),\nu(x,y,z)$ and $w(x,y,z)$ are the displacements of a point within the block in the X, Y and Z directions; x,y and z are the coordinates of a point within the block; $u_i(i=1,2,\cdots,10)$, $\nu_i(i=1,2,\cdots,10)$ and $w_i(i=1,2,\cdots,10)$ are unknown parameters.

Writing in matrix form, the displacement field can be expressed as:

$$\begin{bmatrix} u(x,y,z)\\ \nu(x,y,z)\\ w(x,y,z)\end{bmatrix}_{3\times 1}=[C(x,y,z)]_{3\times 30}\{D\}_{30\times 1}, \tag{17}$$

in which $[C(x,y,z)]$ may be expressed as:

$$[C(x,y,z)]=[C_1\quad C_2], \tag{18}$$

that:

$$[C_1]=\begin{bmatrix} 1 & 0 & 0 & x & 0 & 0 & y\\ 0 & 1 & 0 & 0 & x & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & x & 0\end{bmatrix}$$

$$\begin{matrix} 0 & 0 & z & 0 & 0 & xy & 0 & 0\\ y & 0 & 0 & z & 0 & 0 & xy & 0\\ 0 & y & 0 & 0 & z & 0 & 0 & xy\end{matrix}\Bigg],$$

$$[C_2]=\begin{bmatrix} yz & 0 & 0 & xz & 0 & 0 & x^2\\ 0 & yz & 0 & 0 & xz & 0 & 0\\ 0 & 0 & yz & 0 & 0 & xz & 0\end{bmatrix}$$

$$\begin{matrix} 0 & 0 & y^2 & 0 & 0 & z^2 & 0 & 0\\ x^2 & 0 & 0 & y^2 & 0 & 0 & z^2 & 0\\ 0 & x^2 & 0 & 0 & y^2 & 0 & 0 & z^2\end{matrix}\Bigg],$$

and the displacement variable vector, $\{D\}$, is:

$$\{D\}=$$

$$\left\{u_1\quad \nu_1\quad w_1\quad u_2\quad \nu_2\quad w_2\quad \cdots\quad u_{10}\quad \nu_{10}\quad w_{10}\right\}^T. \tag{19}$$

It is necessary to point out that $[C(x,y,z)]$ and $\{D\}$ matrices in the second-order are much different from $[T(x,y,z)]$ and $\{D\}$ in the first-order 3-D DDA.

Using the second-order displacement functions, it is possible to approximate the stress field by the linear stress.

SUBMATRICES OF EQUILIBRIUM EQUATION DERIVATIONS

In this section, the sub matrices of the equilibrium equation using the second-order displacement functions are derived as follows.

Submatrix of Block Stiffness

The strain energy of the elastic stresses of block i is:

$$\pi_e=\iiint\limits_V \frac{1}{2}\{\varepsilon_i\}^T.\{\sigma_i\}dxdydz, \tag{20}$$

where:

$$\{\varepsilon\}=\left\{\varepsilon_x\quad \varepsilon_y\quad \varepsilon_z\quad \gamma_{yz}\quad \gamma_{zx}\quad \gamma_{xy}\right\}^T, \tag{21}$$

and:

$$\{\sigma\}=\left\{\sigma_x\quad \sigma_y\quad \sigma_z\quad \tau_{yz}\quad \tau_{zx}\quad \tau_{xy}\right\}^T. \tag{22}$$

Assume the blocks are linearly elastic. Denote:

$$[E_i] = \frac{E_i}{(1-\nu_i^2)(1-2\nu_i)} \times \begin{bmatrix} 1-\nu_i & \nu_i & \nu_i & 0 & 0 & 0 \\ \nu_i & 1-\nu_i & \nu_i & 0 & 0 & 0 \\ \nu_i & \nu_i & 1-\nu_i & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}-\nu_i & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}-\nu_i & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}-\nu_i \end{bmatrix}, \quad (23)$$

that E_i is Young's modulus and ν_i is Poisson's ratio. Therefore, the relationship between stress $\{\sigma_i\}$ and strain $\{\varepsilon_i\}$ is expressed as Equation 24:

$$\{\sigma_i\} = [E_i].\{\varepsilon_i\}. \quad (24)$$

Hence, the strain energy of Equation 20 can be expressed as:

$$\pi_e = \iiint_V \frac{1}{2}\{\varepsilon_i\}^T.[E_i].\{\varepsilon_i\}dxdydz, \quad (25)$$

the strains can be approximated by:

$$\{\varepsilon_i\} = [B_i(x,y,z)]_{6\times 30}.\{D_i\}_{30\times 1}. \quad (26)$$

