Valuations on Simple Artinian Rings

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A value function μ (called a simple valuation) on a simple Artinian ring $A = M_n(D)$ is defined which gives rise to a (Krull) valuation on the division ring D. In fact, it is shown that there is a bijection between simple valuations on $M_n(D)$ and valuations on D. Using the notion of a simple valuation, it is also shown that if $F = Z(D)$ is valuated by v and v extends to a Krull valuation W on D, then v has a unique extension to each finite dimensional division F-subalgebra K with $F \subseteq K \subseteq D$. In particular, when A is an F-central simple algebra with skew field component D, and v is a valuation on F, it is proved that v extends to a valuation w on D if, and only if, the simple valuation μ obtained from v extends to a simple valuation $\overline{\mu}$ on A. When A_i is an F-central simple algebra with simple valuation μ_i, $i = 1, 2$, then it is shown that under suitable conditions on the skew field component D_i, there exists a simple valuation μ on $A_1 \otimes_F A_2$ extending μ_i on A_i.

INTRODUCTION

Let D be a division ring and put $A = M_n(D)$. It is well known that for $n > 1$, A as a simple left (right) Artinian ring does not admit a (semi) valuation. For if v is a valuation on A, then the existence of zero-divisors in A forces $v(X) = \infty$ for some $0 \neq X \in M_n(D) = A$. Thus $P = \{X \in A | v(X) = \infty\}$ is a proper two-sided ideal of A, which is absurd. To avoid this situation, we exclude one of the operations on matrices in A, namely addition, and replace it with another operation called the determinantal sum. Given two matrices $X = (x_1, x_2, \dots, x_n)$, $Y = (y_1, x_2 \dots, x_n)$ in A, we define the determinantal sum of X and Y with respect to the first column as:

$$X \bigtriangledown Y = (x_1 + y_1, x_2, \dots, x_n) \ .$$

The determinantal sum with respect to another column, or a row, is defined for a suitable pair of matrices.

Let Γ be a totally ordered abelian group. A simple valuation on $A = M_n(D)$ is a function $\mu : A \to \Gamma \cup \{\infty\}$ satisfying:

SV 1

$\mu(XY) = \mu(X) + \mu(Y)$, for all $X, Y \in A$.

SV 2

$\mu(X \bigtriangledown Y) \geq \min\{\mu(X), \mu(Y)\}$, for all $X, Y \in A$ such that $X \bigtriangledown Y$ is defined.

SV 3

$\mu(X)$ is unchanged if any row or column of X is multiplied by -1.

SV 4

$\mu(X) = \infty$ for any singular matrix $X \in A$.

We observe that when $n = 1$, i.e., A is a division ring, then SV 1-4 simply say that μ is a valuation on A. We collect some of the consequences of the above axioms in the following:

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Proposition 1

Let μ be a simple valuation on $A = M_n(D)$. Then we have .

SV 5

If $\mu(X) \neq \mu(Y)$, then $\mu(X \bigtriangledown Y) = \min\{\mu(X), \mu(Y)\}$, whenever $X \bigtriangledown Y$ is defined in A.

SV 6

$\mu(E) = 0$ for any elementary matrix E in A.

SV 7

$\mu(X)$ is unchanged if X is multiplied on the left (or right) by an elementary matrix.

SV 8

$\mu(X)$ remains unchanged under any permutation of rows (or columns).

Proof

If $\mu(X) \neq \mu(Y)$, without loss of generality, suppose that $\mu(X) < \mu(Y)$ and $X \bigtriangledown Y$ is defined, say, with respect to the first column. Put $Y = (y_1, \ldots, y_n)$, $Y_1 = (-y_1, y_2, \ldots, y_n)$. We note that $\mu(Y) = \mu(Y_1)$, by SV 3. Suppose that $\mu(X \bigtriangledown Y) > \min\{\mu(X), \mu(Y)\}$ and $\mu(X) < \mu(Y)$, i.e., $\mu(X) < \mu(X \bigtriangledown Y)$. Then we have the contradiction:

$$\begin{aligned}\mu(X) = \mu[X \bigtriangledown (Y \bigtriangledown Y_1)] &= \mu[(X \bigtriangledown Y) \bigtriangledown Y_1] \\ &\geq \min\{\mu(X \bigtriangledown Y), \mu(Y_1)\} \\ &= \min\{\mu(X \bigtriangledown Y), \mu(Y)\} > \mu(X) \ .\end{aligned}$$

This establishes SV 5 and SV 6–8 are proved easily.

