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Research Note

On the Poincaré Index of Isolated Invariant Sets

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In this paper, the Conley index theory is used to examine the Poincaré index of an isolated invariant set. Some limiting conditions on a critical point of a planar vector field are obtained to be an isolated invariant set. As a result, the existence of infinitely many homoclinic orbits for a critical point with the Poincaré index greater than one is shown.

Keywords: Conley index; Homoclinic orbit; Poincaré-Lefchetz duality; Poincaré index.

INTRODUCTION

The Conley index has proved to be a useful tool in the investigation of the qualitative properties of dynamical systems. It has generalized the Morse theory for an isolated invariant set of a continuous flow on a locally compact metric space [1,2]. In the Conley index theory, we deal with a pair of closed sets, namely (N, L), called an index pair, for an isolated invariant set, I. The homotopy type of $(N/L, [L])$ is independent of the index pair chosen, which is called the Conley index of I and denoted by $h(I)$. It is a topological index, which uses the dynamics around an isolated invariant set and gives information about the dynamics within the invariant set.

A well-known topological index in dynamical systems is the Poicaré index, which is defined for isolated critical points of a smooth vector field on a manifold. When the vector field does not vanish on the boundary of the manifold, then, by using the Poincaré-Hopf theorem, the values of the vector filed on the boundary give some information about critical points in the interior of the manifold.

This paper mainly deals with the relation between the Conley index theory and the Poincaré-Hopf theorem [3]. The Poincaré index of an isolated invariant set I, is defined as $\chi(h(I))$. This definition coincides with the classical Poincaré index when the invariant set is a single point. Some topological properties of the Conley index are used to obtain restrictions on the Poincaré index of isolated invariant sets in dimension two. It is well-known that, on a two-dimension manifold, M, the Poincaré index of an isolated critical point of a gradient vector field is not greater than one [4]. Here, it is shown that a critical point, x, with the Poincaré index greater than one, cannot be an isolated invariant set. This concludes the existence of infinitely many homoclinic orbits for such a critical point.

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CONLEY INDEX

Let φ^t be a C^1-flow on a smooth manifold, M. A subset, $I \subset M$, is called an isolated invariant set, if it is the maximal invariant set in some compact neighborhood of itself. Such a neighborhood is called an isolating neighborhood.

Definition 1

A closed pair (N, L) is called an *index pair* for I if:

1. $\overline{N - L}$ is an isolating neighborhood for I,
2. L is positively invariant relative to N, i.e., if $x \in L$, $t \geq 0$, $\varphi^{[0,t]}(x) \subset N$, then $\varphi^{[0,t]}(x) \subset L$,
3. L is the exit set of N, i.e., if $x \in N$, $t \in \mathbb{R}^+$ and $\varphi^t(x) \notin N$, then there is a $t' \in [0, t]$, such that $\varphi^{t'}(x) \in L$.

In [1,2,5] it has been shown that every isolated invariant set, I, admits an index pair, (N, L), and the homotopy type of $(N/L, [L])$ is independent of the index pair chosen. The homotopy type of $(N/L, [L])$ is denoted by $h(I)$ and called the *Conley index* of I. The *homology Conley index* of I is defined by $H_*(h(I)) = H_*(N/L, [L])$.

Example 1

Let $x \in M$ be a nondegenerate critical point for $f : M \xrightarrow{C^2} \mathbb{R}$. Then, $\{x\}$ is an isolated invariant set for $-\nabla f$ and by Morse Lemma [6], it is easy to show that $h(\{p\})$ is a pointed k-sphere, where k is the number of positive eigenvalues of Hessian matrix f at p. Therefore, the Conley index can be considered as a generalization of the Morse index.

In general, It is not true that $H_*(N,L) \cong H_*(N/L,[L])$ for every index pair (N,L). In [5], Salamon introduced a class of index pairs for which the above isomorphism holds.

Definition 2

An index pair, (N,L), is called *regular* if the exit time map:

$$\tau_+ : N \longrightarrow [0,+\infty],$$

$$\tau_+(x) = \begin{cases} sup\{t|\varphi^{[0,t]}(x) \subset N-L\} & \text{if } x \in N-L, \\ 0 & \text{if } x \in L, \end{cases}$$

is continuous (see [5] for more details about regular index pairs). For every regular index pair, (N,L), the induced semi-flow on N is defined by:

$$\varphi_\natural^t : N \times \mathbb{R}^+ \longrightarrow N,$$

$$\varphi_\natural^t(x) = \varphi^{\min\{t,\tau_+(x)\}}(x).$$

Proposition 1

If (N,L) be a regular index pair for a continuous flow φ^t, then L is a neighborhood deformation retract in N. In particular, the natural map $\pi : N \longrightarrow N/L$ induces an isomorphism, $H_*(N,L) \cong H_*(N/L,[L])$.

