Scientia Iranica, Vol. 15, No. 6, pp. 553–557

SCIENTIA
IRANICA

Research Note

Boundedness and Regularity with Nonlinear Dependence of Hessian and Gradient

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Sufficient conditions for the boundedness and regularity of a function, whose partial derivatives satisfy a certain set of equations, are presented. Energy methods are used to establish these results. The asymptotic behavior of the gradient toward a constant function is also investigated.

Keywords: Boundedness; Regularity; Asymptotic behavior; Nonlinear ODE; Hessian; Gradient.

INTRODUCTION

Recently, boundedness, convergence and the asymptotic behavior of solutions of partial differential equations have been considered [1-3]. Therefore, some special class of PDE's is considered. The aim of this paper is to present sufficient conditions for the boundedness and regularity of a function $u : (0,\infty)^2 \to \mathbb{R}$ whose partial derivatives satisfy:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + r_1(x,y)f_1\left(\frac{\partial u}{\partial x}\right) + r_2(x,y)f_2\left(\frac{\partial u}{\partial y}\right) \\ \qquad + r_3(x,y)f_3(u) = \xi(x,y) \\ \frac{\partial^2 u}{\partial x \partial y} + s_1(x,y)g_1\left(\frac{\partial u}{\partial x}\right) + s_2(x,y)g_2\left(\frac{\partial u}{\partial y}\right) \\ \qquad + s_3(x,y)g_3(u) = \eta(x,y) \\ \frac{\partial^2 u}{\partial y^2} + t_1(x,y)h_1\left(\frac{\partial u}{\partial x}\right) + t_2(x,y)h_2\left(\frac{\partial u}{\partial y}\right) \\ \qquad + t_3(x,y)h_3(u) = \theta(x,y) \end{cases} \tag{1}$$

with:

$$r_1, r_2, r_3, s_1, s_2, s_3, t_1, t_2, t_3 \in C(0,+\infty)^2,$$

$$f_1, f_2, f_3, g_1, g_2, g_3, h_1, h_2, h_3 \in C(\mathbb{R}),$$

$$\xi, \eta, \theta \in C(0,+\infty)^2.$$

Energy methods [4,5] are used to arrive at the boundedness and regularity results. It will also be shown, in regard to the asymptotic behavior, that, under certain conditions, function satisfying Equations 1 behaves asymptotically, like a constant-gradient function. Equations 1 can be considered as a nonlinear relation between the Hessian and the gradient of u. The boundedness and regularity behavior is a key issue in many theoretical and applied areas including dynamical systems control systems, and mathematical physics [6-8].

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RESULTS ON BOUNDEDNESS

Theorem 1

If the following conditions hold:

(i) r_1, r_3, s_3, t_2 and t_3 are nonnegative on $(0,+\infty)^2$,

(ii) For every $y \in [0,\infty)$, functions $r_3(.,y)$ and $s_3(.,y)$ are increasing on $(0,+\infty)$,

(iii) For every $x \in [0,\infty)$, functions $s_3(x,.)$ and $t_3(x,.)$ are increasing on $(0,+\infty)$,

(iv) For all $\lambda \in \mathbb{R}$, $\lambda f_1(\lambda) \geq 0$ and $\lambda h_2(\lambda) \geq 0$,

(v) There exists a constant $K > 0$ such that for all $\lambda \in \mathbb{R}$;

$$|f_2(\lambda)| \leq K|\lambda|, \qquad |h_1(\lambda)| \leq K|\lambda|,$$

$$|g_i(\lambda)| \leq K|\lambda|, \qquad \text{for } i = 1,2,3,$$

(vi) s_1, s_2, $r_2\sqrt{\frac{s_3}{r_3}}$, $s_1\sqrt{\frac{r_3}{s_3}}$, $s_2\sqrt{\frac{t_3}{s_3}}$, $t_1\sqrt{\frac{s_3}{t_3}} \in L^1(0,+\infty)^2$,

(vii) $\frac{\xi}{\sqrt{r_3}}$, $\frac{\eta}{\sqrt{s_3}}$, $\frac{\theta}{\sqrt{t_3}} \in L^1(0,+\infty)^2$,

