On the Periodic Solutions for the Liénard Equation

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In this paper, a sufficient and necessary condition and some criteria which judge the existence of the periodic solutions for the Liénard equation are given. One of the results contains a main theorem (IIB) of J.G. Wendel [1].

INTRODUCTION

In this paper, we consider the Liénard equation:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 , \tag{1}$$

with $f, g : \mathbb{R} \to \mathbb{R}$, or its equivalent representation:

$$\begin{cases} \dfrac{dx}{dt} = y - F(x) \\ \dfrac{dy}{dt} = -g(x) \end{cases} , \tag{2}$$

where $F(x) = \int_0^x f(u)du$, and let $G(x) = \int_0^x g(u)du$. We assume that $f(x)$ and $g(x)$ are all continuous functions and satisfy the conditions of existence and uniqueness for the solutions having initial problems. Let $xg(x) > 0$ for $x \neq 0$, i.e., the origin is the unique critical point for the system in Equation 2. Many authors have studied the existence of periodic solutions of Equations 1 or 2 [1-5]. In this paper, we further study this problem. We give a sufficient and necessary condition that Equation 1 has the periodic solutions (see Theorem 1). This result contains a main theorem (IIB) of Wendel [1]. In addition, we also give some criterions that Equation 1 has the periodic solutions.

We call the curve $y = F(x)$ the characteristic curve of Equation 2 and define regions $D_1 = \{(x, y);\ x \geqslant 0 \text{ and } y > F(x)\}$, $D_2 = \{(x, y); x \geqslant 0 \text{ and } y < F(x)\}$.

Lemma 1

Suppose that γ is an orbit of Equation 2 passing through a point $B(x_0, F(x_0))(x_0 \neq 0)$ on the characteristic curve. Then γ must traverse the y-axis at two points $A(0, y_A)(y_A \geqslant 0)$ and $C(0, y_c)(y_c \leqslant 0)$. For the proof of Lemma 1, see [6], Lemma 3.2.

THE PERIODIC SOLUTIONS OF EQUATION 1

Theorem 1

Suppose that (1) $f(-x) = -f(x)$, $g(-x) = -g(x)$; (2) $g(x) > 0$ for $x > 0$; (3) there exist that $k > 0, x_1 > 0$ such that $F(x) \leqslant 0$ for $0 \leqslant x \leqslant x_1, F(x) > -k > -\infty$ for $x \geqslant 0$. Then, all solutions of Equation 1 having initial conditions $x = 0$, $\dot{x} = y_A > 0$ are periodic if

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Scientia Iranica, Vol. 1, No. 4, © Sharif University of Technology, January 1995.

and only if:

$$\overline{\lim_{x\to\infty}}[G(x)+F(x)]=\infty \ . \tag{3}$$

Proof

We first prove the necessary by reduction to absurdity. Assume that Equation 3 does not hold. This implies that there exist M, $N>0$ such that $G(x)<M$, $F(x)<N$ for $x\geqslant 0$. Thus, by Lemma 1 in [7] (or see [8]), there exists a point $A(0,y_A)(y_A>0)$ on the positive semi-trajectory of Equation 2 leaving the point $A(0,y_A)(y_A>0)$ which will stay above the characteristic curve and go to infinity. Therefore, Equation 1 cannot have the periodic solutions.

Assume now that the condition in Equation 3 holds. Let γ^+ denote the positive semi-trajectory passing through a point $A(0,y_A)(y_A>0)$.

If $\lim_{x\to\infty} G(x)=\infty$ for $x>0$, we consider the curves defined by:

$$\lambda(x,y)=\frac{1}{2}(y+k)^2+G(x)=\text{constant.}$$

The total derivative of λ along a solution of Equation 2 is given by $\dot{\lambda}=-g(x)[F(x)+k]\leqslant 0$ for $x\geqslant 0$. This shows that there exists a $t_1>0$ such that γ^+ enters the interior of $\lambda(x,y)=c$. Because $x(t)$ increases, $y(t)$ decreases in D_1, γ^+ must cross the characteristic curve.

If $\overline{\lim_{x\to\infty}} F(x)=\infty$, obviously γ^+ traverses the characteristic curve. By Lemma 1, γ^+ crosses the y-axis at a point $C(0,y_c)(y_c\leqslant 0)$. Next, by Condition 3, γ^+ cannot tend to the origin. It follows that $y_c\neq 0$. Condition 1 shows that γ has mirror symmetry about the y-axis. Thus, γ is an oval surrounding the origin. Therefore, we have proved that Theorem 1 is correct.

Corollary 1

Assume that (1) $f(-x)=-f(x)$, $g(-x)=-g(x)$; (2) $g(x)>0$ for $x>0$; (3) there exist that $k>0$, $x_1>0$ such that $F(x)\leqslant 0$ for $0\leqslant x\leqslant x_1$ and $|F(x)|\leqslant k$ for all x; (4) $G(\infty)=\infty$. Then, all solutions of Equation 1 having initial conditions $x=0$, $\dot{x}=y_A>0$ are periodic.

Remark 1

A main result of Wendel [1] is a special case of Theorem 1.

