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Research Note

On the Solvability of $\bar{\partial}$ in a Class of Hypo-Analytic Manifolds

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The solvability of $\bar{\partial}$ operator in a class of hypo-analytic manifolds in complex dimension 2 is studied. Suitable weighted L^2 spaces are introduced for establishing an a priori inequality. The regularity of the solutions is shown by using a theory of degenerate elliptic operators, developed by Grusin and Visik. The theorem obtained is a degenerate version for the $\bar{\partial}$ problem in strictly pseudo-convex domains.

INTRODUCTION

Mirror symmetry establishes a duality between Calabi-Yau manifolds i.e. compact simply-connected Kähler manifolds with trivial canonical bundles [1]. Thirty years ago, S-T Yau [2] proved a conjecture of Calabi about the existence of Ricci-flat Kähler metrics on these manifolds. Calabi-Yau metrics are obtained by solving a complex Monge-Ampère equation and provide suitable backgrounds for defining meaningful theories of sigma-models [1]. Dual Calabi-Yau manifolds provide equivalent superconforml theories in Physics. In the mathematical context, this leads to an equivalence between two types of apparently different objects on dual manifolds. One is defined in holomorphic terms and the other is defined in terms of symplectic geometry. To obtain an image about how this duality may proceed in complex dimension 2, we, first, recall some definitions about quaternionic structures [3]. Let M be a differentiable manifold. By an almost quaternionic structure on M, we mean two almost complex structures, I and J, on M, such that $I \circ J + J \circ I = 0$. An almost quaternionic structure, $\{I, J\}$, defines an action of the algebra of quaternions in each tangent space, $T_pM, p \in M$, by letting a number, $q = a+bi+cj+dk$, acting as $Q = a+bI+cJ+dK$, where $K = I \circ J$. Now, Q is an almost complex structure if, and only if, $a = 0$ and $b^2 + c^2 + d^2 = 1$. So, to every almost quaternionic structure, $\{I, J\}$ is associated a 2-sphere, $S(I, J)$, of almost complex structures named the twistor sphere. If $\{M, g\}$ is a Riemannian manifold, an almost quaternionic structure, $\{I, J\}$, on M, is called quaternionic, if g is Kähler with respect to each of the almost complex structures of $S(I, J)$. In particular all the structures in $S(I, J)$ are integrable and we obtain a sphere of complex structures. For Calabi-Yau manifolds in complex dimension 2 or, more precisely, $K3$ surfaces, one can naively say that mirror symmetry interchanges I and J at the level of complex structures (a more precise statement can be found in [4]). The study of singularities of Calabi-Yau metrics [2,5] has motivated us to study quaternionic structures, which are singular along a sub-manifold. In [5], a class of degenerate complex surfaces has been introduced, which is hoped to be mirrors for complex surfaces of general type. These manifolds describe a special subclass of the manifolds known in the current literature as hypo-analytic manifolds [6]. The existence of a degenerate Hodge theory is shown in [5,7]. In this note, it is desired to study the global solvability of a $\bar{\partial}$ operator for the same type of hypo-analytic structures. Let us first recall a definition from [6].

Defenition 1

Let M be a C^∞ manifold. By a hypo-analytic structure on M, we mean the data of an open covering $\{\Omega_i\}$ of M and for each index i, of a C^∞ map $Z_i = (Z_{i,1}, \cdots, Z_{i,m}) : \Omega_i \to \mathbb{C}^m$ with $m \geq 1$ independent of i, such that the following is true:

1. $dZ_{i,1}, \cdots, dZ_{i,m}$ are $\mathbb{C}$-linearly independent at each point;
2. if $i \neq k$ for each point p of $\Omega_i \cap \Omega_k$, there is a

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holomorphic map $F^i_{k,p}$ of an open neighborhood of $Z_i(p)$ in $\mathbb{C}^m$ into $\mathbb{C}^m$, such that $Z_k = F^i_{k,p} o Z_i$ in a neighborhood of p in $\Omega_i \cap \Omega_k$.