(viii) Functions f_3 and h_3 are nonnegative on $\mathbb{R}$,

(ix) Functions F_3 and H_3 defined by:

$$F_3(w) := \int_0^w f_3(\lambda)d\lambda,$$

$$H_3(w) := \int_0^w h_3(\lambda)d\lambda,$$

have the property:

$$\lim_{w\to\pm\infty} F_3(w) = +\infty,$$

$$\lim_{w\to\pm\infty} H_3(w) = +\infty,$$

then, for every set, functions $a(x)$, $\alpha(x)$, $b(y)$ and $\beta(y)$ are defined on $(0,+\infty)$ with the following properties:

(x) Functions a and b nonnegative and decreasing on $(0,+\infty)$,

(xi) Functions α and β nonnegative and decreasing on $(0,+\infty)$,

(xii) $\sqrt{\frac{\alpha r_3}{a}}$, $\sqrt{\frac{a s_3}{\alpha}}$, $\sqrt{\frac{b s_3}{\beta}}$, $\sqrt{\frac{\beta t_3}{b}} \in L^1(0,+\infty)^2$,

the functions

$$u, \qquad \frac{u}{\sqrt{a/\alpha}}, \qquad \frac{u}{\sqrt{b/\beta}}, \qquad \frac{1}{r_3}\frac{\partial u}{\partial x}, \qquad \frac{1}{s_3}\frac{\partial u}{\partial x},$$

$$\frac{1}{s_3}\frac{\partial u}{\partial y}, \qquad \frac{1}{t_3}\frac{\partial u}{\partial y},$$

are all bounded on $(0,+\infty)^2$.

Proof

Introducing:

$$v_{ij} := \frac{\partial^{i+j}u}{\partial x^i \partial y^j}, \qquad \text{for } i,j = 0,1,2, \quad i+j \leq 2,$$

we transform Equations 1 into:

$$\begin{cases} v_{20} = \xi - r_1 f_1(v_{10}) - r_2 f_2(v_{01}) - r_3 f_3(v_{00}) \\ v_{11} = \eta - s_1 g_1(v_{10}) - s_2 g_2(v_{01}) - s_3 g_3(v_{00}) \\ v_{02} = \theta - t_1 h_1(v_{10}) - t_2 h_2(v_{01}) - t_3 h_3(v_{00}) \end{cases} \tag{2}$$

Defining two "energy functions":

$$E_F := \frac{\alpha}{a} v_{00}^2 + \frac{1}{r_3} v_{10}^2 + \frac{1}{s_3} v_{01}^2 + 2F_3(v_{00}),$$

$$E_H := \frac{\beta}{b} v_{00}^2 + \frac{1}{s_3} v_{10}^2 + \frac{1}{t_3} v_{01}^2 + 2H_3(v_{00}),$$

we have:

$$\begin{aligned} \frac{\partial E_F}{\partial x} &= \left(\frac{1}{a}\frac{d\alpha}{dx} - \frac{\alpha}{a^2}\frac{da}{dx}\right) v_{00}^2 - \frac{1}{r_3^2}\frac{\partial r_3}{\partial x} v_{10}^2 \\ &- \frac{1}{s_3^2}\frac{\partial s_3}{\partial x} v_{01}^2 + 2\frac{\alpha}{a} v_{00} v_{10} + 2\frac{1}{r_3} v_{10}\xi + 2\frac{1}{s_3} v_{01}\eta \\ &- 2\frac{r_1}{r_3} v_{10} f_1(v_{10}) - 2\frac{s_2}{s_3} v_{01} g_2(v_{01}) - 2 v_{01} g_3(v_{00}) \\ &- 2\frac{r_2}{r_3} v_{10} f_2(v_{01}) - 2\frac{s_1}{s_3} v_{01} g_1(v_{10}), \end{aligned} \tag{3}$$

$$\begin{aligned} \frac{\partial E_H}{\partial y} &= \left(\frac{1}{b}\frac{d\beta}{dy} - \frac{\beta}{b^2}\frac{db}{dy}\right) v_{00}^2 - \frac{1}{s_3^2}\frac{\partial s_3}{\partial y} v_{10}^2 \\ &- \frac{1}{t_3^2}\frac{\partial t_3}{\partial y} v_{01}^2 + 2\frac{\beta}{b} v_{00} v_{01} + 2\frac{1}{s_3} v_{10}\eta + 2\frac{1}{t_3} v_{01}\theta \\ &- 2\frac{t_2}{t_3} v_{01} h_2(v_{01}) - 2\frac{s_1}{s_3} v_{01} g_1(v_{01}) - 2 v_{10} g_3(v_{00}) \\ &- 2\frac{s_2}{s_3} v_{10} g_2(v_{01}) - 2\frac{t_1}{t_3} v_{10} h_1(v_{10}). \end{aligned} \tag{4}$$