Proof
Consider the induced semi-flow, $\varphi_\natural$ on N, and the neighborhood, $U := \tau_+^{-1}\ [0,1]$, of L. Now, $\varphi_\natural|_{U\times\ [0,1]}$ gives the desired deformation retraction.□

In [7], Robbin and Salamon proved that every isolated invariant set admits a regular index pair, which is stable under perturbation. They first showed the existence of a smooth Liapunov function on a neighborhood of the isolated invariant set.

Theorem 1

Let N be an isolating neighborhood of I. Then, there is a neighborhood U of N and a smooth function $f : U \longrightarrow \mathbb{R}$ satisfying:

(i) $f(x) = 0$ for all $x \in I$,

(ii) $\frac{d}{dt}|_{t=0} f(\varphi^t(x)) < 0$ for all $x \in N - I$. (f decreases along orbits in $U - I$.)

Then, they used this Liapunov function to construct a triple (N,L^-,L^+), such that (N,L^+) is a regular index pair for I, with respect to the forward flow, and (N,L^-) is a regular index pair for I, with respect to the reverse flow. Furthermore, L^+ and L^- can be chosen to be $(n-1)$-manifolds with a boundary, so that N is a manifold with corners, which are contained in $L^- \cap L^+$ and $N = L^- \cup L^+$. Such a triple is called (N,L^-,L^+) as a *regular index triple* for I in M. The Conley indices of I related to the forward and reverse flows are represented by $h^+(I)$ and $h^-(I)$, respectively. If M is orientable in a neighborhood of I, then, by the Poincaré-Lefschetz duality, $H_*(N,L^+) \simeq H^{m-*}(N,L^-)$, where $m = dim M$ (see [8,9,10]). Thus, the indices for the forward and reverse flows are related by $H^*(h^+(I)) = H_{m-*}(h^-(I))$. If we consider the homology with coefficients in $\mathbb{Z}_2$, the Poincaré-Lefschetz duality is valid without the assumption of orientability.

Definition 3

$A \subset M$ is called an attractor set if it is the ω-limit set of a compact neighborhood of itself. A repeller set is an attractor set for the reverse flow.

Proposition 2

I is an attractor set for φ^t if, and only if, there is an index pair (N,L), for I which $L = \varnothing$.

Proof
Let I be an attractor and V be a compact neighborhood of I, such that $\omega(V) = I$. Then, there is a $T > 0$, such that $\varphi^{[T,\infty)}(V) \subset int(V)$. If we set $N := \bigcap_{0 \le s \le T} \varphi^s(V)$, it is not difficult to show that N is a positively invariant isolating neighborhood for I (see [11]). Therefore, $(N,\varnothing)$ is an index pair for I. Now, assume that $(N,\varnothing)$ is an index pair for I. According to the property (iii) of the definition of the index pair, it is implied that N is positively invariant, hence, $\omega(N) \subset N$. Since N is an isolating neighborhood for I, it follows that $\omega(N) \subseteq I$. Since I is an invariant set, we conclude that $\omega(N) = I$.

Theorem 2

Suppose that $I \subset M$ is a connected isolated invariant set.

(i) If I is not an attractor, then $H_0(h^+(I)) = 0$,

(ii) If I is not a repeller, then $H_m(h^+(I);\mathbb{Z}_2) = 0$. Moreover, if M is orientable in a neighborhood of I, then $H_m(h^+(I)) = 0$ $(m = \dim M)$.

Proof

Consider a regular index triple (N, L^+, L^-) for I. One may assume that N is connected (otherwise replace N by the connected component of N that contains I). Since I is not an attractor set, $L^+ \neq \varnothing$ by Proposition 2. Thus:

$$H_0(h^+(I)) = H_0(N, L^+) = 0.$$

Similarly we have $L^- \neq \varnothing$ and $H^0(h^-(I)) = 0$. Now, by the Poincaré-Lefschetz duality $H_m(h^+(I)) \simeq H^0(h^-(I)) = 0$.

POINCARÉ INDEX

The Poincaré index is defined for isolated critical points of a smooth vector field. Here, the Conley index is used to extend its definition for any isolated set.

Definition 4

The Poincaré index of an isolated invariant set, I, is defined as being the Euler characteristic of the Conley index of I, i.e. $ind_p(I) := \chi(h(I))$.

Suppose that the flow, φ^t, is associated with a vector field, X on M. If $\{x\}$ is a critical point of X and an isolated invariant set for φ^t, then, $ind_p(x)$ coincides with the classical definition of the Poincaré index of x (up to a sign). This is a special case of the results of [3], in which McCord developed the Poincaré-Hopf theorem and showed that $ind_p(I) = (-1)^m \sum ind(x)$, where the sum is taken over all critical points in I, $ind(x)$ is the Poincaré index of x relative to vector field X and $m = dimM$.