Now, let:

$$\Omega \subset \mathbb{C}^2 = \{(w,z)|w = u+iv, z = x+iy)\},$$

and let $\mathbb{H} = \{(U+iV, X+iY)|U,V,X,Y \in \mathbb{R}\}$ be equipped with its standard quaternionic structure, $\{I, J\}$, and let $\pi : \mathbb{C}^2 \to \mathbb{H}$ be the application defined by:

$$\pi(w,z) = (w, z^2).$$

Let the complex coordinates associated to J be given by $(U+iX, V+iY)$. Let $\mathcal{W}$ be an open subset of $\mathbb{H}$, which is strongly pseudo-convex, with respect to this new complex structure, J, and such that the application, $\pi|_\Omega : \Omega \to \mathcal{W}$, be a double covering space ramified along $\mathbb{R}^2 \times \{0\}$. In this way, π defines a hypo-analytic structure on Ω, as in Definition 1, which is given by the following holomorphic coordinates:

$$\tau = u + i(x^2 - y^2), \qquad \rho = v - i(2xy).$$

The pullback $J' = \pi^*(J)$ of the complex structure, J, is in fact, a degenerate complex structure on Ω. If $r^2 = x^2 + y^2$, then, J' can be described as:

$$J'(dx) = \frac{1}{2r^2}(xdu + ydv),$$

$$J'(dy) = \frac{1}{2r^2}(xdv - ydu),$$

$$J'(du) = -2xdx + 2ydy,$$

$$J'(dv) = 2xdy + 2ydx.$$

Following André Weil [8], we can write:

$$\bar{\partial}_{J'} = \frac{1}{2}(d + i(J')^{-1}dJ).$$

If $A^k(\Omega)$ is the space of C^∞ sections of the vector bundle $\pi^*(\Lambda^k(T^*U))$, then, we put:

$$A^{p,q}(\Omega) := A^{p+q}(\Omega) \cap A^{p,q}(\Omega \setminus \mathbb{R}^2 \times \{0\}).$$

If $\pi' : \Omega \to \mathcal{W}$ is a ramifide double covering, which coincides with π upto order 2 along $\mathbb{R}^2 \times \{0\}$, then, π' defines, as well, a hypo-analytic structure on Ω. The degenerate complex structure, J', the operator, $\bar{\partial}_{J'}$, and the space of (p,q)-forms, $A^{p,q}(\Omega)$, can also be defined similarly.

Our aim in this note is to establish a solvability theorem as follows.

Theorem 1 (Main Theorem)

If Ω is as above, then the equation $\bar{\partial}_{J'} u = f$ has a solution, $u \in A^{p,q}(\Omega)$, for every $f \in A^{p,q+1}(\Omega)$ such that $\bar{\partial}_{J'} f = 0$.

For the proof, first, we show the solvability in appropriate weighted L^2 spaces by using the approximation method of Hörmander. Then, the theory of Grusin and Visik, on a class of degenerate elliptic operators, is appealed to, in order to obtain regularity.

In what follows, we start the proof of this theorem for the case where the hypoanalytic structure in Ω is induced by π. The proof of the theorem in the more general case obtained via π', can be accomplished by introducing slight modifications to this argument. These modifications will be listed at the end.

EXISTENCE THEOREM IN WEIGHTED L^2 SPACES

Let us start by the observation that, if we set;

$$\omega'_1 = du + i(2xdx - 2ydy),$$

$$\omega'_2 = dv - i(2xdy + 2ydx),$$

then, $B = \{\omega'_1, \omega'_2, \bar{\omega}'_1, \bar{\omega}'_2\}$ forms a basis for $A^1(\Omega)$ over $C^\infty(\Omega)$. The dual of this basis, with respect to the standard metric, is denoted, respectively, by $B^* = \{v'_1, v'_2, \bar{v}'_1, \bar{v}'_2\}$ and can be described as:

$$v'_1 = \frac{1}{2}\left(\partial_u + \frac{1}{2i}\frac{x\partial_x - y\partial_y}{x^2+y^2}\right),$$

$$v'_2 = \frac{1}{2}\left(\partial_v - \frac{1}{2i}\frac{y\partial_x + x\partial_y}{x^2+y^2}\right).$$