Assumptions (i), (ii), (iii), (iv), (v), (x) and (xi) yield:

$$\begin{aligned} \frac{\partial E_F}{\partial x} &\leq 2\frac{\alpha}{a}|v_{00}||v_{10}| + 2\frac{1}{r_3}|v_{10}||\xi| + 2\frac{1}{s_3}|v_{01}||\eta| \\ &+ 2K|s_2|\frac{v_{01}^2}{s_3} + 2K|v_{01}||v_{00}| \\ &+ 2K\left(\frac{|r_2|}{r_3} + \frac{|s_1|}{s_3}\right)|v_{10}||v_{01}|, \end{aligned}$$

and:

$$\begin{aligned} \frac{\partial E_H}{\partial y} &\leq 2\frac{\beta}{b}|v_{00}||v_{01}| + 2\frac{1}{s_3}|v_{10}||\eta| + 2\frac{1}{t_3}|v_{01}||\theta| \\ &+ 2K|s_1|\frac{v_{10}^2}{s_3} + 2K|v_{10}||v_{00}| \\ &+ 2K\left(\frac{|s_2|}{s_3} + \frac{|t_1|}{t_3}\right)|v_{10}||v_{01}|. \end{aligned}$$

By assumption (viii) and the inequality, $2|AB| \leq A^2 + B^2$, one obtains:

$$2\frac{\alpha}{a}|v_{00}||v_{10}| \leq \sqrt{\frac{\alpha r_3}{a}} E_F,$$

$$2\frac{\beta}{b}|v_{00}||v_{01}| \leq \sqrt{\frac{\beta t_3}{b}} E_H,$$

$$2\frac{1}{r_3}|v_{10}||\xi| \leq \frac{|\xi|}{\sqrt{r_3}} + \frac{|\xi|}{\sqrt{r_3}} E_F,$$

$$2\frac{1}{s_3}|v_{10}||\eta| \leq \frac{|\eta|}{\sqrt{s_3}} + \frac{|\eta|}{\sqrt{s_3}}E_H,$$

$$2\frac{1}{s_3}|v_{01}||\eta| \leq \frac{|\eta|}{\sqrt{s_3}} + \frac{|\eta|}{\sqrt{s_3}}E_F,$$

$$2\frac{1}{t_3}|v_{01}||\theta| \leq \frac{|\theta|}{\sqrt{t_3}} + \frac{|\theta|}{\sqrt{t_3}}E_H,$$

$$2K|v_{01}||v_{00}| \leq K\sqrt{\frac{as_3}{\alpha}}E_F,$$

$$2K|v_{10}||v_{00}| \leq K\sqrt{\frac{bs_3}{\beta}}E_H,$$

and:

$$2K\left(\frac{|r_2|}{r_3} + \frac{|s_1|}{s_3}\right)|v_{10}||v_{01}| \leq K\left(|r_2|\sqrt{\frac{s_3}{r_3}} + |s_1|\sqrt{\frac{r_3}{s_3}}\right)E_F,$$

$$2K\left(\frac{|s_2|}{s_3} + \frac{|t_1|}{t_3}\right)|v_{10}||v_{01}| \leq K\left(|s_2|\sqrt{\frac{t_3}{s_3}} + |t_1|\sqrt{\frac{s_3}{t_3}}\right)E_H.$$