Theorem 2

Suppose that (1) $f(-x)=-f(x)$, $g(-x)=-g(x)$; (2) $g(x)>0$ for $x>0$; (3) there exist a $k>0$, $x_1>0$ such that $F(x)\leqslant 0$ for $0\leqslant x\leqslant x_1$, $F(x)\leqslant k<\infty$ for all x; (4) $\overline{\lim_{x\to\infty}}\left[\int_0^x \frac{g(u)}{c-F(u)}du+F(x)\right]=\infty(c>0)$. Then all solutions of Equation 1 having initial conditions $x=0$, $\dot{x}=y_A>0$ in Equation 2 are periodic.

Proof

We consider the system in Equation 2. Take a point $A(0,y_A)$ on the positive y-axis. Let γ^+ be the positive semi-trajectory passing through A.

If $\sup_{x>0} F(x)\geqslant y_A$, then there exists a $x'>0$ such that $F(x')\geqslant y_A$. Let the straight line $x=x'$ traverse the characteristic curve at $B(x',y')$. It follows that $y'\geqslant y_A$. Because $x(t)$ increases and $y(t)$ decreases in D_1 as t increases, it is obvious that γ^+ crosses the characteristic curve. Thus, by Lemma 1, γ^+ must cross the negative y-axis at $C(0,y_c)(y_c\leqslant 0)$. By Condition 3, γ^+ cannot tend to the origin. It follows that $y_c\neq 0$. Therefore, by Condition 1 and symmetry, γ is a closed curve.

If $\sup_{x>0} F(x)<y_A$, take a point $M(0,y_0)$ on the positive y-axis such that $y_0>y_A$. Then, by Condition 4, there exists a $x_2>0$ such that:

$$\int_0^{x_2}\frac{g(x)}{y_0-F(x)}dx+F(x_2)>y_0 \ . \tag{4}$$

We consider the comparison equation:

$$\begin{cases}\dot{x}=y_0-F(x)\\ \dot{y}=-g(x)\end{cases} \tag{5}$$

Let $y=y(x)$ denote the positive semi-trajectory γ_M^+ passing through the point M. Thus, by Equation 4, $y(x_2)<F(x_2)$. This implies that γ_M^+ crosses the characteristic curve for $0\leqslant x\leqslant x_2$. Because, for $0\leqslant x\leqslant x_2$, $y\leqslant y_0$, we have:

$$\frac{dy}{dx}\Big|_{(2)}\leqslant\frac{dy}{dx}\Big|_{(5)}<0 \ ,$$

this shows that γ^+ must cross the characteristic curve for $0 \leqslant x \leqslant x_2$. It follows that γ_M^+ will cross the negative y-axis at $C(0, y_c)(y_c < 0)$ by Lemma 1 and Condition 3 as t increases. By symmetry, γ is a closed curve. Therefore, we have proved that the result of Theorem 2 is correct. The proof is complete.

We can show the following theorem in a way similar to the proof of Theorem 2.

Theorem 3

Suppose that (1) $f(-x) = -f(x)$, $g(-x) = -g(x)$; (2) $g(x) > 0$ for $x > 0$; (3) there exist that $k > 0$, $x_1 > 0$ such that $F(x) \geqslant 0$ for $0 \leqslant x \leqslant x_1$ and $F(x) \geqslant -k > -\infty$ for all x; (4) $\overline{\lim_{x\to\infty}} \left[\int_0^x \frac{g(u)}{c + F(u)} du - F(x)\right] = \infty (c > 0)$. Then, all solutions of Equation 1 having initial conditions $x = 0$, $\dot{x} = y_c < 0$, are periodic.

In the following, we give two examples to show the utility of our theorems.

Example 1

Take, in the system in Equation 2, $F(x) = \cos x - 1$, $g(x) = x$. It is easy to check that $F(x)$ and $g(x)$ satisfy the all conditions of Corollary 1. Thus, by Corollary 1 all the orbits of Equation 2 leaving the positive y-axis are closed. Therefore, those orbits are the periodic solutions of Equation 1.

Example 2

Take, in the system in Equation 2, $F(x) = \begin{cases} \cos x - 1, & 0 \leqslant x \leqslant 2\pi \\ -(x^2 - 4\pi^2)^2, & x \geqslant 2\pi \end{cases}$, $g(x) = xe^{x^2}$. It is obvious that Conditions 1, 2 and 3 of Theorem 2 are satisfied. We now check that Condition 4 is also satisfied. Because:

$$\int_{2\pi}^{x} \frac{xe^{x^2}}{c + (x^2 - 4\pi^2)^2} dx - (x^2 - 4\pi^2)^2$$
$$= \frac{1}{2}\frac{e^{x^2}}{c + (x^2 - 4\pi^2)^2} - \frac{1}{2}e^{4\pi^2} - (x^2 - 4\pi^2)^2$$
$$+ \; 2\int_{2\pi}^{x} \frac{x(x^2 - 4\pi^2)e^{x^2}}{[c + (x^2 - 4\pi^2)]^2} dx$$
$$> \frac{1}{2}\frac{e^{x^2}}{c + (x^2 - 4\pi^2)^2} - \frac{1}{2c}e^{4\pi^2} - (x^2 - 4\pi^2)^2$$
$$\to \infty (x \to \infty) \, ,$$

Condition 4 is satisfied. Therefore, the solutions of Equation 1 passing through any points on the positive y-axis are all periodic.

ACKNOWLEDGEMENT

This paper was completed during the author's visit to the Institute of Mathematics, Academia Sinica. The author wishes to acknowledge his gratitute to Professor Yu Shu-Xiang for his ardent help and direction.

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