We shall define weighted L^2 spaces using the measure $d\lambda = (x^2+y^2)dV$, where dV is the standard Lebesgue measure on $\mathbb{R}^4$. Let ϕ be a continuous function in Ω, and $L^2(\Omega, \phi, d\lambda)$ be the space of functions, which are square integrable, with respect to the measure, $e^{-\phi}d\lambda$. By $L^2_{p,q}(\Omega, \phi, d\lambda)$ we mean the space of (p,q)-forms with coefficients in $L^2(\Omega, \phi, d\lambda)$ in the basis B. Similarly, for an open subset Ω' of Ω, the notation $D_{(p,q)}(\Omega')$ is used when the coefficients lie in $C_0^\infty(\Omega')$ consisting of smooth functions with compact support in Ω'. If ϕ_1 and ϕ_2 are two continuous functions in Ω, then, the operator, $\bar{\partial}_{J'}$, defines a linear, closed densely defined operator:

$$T_{J'} : L^2_{(p,q)}(\Omega, \phi_1, d\lambda) \to L^2_{(p,q+1)}(\Omega, \phi_2, d\lambda).$$

Here, we take the derivative in the sense of the current theory of deRham. The density of the domain of $T_{J'}$ follows from the fact that it contains $D_{(p,q)}(\Omega \setminus \mathbb{R}^2 \times \{0\})$.

Choose also another continuous function, ϕ_3, and let $S_{J'}$ denotes the operator from $L^2_{(p,q)}(\Omega, \phi_2, d\lambda)$ to $L^2_{(p,q)}(\Omega, \phi_3, d\lambda)$ defined by $\bar{\partial}_{J'}$. As is shown, for example, in [9], the existence of weak solutions for the $\bar{\partial}_{J'}$ operator can be deduced from the following inequality for suitable choices of ϕ_i, $i = 1, 2, 3$:

$$\|f\|^2_{\phi_2} \leq C^2(\|T^*_{J'} f\|^2_{\phi_1} + \|S_{J'} f\|^2_{\phi_3}). \tag{1}$$

The Adjoint Operator $\mathbf{T^*_{J'}}$

Using the pullback, $g' = \pi^*(g)$, of the standard metric g on $\mathcal{W}$, we can define an operator $* : A^{p,q}(\Omega) \rightarrow A^{2-p,2-q}(\Omega)$ by:

$$(\psi, \eta)_{g'} \Phi_{g'} = \psi \wedge *\eta \quad \text{for all } \psi \in A^{p,q}(\Omega),$$

where $\Phi_{g'}$ is the volume form associated to the degenerate metric g'. For multi-indices, $I = (i_1, \cdots, i_p)$ and $J = (j_1, \cdots, j_q)$, we use the notation:

$$w'^I \wedge \bar{w}'^J = w'_{i_1} \wedge \cdots \wedge w'_{i_p} \wedge \bar{w}'_{j_1} \wedge \cdots \wedge \bar{w}'_{j_q}.$$

Let $f \in D_{T^*_{J'}}$ be given by:

$$f = {\sum_{|I|=p}}' {\sum_{|J|=q+1}}' f_{I,J} \omega'^I \wedge \bar{\omega}'^J,$$

where $\sum'$ means the summation is performed only over strictly increasing multi-indices. Now, for $u \in D_{(p,q)}(\Omega)$, we can write:

$$\begin{aligned}(f, T_{J'} u)_{\phi_2} &= \int (\bar{\partial}_{J'} u \wedge *f) e^{-\phi_2} \\ &= \int \bar{\partial}_{J'}(u \wedge e^{-\phi_2} * f) \\ &\quad + (-1)^{(p+q)} \int u \wedge \bar{\partial}_{J'}(e^{-\phi_2} * f) \\ &= -\int (u \wedge *e^{\phi_1}(*\bar{\partial}_{J'}(e^{-\phi_2} * f))) e^{-\phi_1} \\ &= (u, -e^{\phi_1}(*\bar{\partial}_{J'}(e^{-\phi_2} * f))))_{\phi_1},\end{aligned}$$

which means that $T^*_{J'}$ is given by:

$$T^*_{J'} f = (-1)^{p-1} {\sum_{I,K}}' {\sum_{1 \leq j \leq 2}}' e^{\phi_1} v'_j.(e^{-\phi_2} f_{I,jK}) \omega'^I \wedge \bar{\omega}'^K.$$

Here $f_{I,jK}$ is non-zero only when j does not belong to K and in this case the index jK is equal to $j \cup K$.