Therefore, the elastic strain energy of the ith block can be written in matrix form as:

$$\pi_e = \frac{1}{2}\iiint_V \{D_i\}^T .[B_i]^T.[E_i].[B_i].\{D_i\}dxdydz. \quad (27)$$

By minimizing the strain energy, it leads to the stiffness matrix of the ith block:

$$[K_{ii}] = \iiint_V [B_i(x,y,z)]^T .[E_i].[B_i(x,y,z)]dxdydz. \quad (28)$$

$[K_{ii}]$ is a 30×30 matrix, which is added to the global stiffness matrix.

Submatrix of Initial Stress

In DDA, the computed stresses of the previous time step will be transferred to the next step as initial stress loading. For block i, the initial stresses are given by:

$$\{\sigma_i^0\} = [E_i].\{\varepsilon_i^0\}, \quad (29)$$

then:

$$\{\sigma_i^0\} = [E_i].[B_i(x,y,z)].\{D_i^0\}. \quad (30)$$

Therefore, the potential energy of the initial stress, π_{σ^0}, is given by:

$$\pi_{\sigma^0} = \iiint_V \{\varepsilon_i\}^T.\{\sigma_i^0\}dxdydz = \{D_i\}^T \left(\iiint_V [B_i]^T.[E_i].[B_i]dxdydz\right)\{D_i^0\}. \quad (31)$$

After minimizing π_{σ^0}, the following 30×1 vector is calculated as follows, then, added to vector F_i in the global force vector:

$$\{F_i\} = -\iiint_V [B_i]^T.[E_i].[B_i]dxdydz\{D_i^0\}. \quad (32)$$

Submatrix of Point Loading

For 3-D DDA, the point loading force (F_x, F_y, F_z) can act on any point (x_0, y_0, z_0) of block i. The potential energy of the point loading (F_x, F_y, F_z) is simply:

$$\pi_p = -(F_x u + F_y \nu + F_z w) = -\{D_i\}^T.[C_i]^T.\begin{Bmatrix} f_{x_i} \\ f_{y_i}. \\ f_{z_i} \end{Bmatrix}. \quad (33)$$

After minimizing π_p, the following 30×1 submatrix is added to submatrix $\{F_i\}$ in the global equation:

$$F_i = [C_i(x_0, y_0, z_0)]^T \begin{Bmatrix} f_{x_i} \\ f_{y_i} \\ f_{z_i} \end{Bmatrix}. \quad (34)$$

Submatrix of Fixed Points

At point (x_0, y_0, z_0) of the block i, the computed displacements from the displacement variable $\{D_i\}$ of block i are u_c, ν_c, w_c. The strain energy, π_f, is:

$$\pi_f = \frac{K_f}{2}(u_c^2 + \nu_c^2 + w_c^2), \quad (35)$$

therefore:

$$\pi_f = \frac{K_f}{2}\{D_i\}^T[C_i(x_0, y_0, z_0)]^T[C_i(x_0, y_0, z_0)]\{D_i\}. \tag{36}$$

By minimizing the potential energy, π_f, the following matrix can then be added to submatrix $[K_{ii}]$ in the global stiffness matrix:

$$[K_{ii}]_{30\times 30} = K_f[C_i(x_0, y_0, z_0)]^T[C_i(x_0, y_0, z_0)]. \tag{37}$$

Submatrix of Inertia Forces

Denote $(u(t), \nu(t), w(t))$ as the time dependent displacement of any point (x, y, z) of the ith block. The force of inertia is:

$$\begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} = -M \begin{pmatrix} \frac{\partial^2 u(t)}{\partial t^2} \\ \frac{\partial^2 \nu(t)}{\partial t^2} \\ \frac{\partial^2 w(t)}{\partial t^2} \end{pmatrix}, \tag{38}$$

where M is the unit mass. The potential energy of the inertia force of the ith block, π_i, is given by:

$$\pi_i = -\iiint_V (u, \nu, w) \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} dxdydz$$
$$= \iiint_V M \frac{\partial^2 \{D_i(t)\}}{\partial t^2}\{D_i\}^T[C_i]^T[C_i]dxdydz. \tag{39}$$