Theorem 3

Let I be an isolated invariant set. Then;

$$ind_p(I) = (-1)^m \sum_{x \in I} ind(x).$$

In particular, if $ind_p(I) \neq 0$, then there exists a critical point in I.

The following proposition examines the Poincaré index of attractor and repeller sets.

Proposition 3

Suppose that $I \subset M$ is an NDR (Neighborhood Deformation Retract) isolated invariant set. Then;

(i) If I is an attractor, then $ind_p(I) = \chi(I)$,

(ii) If I is a repeller, then $ind_p(I) = (-1)^m \chi(I)$. ($m = \dim M$.)

Proof

When I is an attractor, there is an index pair $(N, \varnothing)$ for I by Proposition 2. Since I is an NDR, there exists a neighborhood, $U \subset N$, such that I is the deformation retract of U. By the definition of an index pair, N is positively invariant and $\omega(N) = I$. So there is a $T > 0$ such that $\varphi^T(N) \subset U$. Therefore N can be deformed to I and $H_i(N, \varnothing) = H_i(I)$ for every i. Thus $\chi(h(I)) = \chi(I)$ which proves (i). Notice that for a finite CW-complex, the Euler characteristic does not depend on the coefficients field. Since I is assumed to be an NDR, it has the homotopy type of a finite CW-complex. If we consider the homology with coefficients in $\mathbb{Z}_2$, we obtain $\chi(h^+(I)) = (-1)^m \chi(h^-(I))$ by the duality theorem. So if I is an NDR repeller, then $ind_p(I) = (-1)^m \chi(I)$. $\square$

APPLICATIONS

In this section, a smooth vector field is considered on a surface, M, with an isolated critical point, x. It is desired to show that if $ind(x) > 1$, then, x is accumulated by infinitely many homoclinic orbits. Since isolated invariant sets have been defined to be compact, one may assume that the vector fields are complete. This is achieved by multiplying the vector field by a smooth compact support function.

Lemma 1

Let $I \subset M$ be a connected NDR isolated invariant set, such that $ind_p(I) > 0$. Then, I is either an attractor or a repeller and $ind_p(I) = \chi(I)$.

Proof

Suppose that I is neither an attractor nor a repeller. By Theorem 2, $H_2(h(I); \mathbb{Z}_2) \simeq H_0(h(I); \mathbb{Z}_2) = 0$. Now we conclude that;

$$\begin{aligned} ind_p(I) = \chi(h(I)) &= rank(H_2(h(I)); \mathbb{Z}_2) \\ &\quad - rank(H_1(h(I)); \mathbb{Z}_2) + rank(H_0(h(I)); \mathbb{Z}_2) \\ &= -rank(H_1(h(I)); \mathbb{Z}_2) \leq 0. \end{aligned}$$

Since $m = 2$, the proof is complete by Proposition 3.

Theorem 4

Let x be a critical point for a vector field on surface M. If $ind(x) > 1$, then there exists a homoclinic orbit in any neighborhood of x.

Proof

It is first shown that $\{x\}$ cannot be an isolated invariant set. Suppose the contrary, then according to Theorem 3 and the above lemma, $ind(x) = \chi(\{x\}) = 1$ which is a contradiction. Consider a closed neighborhood V of x with no critical points rather than x. Let $I(V)$ be the maximal invariant set in V. The above argument says that there is a point, $y \neq x$ in $I(V)$, hence, $\omega(y) \subset I(V)$. Notice that there cannot be any cycle in V. To see this, suppose that γ is a cycle in V. Then, the Poincaré index of γ must be one [12], thus, there exists a critical point inside γ. Since the only critical point in V is x, we get $ind(x) = 1$, which is a contradiction. Now, according to the Poincaré-Bendixon theorem [12,13], $\omega(y)$ and $\alpha(y)$ are critical points or homoclinic orbits. If neither of $\omega(y)$ and $\alpha(y)$ are homoclinic orbits, then, $\omega(y) = \alpha(y) = x$. So, there exists a homoclinic orbit in V.

Proposition 4

Let γ be a homoclinic orbit with no critical point inside of it. Then, all the orbits inside γ are homoclinic.

Proof

Let $x := \omega(\gamma) = \alpha(x)$ and Ω be the region surrounded by γ. Similar to the above argument, there is no cycles in Ω and the limit sets of any orbit in Ω are either $\{x\}$ or homoclinic orbits. Since x is the only critical point in Ω, it belongs to all limit sets. On the other hand, it is known that, if one of the limit sets is not a critical point, then, the limit sets are disjoint [12,13]. Therefore, the limit sets of any orbit in Ω must be $\{x\}$.

Remark 1

It is well-known that, if x is a critical point of a gradient vector field, then, $ind(x) \leq 1$. The above theorem clearly shows why this result is true.

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