Density of $\mathbf{D_{p,q}(\Omega)}$

In order to prove that smooth differential forms with compact support are dense in $D_{T^*_{J'}} \cap D_{S_{J'}}$ w.r.t the graph norm (Relation 3), the following lemma shall be used, which can be proved by the same method as in [9] (page 80).

Lemma 1

Let η_ν $\nu = 1, 2, \cdots$ be a sequence of functions in $C^\infty_0(\Omega)$, such that $0 \leq \eta_\nu \leq 1$, and for any compact set C in Ω, there exists an integer N, such that $\eta_\nu|_C = 1$ for all ν larger than N. Suppose that $\phi_2 \in C^1(\Omega)$ and that:

$$e^{-\phi_{j+1}} \sum_{k=1}^{2} |\bar{v}'_k . \eta_\nu|^2 \leq e^{-\phi_j} \quad j = 1, 2; \; \nu = 1, 2, \cdots. \tag{2}$$

Then, $D_{(p,q+1)}(\Omega)$ is dense in $D_{T^*_{J'}} \cap D_{S_{J'}}$ for the graph norm:

$$f \rightarrow \|f\|_{\phi_2} + \|T^*_{J'} f\|_{\phi_1} + \|S_{J'} f\|_{\phi_3}. \tag{3}$$

In particular, if the function, $\psi \in C^\infty(\Omega)$, is chosen, such that:

$$\sum_{k=1}^{2} |\bar{v}'_k . \eta_\nu|^2 \leq e^\psi \text{ in } \Omega, \; \nu = 1, 2, \cdots,$$

then, the functions, $\phi_1 = \phi - 2\psi$, $\phi_2 = \phi - \psi$ and $\phi_3 = \phi$, satisfy Relation 2, for any choice of ϕ.

On the other hand, given a $(p, q+1)$-form $f \in D_{T^*_{J'}} \cap D_{S_{J'}}$ and a real number $\epsilon > 0$, one can find a neighborhood U of $\Omega \cap \{\mathbb{R}^2 \times 0\}$, such that the $(p, q+1)$-form $\tilde{f}$, defined by:

$$\tilde{f}(x) = \begin{cases} 0 & \text{if } x \in U, \\ f(x) & \text{otherwise}, \end{cases}$$

lies in the ϵ neighborhood of f, with respect to the graph norm (Relation 3). Now, it is clear that, if f has compact support, then, $\tilde{f}$ can be approximated by smooth $(p, q+1)$-forms of $D_{(p,q+1)}(\Omega \setminus \mathbb{R}^2 \times 0)$ and this shows that the latter is dense in $D_{T^*_{J'}} \cap D_{S_{J'}}$.

Proof of Inequality 1

For $j = 1, 2$ and $w \in C^\infty(\Omega)$, δ_j is defined as follows:

$$\delta_j w = e^\phi(v'_j.(we^{-\phi})) = v'_j.w - w(v'_j.(\phi)).$$

Thanks to the special choice of $d\lambda$, we still can use a Stokes type formula:

$$\int w_1(v'_k.\bar{w}_2) e^{-\phi} d\lambda = -\int (\delta_k w_1) \bar{w}_2 e^{-\phi} d\lambda$$

$$w_1, w_2 \in C^\infty_0(\Omega).$$

The proof of this relation is a simple application of a divergence formula, which is performed here for $k = 1$:

$$\int w_1(v_1'.\bar{w}_2)e^{-\phi}d\lambda$$

$$=\frac{1}{2}\int w_1\left(\partial_u\bar{w}_2+\frac{1}{i}\frac{x\partial_x\bar{w}_2-y\partial_y\bar{w}_2}{x^2+y^2}\right)e^{-\phi}(x^2+y^2)dV$$