Using the Taylor approximation, the following equation can be obtained:

$$\{D_i\} = \{D_i(\Delta)\} = D_i(0) + \Delta\frac{\partial D_i(0)}{\partial t} + \frac{\Delta^2}{2}\frac{\partial^2 D_i(0)}{\partial t^2}, \tag{40}$$

where Δ is the time interval of this time step and $\{D_i\}$ is the displacement at the end of this time step. The displacement at the beginning of time step $D_i(0)$ is zero. This equation becomes:

$$\{D_i\} = \Delta\frac{\partial^2 D_i(0)}{\partial t} + \frac{\Delta^2}{2}\frac{\partial^2 D_i(0)}{\partial t^2}. \tag{41}$$

Assuming a constant acceleration at each time step, we have:

$$\frac{\partial^2 D(t)}{\partial t^2} = \frac{2}{\Delta^2}\{D_i\} - \frac{2}{\Delta}\frac{\partial D_i(t)}{\partial t} = \frac{2}{\Delta^2}\{D_i\} - \frac{2}{\Delta}[V_0], \tag{42}$$

where $\frac{\partial^2 D_i(t)}{\partial t^2}$ is the velocity, $[V_0]$, at the beginning of the time step. Thus, Equation 37 becomes:

$$\pi_i = M\frac{2}{\Delta^2}\{D_i\}\{D_i\}^T \left(\iiint_V [C_i]^T[C_i]dxdydz\right)$$
$$- M\frac{2}{\Delta}\{D_i\}^T \left(\iiint_V [C_i]^T[C_i]dxdydz\right)[V_0]. \tag{43}$$

Hence, the contribution to the global matrix is presented as:

$$[K_{ii}]_{30\times 30} = \frac{2M}{\Delta^2}\iiint_V [C_i]^T[C_i]dxdydz, \tag{44}$$

and:

$$[F_i]_{30\times 1} = \frac{2M}{\Delta}\left(\iiint_V [C_i]^T[C_i]dxdydz\right)[V_0]. \tag{45}$$

It is worth noting that the integration;

$$\iiint_V [C_i(x, y, z)]^T[C_i(x, y, z)]dxdydz,$$

is much different from the first-order 3-D DDA.

VALIDATION

Four problems are chosen to verify the modified method. Three of them are related to the behavior of a single beam under various loading conditions and one of them is an example of block sliding.

Beam Subjected to Three Loads

As shown in Figure 2, an 8-m long and 1-m deep beam with unit thickness subjected to three loads at the end of the beam was used for the validation test. The material properties of the beam were assumed to be $E = 10^8$ ton/m^2 and $\nu = 0.2$.

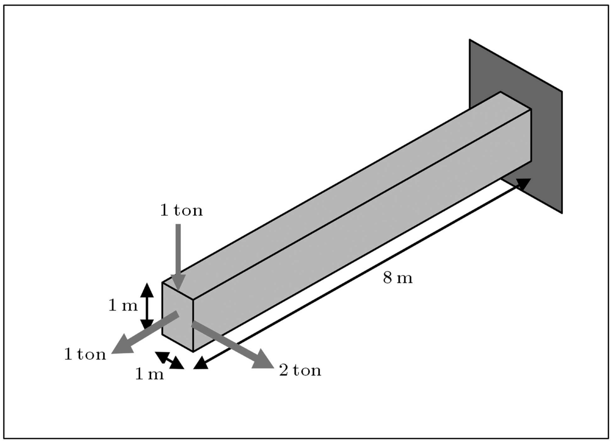

Figure 2. Beam subjected to three forces.

The axial deformation along the axis of a beam subjected to an axial load is [19]:

$$\Delta = \frac{Px}{AE}, \tag{46}$$

where Δ is the axial deformation, x is the distance from the fixed end, P is a concentrated load acting at the middle point of the free end, A is the cross-sectional area, and E is the Young's modulus of the beam. The problem was solved by a 1st-order and the new 2nd-order 3D DDA programmed code. The 3-D DDA modeling results and the theoretical solutions along the axis of the beam are presented in Figure 3.

It can be seen from this figure that the DDA modeling with the first order displacement function does not give accurate results until a polynomial displacement function having a second-order is used for modeling.

The approximate deflection of the axis of the beam is:

$$\nu = \frac{P}{6EI}(2L^3 - 3L^2x + x^3), \tag{47}$$

where L is the length of the beam and I is the moment of inertia of the cross section of the cantilever. Deflection of the cantilever axis, calculated using Equation 47, is plotted in Figures 4 and 5 along with a 3-D DDA solution.

As can be observed, the first order approximation for the displacements in the block is not suitable for the modeling of bending in related problems. The modeling results are improved substantially when the second order polynomials are used. The calculated deflection at the free end is about 73% of the theoretical value, which could be further improved if third-order polynomials were used.