The next result shows the connection between simple valuations on a simple Artinian ring A and valuations on its skew field component.

Theorem 1

Let D be a division ring with an abelian valuation v. Then v may be extended to a simple valuation μ on $A = M_n(D)$, for each $n \geq 1$, by the equation

$$\mu(X) = v(DetX), \quad X \in A \ ,$$

where "Det" denotes the Dieudonne' determinant, together with the rule $\mu(X) = \infty$ when X is singular. Moreover, the correspondence $v \leftrightarrow \mu$ is a bijection between abelian valuations on D and simple valuations on A.

Proof

Let Γ be the value group of v. We first observe that the commutativity of the diagram:

$$\begin{array}{ccc} GL_n(D) & \xrightarrow{Det} & D^{*\ ab} \\ \downarrow & \nearrow & \downarrow \\ D^* & \xrightarrow{v} & \Gamma \end{array} \ ,$$

enables us to speak of $v(DetX)$. Now define a function $\mu : M_n(D) \longrightarrow \Gamma \cup \{\infty\}$ by $\mu(X) = v(DetX)$, together with the rule $\mu(X) = \infty$ when X is singular. We show that μ defines a simple valuation on A. The axioms SV 1 and SV 3-4 are easily satisfied by properties of the determinant. To show SV 2, let

$$X = (x_1, x_2, \ldots, x_n), \ Y = (y_1, x_2, \ldots, x_n) \ ,$$

so that $X \bigtriangledown Y$ is defined with respect to the first column. If X and Y are singular, then it is easily seen that $X \bigtriangledown Y$ is also singular. Thus, $\mu(X \bigtriangledown Y) = \min\{\mu(X), \mu(Y)\} = \infty$.

Now, assume that X is non-singular. Thus, the columns $x_1, x_2, \ldots, x_n, y_1$ are left linearly dependent over D so we may have $x_1 = \lambda y_1 + \sum_{i=2}^{n} \lambda_i \, x_i$, where $\lambda \in D^*$ and $\lambda_i \in D$. Therefore:

$$X \bigtriangledown Y = ((1+\lambda)y_1 + \sum_{i=2}^{n} \lambda_i \, x_i, \ x_2, \ldots, x_n) \ .$$

So, $Det(X \bigtriangledown Y) = Det((1+\lambda)y_1, x_2, \ldots, x_n) = (\overline{1+\lambda})DetY$, where $(\overline{1+\lambda}) \in D^{*ab}$. We also have $X = (\lambda y_1 + \sum_{i=2}^{n} \lambda_i \ x_i, x_2, \ldots, x_n)$, so $DetX = \overline{\lambda} DetY$. Thus:

$$\begin{aligned}\mu(X \bigtriangledown Y) = v(DetX \bigtriangledown Y) &= v((\overline{1+\lambda})DetY) \\ &= v(1+\lambda) + \mu(Y) \\ &\geq \min\{v(1), v(\lambda)\} + \mu(Y) \\ &= \min\{\mu(Y), v(\lambda DetY)\}\end{aligned}$$

$$= \min\{\mu(Y), v(\overline{\lambda} DetY)\}$$
$$= \min\{\mu(Y), v(DetX)\}$$
$$= \min\{\mu(Y), \mu(X)\} ,$$

and this establishes SV 2.

To show the converse, let μ be a simple valuation on $M_n(D)$, and consider the embedding $D \xrightarrow{f} DI \subset M_n(D)$, where I is the $n \times n$ unit matrix. Denote by $D_i(d)$ the matrix $I + (d-1)\, e_{ii}$, where e_{ii} differs from the zero matrix only in the (i,i)-th place, which is 1. We first calculate $\mu(dI)$, $d \in D^*$. We have:

$$\mu(dI) = \mu(D_1(d)D_2(d)\dots D_n(d)) = \sum_{i=1}^{n} \mu(D_i(d)) .$$

However, because of SV 8, we have $\mu(D_i(d)) = \mu(D_j(d))$ for all $1 \le i,j \le n$. Thus $\mu(dI) = n\mu(D_n(d))$. Now we have:

$$\begin{aligned}\mu(d_1Id_2I) &= \mu(d_1d_2I) = n\mu(D_n(d_1d_2))\\ &= n\mu(D_n(d_1)D_n(d_2))\\ &= n\mu(D_n(d_1)) + n\mu(D_n(d_2))\\ &= \mu(d_1I) + \mu(d_2I) .\end{aligned}$$