$$=-\frac{1}{2}\int\bar{w}_2(\partial_u(w_1e^{-\phi})+\frac{1}{i}\partial_x(xw_1e^{-\phi})$$

$$-\partial_y(yw_1e^{-\phi}))dV=-\int(\delta_kw_1)\bar{w}_2e^{-\phi}d\lambda.$$

Also, we have:

$$\delta_j\bar{v}_k'-\bar{v}_k'\delta_j=\bar{v}_k'.(v_j'.(\phi)),$$

hence, the proof of [9] (pages 82-84) works also in this case and we obtain:

$$\sum\nolimits_{I,K}'\int\sum\nolimits_{j,k=1}' f_{I,jK}\bar{f}_{I,kK}(v_j'.(\bar{v}_k'.(\phi)))e^{-\phi}d\lambda$$

$$+\sum\nolimits_{I,J}'\sum\nolimits_{j=1}'\int|\bar{v}_j'.f_{I,J}|^2e^{-\phi}d\lambda\leq 2\|T_{J'}^*f\|_{\phi_1}^2$$

$$+\|S_{J'}f\|_{\phi_3}^2+2\int|f|^2|\partial_{J'}\psi|^2e^{-\phi}d\lambda. \qquad (4)$$

If ϕ is taken to be the pullback of a strictly pluri-subharmonic function on $\mathcal{W}$, then, there exists a positive continuous function c in Ω, such that, for all $\gamma\in\mathbb{C}^2$, we have:

$$\sum_{j,k=1}^{\prime}(v_j'.(\bar{v}_k'.(\phi)))\gamma_j\bar{\gamma}_k\geq c\sum_{j=1}^{\prime}|\gamma_j|^2.$$

Thus, it follows from Equation 4 that:

$$\int(c-2|\partial_{J'}\psi|^2)|f|^2e^{-\phi}d\lambda\leq 2\|T_{J'}^*f\|_{\phi_1}^2+\|S_{J'}f\|_{\phi_3}^2$$

$$\forall f\in D_{p,q+1}(\Omega),$$

which proves the following lemma

Lemma 2

If ϕ and $\psi\in C^2(\Omega)$ satisfy:

$$\sum\nolimits_{j,k=1}'(v_j'.(\bar{v}_k'.(\phi)))\gamma_j\bar{\gamma}_k\geq 2(|\bar{\partial}_{J'}\psi|^2+e^{\psi})\sum\nolimits_{j=1}'|\gamma_j|^2,$$

$$\gamma\in\mathbb{C}^2.$$

Then, for $\phi_1=\phi-2\psi$, $\phi_2=\phi-\psi$ and $\phi_3=\phi$, we have:

$$\|f\|_{\phi_2}^2\leq\|T_{J'}^*f\|_{\phi_1}^2+\|S_{J'}f\|_{\phi_3}^2,\qquad f\in D_{T_{J'}^*}\cap D_{S_{J'}}.$$

To complete the proof of Inequality 1, assume that $q\in C^\infty(\mathcal{W})$ is a strictly pluri-subharmonic function and let $s=q\circ\pi$. This implies that there exists a strictly positive $m\in C^0(\Omega)$, such that:

$$\forall\gamma\in\mathbb{C}^2,\qquad\sum\nolimits_{j,k=1}' v_j'.\bar{v}_k'.s\gamma_j\bar{\gamma}_k\geq m\sum\nolimits_1'|\gamma_j|^2.$$

Suppose, also, that ψ is the pullback of some function defined on $\mathcal{W}$ by π. Hence we have $|\bar{\partial}_{J'}\psi(p)|<\infty$, $\forall p\in\Omega$. If χ is a C^∞ convex increasing function satisfying:

$$\chi'(t)\geq\sup_{K_t}2(|\bar{\partial}_{J'}\psi|+e^{\psi})/m,$$

where $K_t:=s^{-1}(-\infty,t]\subset\subset\Omega$, then, $\phi=\chi\circ s$ satisfies:

$$\sum\nolimits_{j,k=1}' v_j'.(\bar{v}_k'.(\phi))\gamma_j\bar{\gamma}_k\geq 2(|\bar{\partial}_{J'}\psi|^2+e^{\psi})\sum\nolimits_1'|\gamma_j|^2,$$

$$\gamma\in\mathbb{C}^2. \qquad (5)$$

It means that, by this choice of ϕ, the hypothesis of the previous lemma is satisfied and Inequality 1 is proved. So, we obtain the following.