So:

$$\frac{\partial E_F}{\partial x} \leq \frac{|\xi|}{\sqrt{r_3}} + \frac{|\eta|}{\sqrt{s_3}} + \Phi E_F, \quad (5)$$

$$\frac{\partial E_H}{\partial y} \leq \frac{|\eta|}{\sqrt{s_3}} + \frac{|\theta|}{\sqrt{t_3}} + \Psi E_H, \quad (6)$$

where:

$$\Phi := \sqrt{\frac{\alpha r_3}{a}} + \frac{|\xi|}{\sqrt{r_3}} + \frac{|\eta|}{\sqrt{s_3}} + K\left[2|s_2| + \sqrt{\frac{as_3}{\alpha}} + |r_2|\sqrt{\frac{s_3}{r_3}} + |s_1|\sqrt{\frac{r_3}{s_3}}\right],$$

$$\Psi := \sqrt{\frac{\beta t_3}{b}} + \frac{|\eta|}{\sqrt{s_3}} + \frac{|\theta|}{\sqrt{t_3}} + K\left[2|s_1| + \sqrt{\frac{bs_3}{\beta}} + |s_2|\sqrt{\frac{t_3}{s_3}} + |t_1|\sqrt{\frac{s_3}{t_3}}\right].$$

Integrating both sides of Relations 5 and 6, and recalling (vii), by the Gronwall Lemma for differential form, we arrive at:

$$E_F(x,y) \leq C_x \exp\left(\int_0^x \Phi(\lambda,y)d\lambda\right)$$

for all $y \in (0,+\infty)$,

$$E_H(x,y) \leq C_y \exp\left(\int_0^y \Psi(x,\lambda)d\lambda\right)$$

for all $x \in (0,+\infty)$,

for some constants, C_x and C_y. Noting (vi), (vii) and (xii), we have:

$$\Phi, \Psi \in L^1(0,+\infty)^2.$$

So, for every $y \in (0,+\infty)$, the energy function, $E_F(.,y)$, is bounded on $(0,+\infty)$ and, hence, so are the functions, $v_{00}/(\sqrt{a/\alpha})$, $v_{10}/\sqrt{r_3}$, $v_{01}/\sqrt{s_3}$, $F_3(v_{00})$ and v_{00}, by (viii). Similarly, for every $x \in (0,+\infty)$, the energy function, $E_H(x,.)$, is bounded on $(0,+\infty)$ and, hence, so are the functions $v_{00}/(\sqrt{b/\beta})$,$v_{10}/\sqrt{s_3}$, $v_{01}/\sqrt{t_3}$, $H_3(v_{00})$ and v_{00}, by (viii). This completes the proof of Theorem 1.

Remark 1

The proof of Theorem 1 is also valid in the case $\xi \equiv \eta \equiv \theta \equiv 0$.

Remark 2

Noting Equations 3 and 4, we can see that the proof of Theorem 1 remains valid, if we relax the assumptions:

$$\frac{\partial r_3}{\partial x} \geq 0, \qquad \frac{\partial s_3}{\partial x} \geq 0, \qquad \frac{\partial s_3}{\partial y} \geq 0, \qquad \frac{\partial t_3}{\partial y} \geq 0,$$

in (ii) and (iii) by the weaker assumptions:

$$\frac{1}{r_3}\frac{\partial r_3}{\partial x}, \qquad \frac{1}{s_3}\frac{\partial s_3}{\partial x}, \qquad \frac{1}{s_3}\frac{\partial s_3}{\partial y},$$

$$\frac{1}{t_3}\frac{\partial t_3}{\partial y} \in L^1(0,+\infty)^2.$$

Theorem 2

If assumptions (i), (ii), (iii), (iv) and (vii) in Theorem 1 are replaced by:

(i') r_1 and r_2 are nonpositive on $(0,+\infty)^2$ and r_3, s_3 and t_s are nonnegative on $(0,+\infty)^2$,

(ii') For every $y \in (0,+\infty)$, the function $s_3(.,y)$ is increasing on $(0,+\infty)$,

(iii') For every $x \in (0,+\infty)$, the function $s_3(x,.)$ is increasing on $(0,+\infty)$

(iv') There exists a constant $M > 0$, such that, for all $\lambda \in I\!R$:

$$0 < \lambda f_1(\lambda) < M\lambda^2, \qquad 0 < \lambda h_2(\lambda) < M\lambda^2,$$

and, for all $(x, y) \in (0, +\infty)$:

$$\frac{\partial r_3}{\partial x}(x,y) + 2Mr_1(x,y)r_3(x,y) > 0,$$

$$\frac{\partial t_3}{\partial y}(x,y) + 2Mt_2(x,y)t_3(x,y) > 0.$$

(vii')

$$\frac{\eta}{\sqrt{s_3}}, \qquad \frac{\xi^2}{(\partial r_3/\partial x) + 2Mr_1r_3},$$

$$\frac{\theta^2}{(\partial t_3/\partial y) + 2Mt_1t_3} \in L^1(0,+\infty)^2,$$

respectively, then, the assertions of Theorem 1 remain valid, provided $\xi \not\equiv 0$ and $\theta \not\equiv 0$ (no restriction on η).