This example shows that the derived formulations and programmed code are working well.

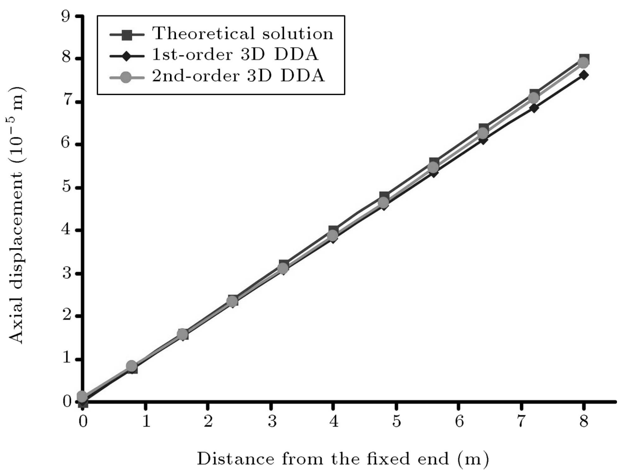

Figure 3. Results of 3-D DDA modeling along the axis of the beam.

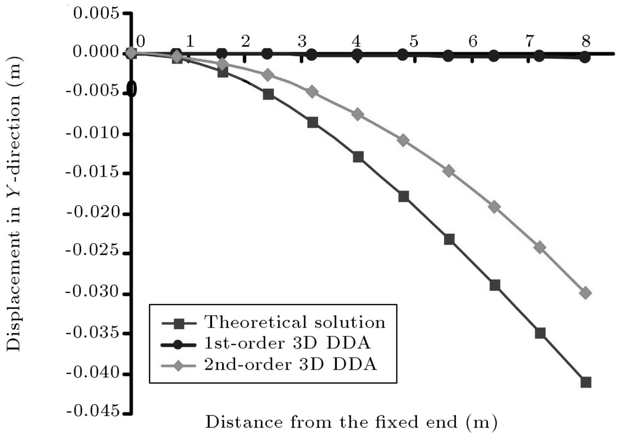

Figure 4. Results of 3-D DDA modeling of beam deflection in Y-direction.

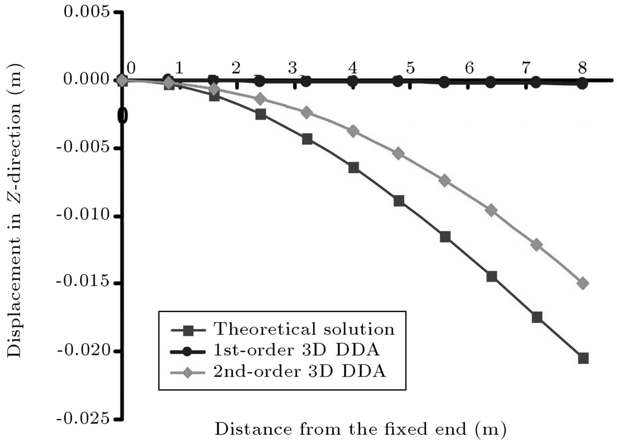

Figure 5. Results of 3-D DDA modeling of beam deflection in Z-direction.

Simply Supported Beam

Figure 6 illustrates a simply supported beam with details on beam length, material properties and the load applied. The height of the beam is 1 m. A concentrated load of 100 tons is applied to the middle span of the beam.

The deflection of the beam axis for this case is:

$$\nu = -\frac{Px}{48EI}(3L^2 - 4x^2), \quad 0 \le x \le \frac{L}{2}, \tag{48}$$

where P is the concentrated load acting at the middle span of the beam, E is the Young's modulus of the beam, I is the moment of inertia of the cross section of the beam and L is the length of the beam.

The deflection of the axis for the simply supported beam computed from Equation 48 is presented in Figure 7.

Also, 3-D DDA results, using the first- and the second-order polynomial displacement functions, are shown in the figure. It can be observed that significant improvement can be made by replacing the original first-order polynomial displacement function with the

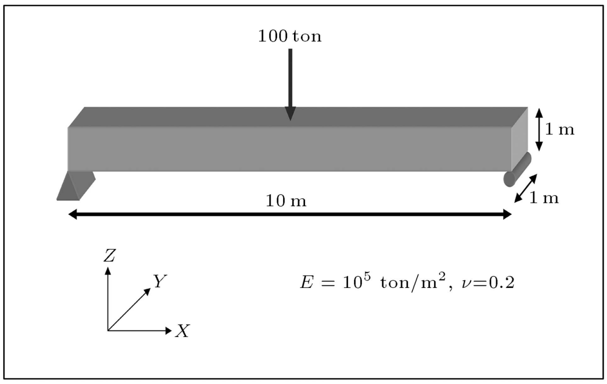

Figure 6. Simply supported beam.