Theorem 2

Let Ω be a degenerate strongly pseudo-convex domain as in the introduction. Then, there exists infinitely many smooth functions ϕ and ψ in Ω, such that the equation, $T_{J'}u=f$, has a solution, $u\in L^2_{(p,q)}(\Omega,\phi-2\psi,d\lambda)$, for every $f\in L^2_{p,q+1}(\Omega,\phi-\psi,d\lambda)$, such that $T_{J'}f=0$.

DEGENERATE ELLIPTIC OPERATORS OF GRUSIN AND VISIK

In this section, first, some results of Grusin and Vishik [10] on a class of degenerate elliptic operators are reviewed, then, they are used to prove the regularity.

Let the variable, $x\in\mathbb{R}^N$, be decomposed into two groups, $x=(x',y)$, where $x'=(x_1,\cdots,x_k)$, $y=(y_1,\cdots,y_n)$ and $N=k+n$. Let $\mathcal{M}$ denote the set of multi-indices (α,β) with $\delta|\beta|$, an integer for a positive number, δ, $|\alpha|+|\beta|\leq m$ and $|\alpha|+(1+\delta)|\beta|\geq m$. Consider a class of operators:

$$p(x,D)=\sum_{(\alpha,\beta)\in\mathcal{M}}\ \sum_{|\gamma|=|\alpha|+(1+\delta)|\beta|-m}a_{\alpha\beta\gamma}(x)y^\gamma D_{x'}^\beta D_y^\alpha, \qquad (6)$$

which satisfies the following condition:

Condition 1

The operator (Equation 6) is elliptic for $y\neq 0$, i.e.:

$$p^0(y,\xi,\eta)=\sum_{|\alpha|+|\beta|=m}\ \sum_{|\gamma|=\delta|\beta|}a_{\alpha\beta\gamma}(0)y^\gamma\xi^\beta\eta^\alpha\neq 0,$$

for $y \neq 0$, $\xi \in \mathbb{R}^k$, $\eta \in \mathbb{R}^n$ and $|\xi| + |\eta| \neq 0$. Consider the operator:

$$p(y,\xi,D_y) = \sum_{(\alpha,\beta)\in M} \sum_{|\gamma|=|\alpha|+(1+\delta)|\beta|-m} a_{\alpha\beta\gamma}(0) y^\gamma \xi^\beta D_y^\alpha,$$

obtained from Equation 6 by taking the Fourier transform in the variable x'.

Theorem 3

Let condition 1 hold. Then, if $\xi = w$ and $|w| = 1$, the space of solutions of the equation:

$$p(y,w,D_y)v(y) = 0,$$

lying in $\mathcal{S}$ is finite dimensional.

For the proof, Grusin [11] first defines on $\mathbb{R}^n_y$ the weighted Sobolev spaces, $H_{(m,\delta)}(\mathbb{R}^n_y)$, consisting of functions $v(y)$, such that:

$$(1+|y|)^{(m-|\alpha|)\delta} D_y^\alpha v(y) \in L_2(\mathbb{R}^n_y), \quad \forall, |\alpha| \leq m,$$

where the derivatives are to be taken in the sense of generalized functions. He then obtains Theorem 3 as a consequence of the following assertation.

Theorem 4

If condition 1 holds, then, the operator, $p(y,w,D_y)$, for $|w| = 1$, is Fredholm from $H_{(m,\delta)}(\mathbb{R}^n) \to L_2(\mathbb{R}^n_y)$.

Let the following condition be introduced also.

Condition 2

For all $w \in \mathbb{R}^n$ and $|w| = 1$ equation:

$$p(y,w,D_y)v(y) = 0,$$

has no nonzero solution in $\mathcal{S}(\mathbb{R}^n_y)$.