We also have:

$$\begin{aligned}\mu(d_1I + d_2I) &= \mu((d_1+d_2)I) = n\mu(D_n(d_1+d_2))\\ &= n\mu(D_n(d_1) \bigtriangledown D_n(d_2))\\ &\ge n\min\{\mu(D_n(d_1)), \mu(D_n(d_2))\}\\ &= \min\{\mu(d_1I), \mu(d_2I)\} ,\end{aligned}$$

where $\bigtriangledown$ is with respect to the n-th column. Thus, the restriction of μ to DI is a valuation. Now, by pullback we obtain a valuation $v = \mu \circ f$ on D, and this completes the proof. We may observe that a simple valuation on $M_n(D)$ may also be extended to a matrix valuation on D which was developed in [1-3]. We also note the following.

Corollary 1

Let D be a division ring with center F, and let v be a valuation on F. If v extends to a valuation w on D, then v has a unique extension to each finite dimensional division F-subalgebra K with $F \subseteq K \subseteq D$.

Proof

If v extends to a valuation w on D, then, by Theorem 1 we obtain a unique simple valuation μ on each $M_n(D)$, $n \ge 1$, given by the formula $\mu(X) = w(DetX)$, $X \in M_n(D)$. Denote the value group of w by $\Gamma_D = \Gamma_F \otimes_Z Q$, where Γ_F is the value group of F, and Q, Z are the rational numbers and the integers respectively. Now, let K be any finite dimensional F-division subalgebra with $F \subseteq K \subseteq D$ and put $r = [K : F]$. It is clear that $w|_K$ is a valuation on K. To show it is unique, we prove that its value is completely determined by v. Consider the simple Artinian ring $M_r(D)$. This contains two isomorphic finite dimensional F-subalgebras KI and K_ρ, where I is the $r \times r$ unit matrix and K_ρ is the image of K under the regular matrix representation ρ. By the Skolem-Noether Theorem, there is an invertible matrix $A \in M_r(D)$ such that:

$$\rho(x) = A^{-1}(xI)A,\ x \in K .$$

Now, using the simple valuation μ on $M_r(D)$, we have:

$$\mu(\rho(x)) = v(\det \rho(x)) = \mu(A^{-1}) + \mu(xI) + \mu(A) = w|_K(DetxI) = rw|_K(x) .$$

Thus, $W|_K$ is given by:

$$w|_K(x) = \frac{1}{r}v(\det \rho(x)) = \frac{1}{r}v(N(x)) ,$$

where $N(x)$ is the norm of K to F. This completes the proof. In the special case where $[D : F] < \infty$, Corollary 1 may be restated as corollary 2.

Corollary 2

Let D be a finite dimensional F-central division algebra and let v be a valuation on F. Then v extends to a valuation w on D if, and only if, v has a unique extension to each finite dimensional division F-subalgebra K with $F \subseteq K \subseteq D$.

As another implication of Corollary 1 we may have the following result which is one side of the theorem proved by A. Wadsworth [4].

Corollary 3

Let D be a finite dimensional F-central division algebra and let v be a valuation on F. If v extends to a valuation w on D, then v has a unique extension to each field K with $F \subseteq K \subseteq D$.

We note that a simple valuation μ on $A = M_n(D)$ does not generally define a valuation on arbitrary subfields of A. However, in some special cases, we obtain the following results.

Theorem 2

Let D be a finite dimensional F-central division algebra and let v be a valuation on F. Then v extends to a valuation w on D if, and only if, the simple valuation on $M_r(F)$, $r = [D : F]$, obtained from v, defines a valuation on the regular matrix representation D_ρ of D.

Proof

Assume v extends to a valuation w, say, on D. By Theorem 1, w gives rise to a unique simple valuation $\overline{\mu}$ on $M_r(D)$. Put $\mu = \overline{\mu}|_{M_r(F)}$; then μ is the simple valuation on $M_r(F)$ extending v. We know that:

$$DI \xrightarrow{f} D \xrightarrow{\rho} D_\rho \subset M_r(D) ,$$

is an isomorphism of F-central simple algebras. Thus, by the Skolem-Noether Theorem we have:

$$\rho f(dI) = T^{-1}(dI)T ,$$

for some unit T in $M_r(D)$. Therefore, $\rho(d) = T^{-1}(dI)T$. So:

$$\overline{\mu}(\rho(d)) = \overline{\mu}(T^{-1}) + \overline{\mu}(dI) + \overline{\mu}(T) = \overline{\mu}(dI) .$$

Thus, $\mu(\rho(d)) = \overline{\mu}(dI)$. Now, since $\overline{\mu}$ defines a valuation on DI, the last equation implies that μ defines a valuation on D_ρ.