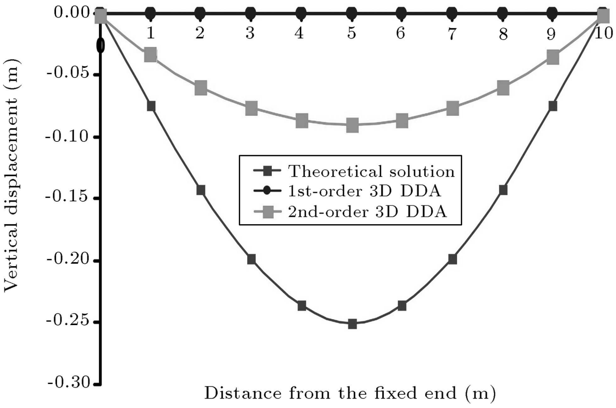

Figure 7. Results of 3-D DDA modeling for simply supported beam.

second-order approximations. The 3-D DDA modeling using a second-order displacement function gives much more precise prediction results. The predicted deflection at the middle span of the beam is 0.0% and 36% of the theoretical value for the first- and second-order polynomials, respectively.

Short Span Beam

The last verification problem is related to a short span beam subjected to a concentrated load perpendicular to the beam axis at the free end. The geometry and material properties of the beam are provided in Figure 8.

The approximate deflection of the axis of the beam at the free end of the beam is:

$$\delta = \frac{PL^3}{3EI}\left(1 + \frac{3E}{10G}\frac{h^2}{L^2}\right), \tag{49}$$

where P is the concentrated load, E is the Young's modulus of the beam, I is the moment of inertia of the cross section of the beam, G is the shear modulus of the beam and h and L are the height and length of the beam, respectively.

In this example, five different short span beams are analyzed using the original and second-order 3-D DDA. The deflection at the free end of the beams, calculated using Equation 49 and obtained using first and second-order 3-D DDA, are plotted in Figure 9. Information regarding the beams and obtained errors for the original and second-order 3-D DDA modeling is presented in Table 1. It would appear that using the second-order displacement functions in 3-D DDA can give much better results.

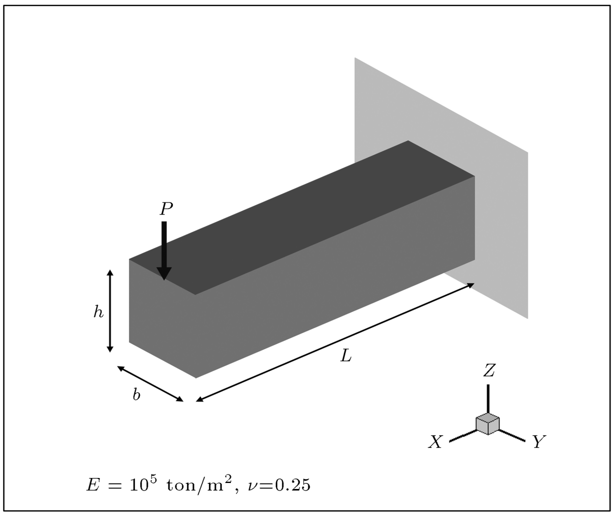

Figure 8. Short span beam.

Sliding of a Block along a Frictionless Inclined Plane

In this example (Figure 10), the sliding of a block along an inclined plane is examined. The interface between the two blocks is assumed to be frictionless. The values for the density, Young's modulus and Poisson ratio of the two blocks are assumed to be 2500 kg/m^3, 10 GPa and 0.2, respectively. The top block is assumed to start its motion from at-rest conditions (i.e., $\nu = 0$). Under the action of gravitational force, the displacement, s, of the block along an inclined plane at an angle, α, is determined analytically as a function of time t, given

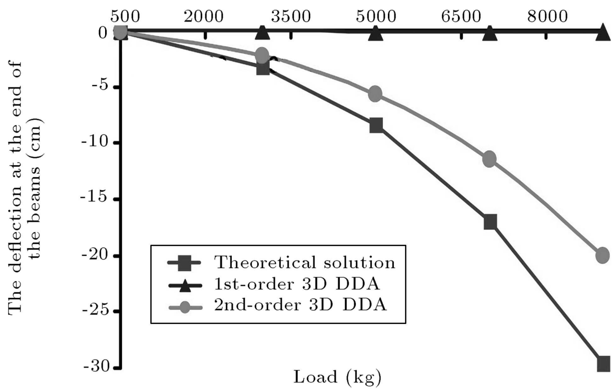

Figure 9. Results of 3-D DDA modeling of the short span beams deflection in Z-direction.