Proof

Similar to the proof of Theorem 1, using the defined energy functions E_F and E_H, and by similar Gronwall-type arguments.

RESULTS ON L^2 REGULARITY

Theorem 3

Under the assumptions of Theorem 2, all functions:

$$\left(\frac{1}{a}\frac{d\alpha}{dx}\right)^{1/2}u, \qquad \left(\frac{\alpha}{a^2}\frac{da}{dx}\right)^{1/2}u, \qquad \left(\frac{1}{b}\frac{d\beta}{dy}\right)^{1/2}u,$$

$$\left(\frac{\beta}{b^2}\frac{db}{dy}\right)^{1/2}u,$$

are in $L^2(0,+\infty)^2$.

Proof

With the hypotheses, Relations 3 and 4 lead to:

$$\left(\frac{1}{a}\frac{d\alpha}{dx} - \frac{\alpha}{a^2}\frac{da}{dx}\right)v_{00}^2 + \frac{(\partial r_3/\partial x) + 2Mr_1r_3}{r_3^2}v_{10}^2$$

$$\leq -\frac{\partial E_F}{\partial x} + \frac{|\xi|}{\sqrt{r_3}} + \frac{|\eta|}{\sqrt{s_3}} + \Phi E_F, \tag{7}$$

and:

$$\left(\frac{1}{b}\frac{d\beta}{dy} - \frac{\beta}{b^2}\frac{db}{dx}\right)v_{00}^2 + \frac{(\partial t_3/\partial y) + 2Mt_2t_3}{t_3^2}v_{01}^2$$

$$\leq -\frac{\partial E_H}{\partial y} + \frac{|\eta|}{\sqrt{s_3}} + \frac{|\theta|}{\sqrt{t_3}} + \Psi E_H. \tag{8}$$

By the boundedness of E_F and $\partial E_F/\partial x$ as functions of x for every fixed y, and the boundedness of E_H and $\partial E_H/\partial y$ as functions of y for every fixed x, by Relations 7 and 8 we observe that the expressions:

$$\int_0^x \frac{1}{a(p)}\frac{d\alpha}{dx}(p)v_{00}(p,y)dp - \int_0^x \frac{\alpha(p)}{a^2(p)}\frac{da}{dx}(p)v_{00}^2(p,y)dp$$

$$+\int_0^x \frac{1}{r_3^2(p,y)}\left[\frac{\partial r_3}{\partial x}(p,y) + 2Mr_2(p,y)r_3(p,y)\right]v_{10}^2(p,y)dp,$$

and:

$$\int_0^y \frac{1}{b(q)}\frac{d\beta}{dy}(p)v_{00}(x,q)dq - \int_0^y \frac{\beta(q)}{b^2(q)}\frac{db}{dy}(q)v_{00}^2(x,q)dq$$

$$+\int_0^y \frac{1}{t_3^2(x,q)}\left[\frac{\partial t_3}{\partial y}(x,q) + 2Mt_2(x,q)t_3(x,q)\right]v_{01}^2(x,q)dq,$$

are bounded for every $(x,y) \in (0,+\infty)^2$, and the proof is complete by the Fubini theorem [9].