Then, according to Corollary 2 (page 467) and Lemma 4.2 (page 468) in [10], the following is obtained.

Proposition 1

If conditions 1 and 2 hold, then, $p(y,D)u \in C^\infty(\Omega)$ implies that $u \in C^\infty(\Omega)$.

The following theorem has also been proved in [10].

Theorem 5 [10,11]

If Condition 1 holds for the operator (Relation 6), then $p(y,D)$ is hypo-elliptic if and only if Condition 2 holds.

Example

for any integers $l > 0$ and $r > 0$, the operator:

$$\Delta_y^l + |y|^{2r} \Delta_{x'}^l,$$

satisfies Conditions 1 and 2. In fact, if $y \neq 0$, this operator is clearly elliptic, so it suffices to check Condition 2. This can be done by the usual energy method. Let $v \in \mathcal{S}(\mathbb{R}^n)$ be a function in the Schwartz space. Multiplying the corresponding equation;

$$((-\Delta_y)^l + |y|^{2r})v(y) = 0,$$

by $\bar{v}(y)$ and integrating over $\mathbb{R}^n$, we find that:

$$\int \bar{v}(y)(-\Delta_y)^l v(y) dy + \int |y|^{2r} |v(y)|^2 dy = 0.$$

Integration by parts shows that the first integral is non-negative. Hence, $v(y) = 0$.

Remark

As mentioned in [10], all these theorems are also valid for systems, i.e. the coefficients, $a_{\alpha\beta\gamma}$, can be taken to be matrices.

Proof of the Regularity for Weak Solutions

If f is a $(p,q+1)$ form with $(p,q \geq 0)$ as in Theorem 1, we set:

$$\theta f = \sum_{I,K}{}' \sum_{j=1}^{\prime} v'_j . f_{I,jK} \omega^I \wedge \bar{\omega}^K.$$

Thus, we have $T^*_{J'} f = (-1)^{(p-1)} e^{\phi_1} \theta(e^{-\phi_2} f)$. Now, let u be a solution of the equation $\bar{\partial}_{J'} u = f$ belonging to the closure of the image of T^*. Using the relation, $\theta^2 = 0$, we obtain:

$$\theta(e^{-\phi_1} u) = 0, \qquad \bar{\partial}_{J'} u = f.$$

This can also be written as:

$$\bar{\partial}_{J'} u = f, \qquad \theta u = au,$$

where a is a differential operator of order 0 with C^∞ coefficients acting on u. If we set $r^2 = x^2 + y^2$, then:

$$r^2 \Delta_{J'} u = r^2 \theta \bar{\partial}_{J'} u + r^2 \bar{\partial}_{J'} \theta u = r^2 \theta f + r^2 \bar{\partial}_{J'}(au)$$

$$= r^2 \theta f + r^2 (\bar{\partial}_{J'} a) u + r^2 a (\bar{\partial}_{J'} u),$$

or:

$$r^2 \Delta_{J'} u - r^2 (\bar{\partial}_{J'} a) u + r^2 a (\bar{\partial}_{J'} u) = r^2 \theta f.$$

The degenerate elliptic operator $u \to r^2 \Delta_{J'} u - r^2(\bar{\partial}_{J'} a)u + r^2 a(\bar{\partial}_{J'} u)$, is of the type considered by Grusin, with parameters $\delta = 1$, $N = 4$ and $m = n = 2$. The operator, $p(y,w,D_y)$, obtained from its leading order part (as considered previously) is $-\Delta_{(x,y)} + x^2 + y^2$ and satisfies Conditions 1 and 2, as shown in the example above. On the other hand, $r^2 \theta f$ is C^∞, hence, the regularity of solutions is obtained.

CONCLUDING REMARK

Note that, in the more general case, where the hypo-analytic structure in Ω is induced by the application π', the author's proof still works with some slight modifications. By choosing the basis B to be again the pullback of the standard basis of $\mathcal{W}$ via π', essentially, all the formulas and estimations hold without any changes. The proof of the Stokes formula in page 6 should be modified. The regularity follows similarly.

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