Conversely, assume that the value group of v is Γ and put $\Delta = \Gamma \otimes_Z Q$, the divisible hull of Γ. Define a function $w : D \longrightarrow \Delta$ by $w(d) = \frac{1}{r}\ \mu(\rho(d))$. Since μ defines a valuation on D_ρ then it is clear that w defines a valuation on D and the uniqueness is obvious.

As a consequence of this theorem, we have the following.

Corollary 4

Let D be a finite dimensional F-central division algebra and let v be a valuation on F. Then v extends to a valuation w on D if, and only if, the simple valuation μ on $M_r(F)$, obtained from v extends to a simple valuation $\overline{\mu}$ on $M_r(D)$, $r = [D : F]$. Moreover, the extensions, when they exist, are unique.

Proof

Assume that v extends to a valuation w on D. By Theorem 1, w gives rise to a unique simple valuation on $M_r(D)$, given by $\overline{\mu}(X) = w(DetX)$, $X \in M_r(D)$, and $\overline{\mu}(X) = \infty$ if X is a singular. To show that the restriction of $\overline{\mu}$ to $M_r(F)$ is μ, we have for each $X \in M_r(F)$:

$$\overline{\mu}(X) = w(DetX) = w(\det X)$$
$$= v(\det X) = \mu(X) .$$

Thus, $\overline{\mu}|_{M_r(F)} = \mu$. Now suppose that μ extends to a simple valuation $\overline{\mu}$ on $M_r(D)$. By Theorem 2, it is enough to show that μ defines a valuation on D_ρ. Using the Skolem-Noether Theorem to $DI \longrightarrow D \longrightarrow D_\rho \subset M_r(D)$, we obtain $\rho(d) = T^{-1}(dI)T$ for some unit T in $M_r(D)$. Thus:

$$\overline{\mu}(\rho(d)) = \mu(\rho(d)) = \overline{\mu}(T^{-1})$$
$$+ \overline{\mu}(dI) + \overline{\mu}(T) = \overline{\mu}(dI) .$$

Now, since $\overline{\mu}$ defines a valuation on DI, we conclude that μ defines a valuation on D_ρ and this completes the proof.

Given a valuation v on D and $E \subseteq D$, denote by Γ_E and Γ_D the value groups of E and D respectively. Denote by $\overline{D}$ the residue class field of D. If $[D : E] < \infty$, D is considered as a left E-vector space, then one has the fundamental inequality:

$$[D : E] \geq |\Gamma_D : \Gamma_E|[\overline{D} : \overline{E}] . \qquad (1)$$

D is called defectless over E if equality holds in Equation 1. We may now prove the last result of this note.

Theorem 3

Let A_i, μ_i be simple valued F-central simple algebras with $\mu_1|_F = \mu_2|_F$ and skew field components D_i, $i = 1, 2$. Suppose that (1) D_1 is defectless over F; (2) $\Gamma_{D_1} \cap \Gamma_{D_2} = \Gamma_F$; (3) $\overline{D}_1 \otimes_F \overline{D}_2$ is a division ring. Then, there is a simple valuation μ on $A_1 \otimes_F A_2$ extending μ_i on A_i and with $\Gamma_{A_1 \otimes A_2} = \Gamma_{A_1} + \Gamma_{A_2}$.

Proof

By Theorem 1, μ_i gives rise to a valuation v_i, say, on D_i, $i = 1, 2$. By a result of [5], $D_1 \otimes_F D_2$ is a division ring with a valuation v extending v_1, v_2 and with $\Gamma_{D_1 \otimes D_2} = \Gamma_{D_1} + \Gamma_{D_2}$. Now, using Theorem 1 again, v gives rise to a simple valuation $\overline{\mu}$, say, on $A_1 \otimes_F A_2$. It is now clearly seen that $\overline{\mu}|A_i = \mu_i$.

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