RESULTS ON ASYMPTOTIC BEHAVIOR

Theorem 4

If assumptions of Theorem 2 hold together, with:

$$r_1, r_2, s_1, s_2, t_1, t_2 \in L^1(0,+\infty)^2,$$

$$r_3, s_3, t_3 \in L^1(0,+\infty)^2 \cap L^2(0,+\infty)^2,$$

then:

$$\lim_{x\to\pm\infty}\frac{\partial u}{\partial x}(x,y) = 0 \quad \text{and} \lim_{x\to\pm\infty}\frac{\partial u}{\partial y}(x,y) = 0,$$

$$\text{for all } y \in (0,+\infty),$$

$$\lim_{y\to\pm\infty}\frac{\partial u}{\partial x}(x,y) = 0 \quad \text{and} \lim_{y\to\pm\infty}\frac{\partial u}{\partial y}(x,y) = 0,$$

$$\text{for all } x \in (0,+\infty).$$

Proof

From Equation 2 we get:

$$|v_{10}||v_{20}| \leq \left(\frac{|v_{10}|}{\sqrt{r_3}}\right)\sqrt{r_3}|\xi|$$

$$+ K\left(\frac{|v_{10}|}{\sqrt{r_3}}\right)\left(\frac{|v_{01}|}{\sqrt{s_3}}\right)|r_2|\sqrt{r_3s_3}$$

$$+ M\left(\frac{|v_{10}|}{\sqrt{r_3}}\right)^2|r_1|r_3 + |f_3(v_{00})|\left(\frac{|v_{10}|}{\sqrt{r_3}}\right)r_3^{3/2},$$

$$|v_{10}||v_{11}| \leq \left(\frac{|v_{10}|}{\sqrt{s_3}}\right)\sqrt{s_3}|\eta| + K\left(\frac{|v_{10}|}{\sqrt{s_3}}\right)\left(\frac{|v_{01}|}{\sqrt{t_3}}\right)|s_2|\sqrt{s_3 t_3} + K\left(\frac{|v_{10}|}{\sqrt{s_3}}\right)^2|s_1||s_3| + K|v_{00}|\left(\frac{|v_{10}|}{\sqrt{s_3}}\right)s_3^{3/2},$$

$$|v_{01}||v_{11}| \leq \left(\frac{|v_{01}|}{\sqrt{s_3}}\right)\sqrt{s_3}|\eta| + K\left(\frac{|v_{10}|}{\sqrt{r_3}}\right)\left(\frac{|v_{01}|}{\sqrt{s_3}}\right)|s_1|\sqrt{r_3 s_3} + K\left(\frac{|v_{01}|}{\sqrt{s_3}}\right)^2|s_2||s_3| + K|v_{00}|\left(\frac{|v_{10}|}{\sqrt{s_3}}\right)s_3^{3/2},$$

$$|v_{01}||v_{02}| \leq \left(\frac{|v_{01}|}{\sqrt{t_3}}\right)\sqrt{t_3}|\theta| + K\left(\frac{|v_{10}|}{\sqrt{s_3}}\right)\left(\frac{|v_{01}|}{\sqrt{t_3}}\right)|t_1|\sqrt{s_3 t_3} + M\left(\frac{|v_{01}|}{\sqrt{t_3}}\right)^2|t_2|t_3 + |h_3(v_{00})|\left(\frac{|v_{01}|}{\sqrt{t_3}}\right)t_3^{3/2}. \quad (9)$$

By Theorem 2, all functions inside the parentheses are bounded and, hence, so are $f_3(v_{00})$ and $h_3(v_{00})$ by the continuity assumptions on f_3 and h_3. Taking this into account, by integrating the first and third inequalities with respect to x, and the second and fourth inequalities with respect to y, using the Hölder inequality, we obtain:

$$\int_0^{+\infty}\left|\frac{\partial u}{\partial x}(p,y)\right|\left|\frac{\partial^2 u}{\partial x^2}(p,y)\right|dp < \infty,$$

for all $y \in (0,+\infty)$,

$$\int_0^{+\infty}\left|\frac{\partial u}{\partial x}(x,q)\right|\left|\frac{\partial^2 u}{\partial x \partial y}(x,q)\right|dq < \infty,$$

for all $x \in (0,+\infty)$,

$$\int_0^{+\infty}\left|\frac{\partial u}{\partial y}(p,y)\right|\left|\frac{\partial^2 u}{\partial x \partial y}(p,y)\right|dp < \infty,$$

for all $y \in (0,+\infty)$,

$$\int_0^{+\infty}\left|\frac{\partial u}{\partial y}(x,q)\right|\left|\frac{\partial^2 u}{\partial y^2}(x,q)\right|dq < \infty,$$

for all $x \in (0,+\infty)$,

and assertions of the theorem followed by a simple lemma which, for instance, can be found in [10].

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