Table 1. Information of the modeled short span beams and obtained errors.

Point no.	Load (kg)	Beam Width (m)	Beam Height (m)	Beam Length (m)	Theoretical Solution (cm)	1st-Order Error (%)	2nd-Order Error (%)
1	500	1	1	4	-0.13	96.04	29.26
2	3000	1	2	8	-3.22	98.84	31.59
3	5000	1	2.5	10	-8.38	99.18	31.87
4	7000	1	3	12	-16.88	99.36	32.02
5	9000	1	3.5	14	-29.55	99.47	32.12

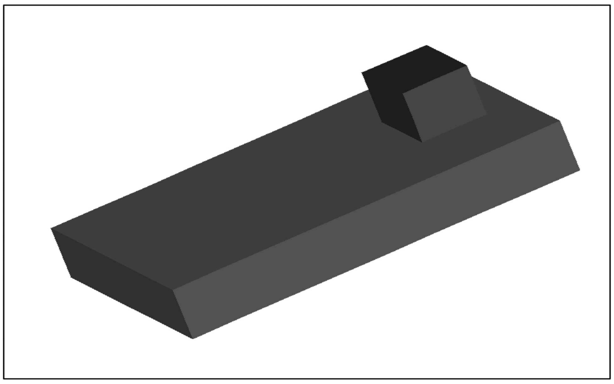

Figure 10. Sliding of a block on a frictionless slope.

as:

$$s(t) = \frac{1}{2}at^2 = \frac{1}{2}(g \sin\alpha).t^2. \tag{50}$$

The path of the centroid of the top block, calculated by the 3-D DDA with second-order displacement functions, is compared with the analytical solution for a slope inclined at 25°. The results are shown in Figure 11. It shows that the analytical solution agrees well with the results computed by the second-order 3-D DDA.

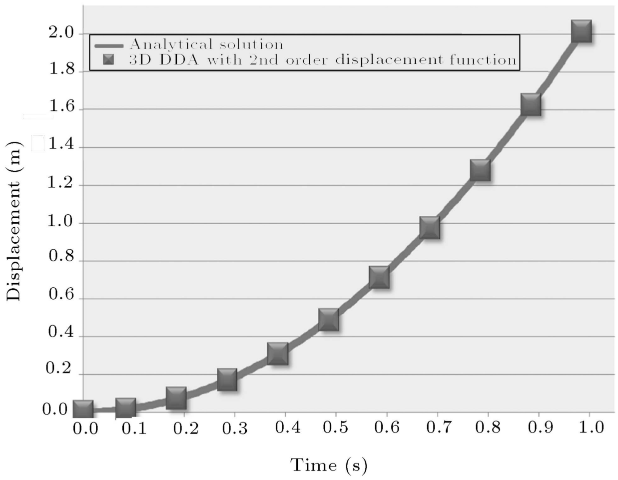

Figure 11. Displacements calculated by analytical solution and 3-D DDA.

CONCLUSIONS

In this paper, 3-D DDA with second-order displacement functions is presented. The formulations of stiffness and force matrices in second-order due to elastic stress, initial stress, point load, body force, inertia force and fixed point are derived. The results of validation tests show that the second-order approximation can obtain much better results than the first order polynomials. Such work will benefit the development of 3-D DDA as well as other numerical methods. This approach is suitable for many problems; however, it may encounter a few difficulties. First, using high-order displacement functions may be inadequate when the size of the block is very large or the variation of stress and strain very rapid. In addition, block faces may deform and not remain as a plane anymore; existing 3-D DDA contact detection schemes cannot be used directly. To deal with this difficulty, there is a simple possible solution. A curved surface can be divided into areas and may be approximated with flat polygons. It is clear that the accuracy of the proposed technique depends on the number of polygons. In fact, a curved face can be defined by some flat polygons, named subfaces here. Each sub-face can be considered as a plane in the original first-order 3-D DDA and its contact may be detected conventionally.

It is clear that more research on the implementation of higher order displacement functions in 3-D DDA is needed to obtain more accurate results.

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