On the Periodic Solutions for the Liénard Equation

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In this paper, a sufficient and necessary condition and some criteria which judge the existence of the periodic solutions for the Liénard equation are given. One of the results contains a main theorem (IIIB) of J.G. Wendel [1].

INTRODUCTION

In this paper, we consider the Liénard equation:
\[ \ddot{x} + f(x) \dot{x} + g(x) = 0, \tag{1} \]
with \( f, g : \mathbb{R} \to \mathbb{R} \), or its equivalent representation:
\[
\begin{align*}
\frac{dx}{dt} &= y - F(x) \\
\frac{dy}{dt} &= -g(x)
\end{align*}
, \tag{2}
\]
where \( F(x) = \int_{x}^{0} f(u) du \), and let \( G(x) = \int_{0}^{x} g(u) du \). We assume that \( f(x) \) and \( g(x) \) are all continuous functions and satisfy the conditions of existence and uniqueness for the solutions having initial problems. Let \( xg(x) > 0 \) for \( x \neq 0 \), i.e., the origin is the unique critical point for the system in Equation 2. Many authors have studied the existence of periodic solutions of Equations 1 or 2 [1-5]. In this paper, we further study this problem. We give a sufficient and necessary condition that Equation 1 has the periodic solutions (see Theorem 1). This result contains a main theorem (IIIB) of Wendel [1]. In addition, we also give some criterions that Equation 1 has the periodic solutions.

We call the curve \( y = F(x) \) the characteristic curve of Equation 2 and define regions \( D_1 = \{(x, y); x > 0 \text{ and } y > F(x)\}, \ D_2 = \{(x, y); x > 0 \text{ and } y < F(x)\}. \)

Lemma 1

Suppose that \( \gamma \) is an orbit of Equation 2 passing through a point \( B(x_0, F(x_0))(x_0 \neq 0) \) on the characteristic curve. Then \( \gamma \) must traverse the \( y \)-axis at two points \( A(0, y_A)(y_A > 0) \) and \( C(0, y_c)(y_c < 0) \). For the proof of Lemma 1, see [6], Lemma 3.2.

THE PERIODIC SOLUTIONS OF EQUATION 1

Theorem 1

Suppose that \( (1) \ f(-x) = -f(x), \ g(-x) = -g(x); \ (2) \ g(x) > 0 \text{ for } x > 0; \ (3) \text{ there exist } k > 0, x_1 > 0 \text{ such that } F(x) \leq 0 \text{ for } 0 \leq x \leq x_1, F(x) > -k > -\infty \text{ for } x > 0. \)

Then, all solutions of Equation 1 having initial conditions \( x = 0, \ \dot{x} = y_A > 0 \) are periodic if

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and only if:
\[ \lim_{x \to \infty} (G(x) + F(x)) = \infty. \] (3)

**Proof**

We first prove the necessary by reduction to absurdity. Assume that Equation 3 does not hold. This implies that there exist \( M, N > 0 \) such that \( G(x) < M, F(x) < N \) for \( x \geq 0 \). Thus, by Lemma 1 in [7] (or see [8]), there exists a point \( A(0, y_A)(y_A > 0) \) on the positive semi-trajectory of Equation 2 leaving the point \( A(0, y_A)(y_A > 0) \) which will stay above the characteristic curve and go to infinity. Therefore, Equation 1 cannot have the periodic solutions.

Assume now that the condition in Equation 3 holds. Let \( \gamma^+ \) denote the positive semi-trajectory passing through a point \( A(0, y_A)(y_A > 0) \).

If \( \lim_{x \to \infty} G(x) = \infty \) for \( x > 0 \), we consider the curves defined by:
\[ \lambda(x, y) = \frac{1}{2}(y + k)^2 + G(x) = \text{constant}. \]

The total derivative of \( \lambda \) along a solution of Equation 2 is given by \( \dot{\lambda} = -g(x)[F(x) + k] \leq 0 \) for \( x \geq 0 \). This shows that there exists a \( t_1 > 0 \) such that \( \gamma^+ \) enters the interior of \( \lambda(x, y) = c \). Because \( x(t) \) increases, \( y(t) \) decreases in \( D_1 \), \( \gamma^+ \) must cross the characteristic curve.

If \( \lim_{x \to \infty} F(x) = \infty \), obviously \( \gamma^+ \) traverses the characteristic curve. By Lemma 1, \( \gamma^+ \) crosses the \( y \)-axis at a point \( C(0, y_e)(y_e \leq 0) \). Next, by Condition 3, \( \gamma^+ \) cannot tend to the origin. It follows that \( y_e \neq 0 \). Condition 1 shows that \( \gamma \) has mirror symmetry about the \( y \)-axis. Thus, \( \gamma \) is an oval surrounding the origin. Therefore, we have proved that Theorem 1 is correct.

**Corollary 1**

Assume that (1) \( f(-x) = -f(x), g(-x) = -g(x) \); (2) \( g(x) > 0 \) for \( x > 0 \); (3) there exist \( k > 0, x_1 > 0 \) such that \( F(x) \leq 0 \) for \( 0 \leq x \leq x_1 \) and \( |F(x)| \leq k \) for all \( x \); (4) \( G(\infty) = \infty \). Then, all solutions of Equation 1 having initial conditions \( x = 0, \dot{x} = y_A > 0 \) are periodic.

**Remark 1**

A main result of Wendel [1] is a special case of Theorem 1.

**Theorem 2**

Suppose that (1) \( f(-x) = -f(x), g(-x) = -g(x) \); (2) \( g(x) > 0 \) for \( x > 0 \); (3) there exist a \( k > 0, x_1 > 0 \) such that \( F(x) \leq 0 \) for \( 0 \leq x \leq x_1 \), \( F(x) \leq k \) for all \( x \); (4) \( \lim_{x \to \infty} \left[ \int_0^x g(u) \right] = \infty(c > 0) \).

Then all solutions of Equation 1 having initial conditions \( x = 0, \dot{x} = y_A > 0 \) in Equation 2 are periodic.

**Proof**

We consider the system in Equation 2. Take a point \( A(0, y_A) \) on the positive \( y \)-axis. Let \( \gamma^+ \) be the positive semi-trajectory passing through \( A \).

If \( \sup_{x > 0} F(x) \geq y_A \), then there exists a \( x' > 0 \) such that \( F(x') \geq y_A \). Let the straight line \( x = x' \) traverse the characteristic curve at \( B(x', y') \).

It follows that \( y' \geq y_A \). Because \( x(t) \) increases and \( y(t) \) decreases in \( D_1 \) as \( t \) increases, it is obvious that \( \gamma^+ \) crosses the characteristic curve. Thus, by Lemma 1, \( \gamma^+ \) must cross the negative \( y \)-axis at \( C(0, y_e)(y_e \leq 0) \). By Condition 3, \( \gamma^+ \) cannot tend to the origin. It follows that \( y_e \neq 0 \).

Therefore, by Condition 1 and symmetry, \( \gamma \) is a closed curve.

If \( \sup_{x > 0} F(x) < y_A \), take a point \( M(0, y_0) \) on the positive \( y \)-axis such that \( y_0 > y_A \). Then, by Condition 4, there exists a \( x_2 > 0 \) such that:
\[ \int_0^{x_2} \frac{g(x)}{y_0 - F(x)} \, dx + F(x_2) > y_0. \] (4)

We consider the comparison equation:
\[ \begin{cases} \dot{x} = y_0 - F(x) \\ \dot{y} = -g(x) \end{cases} \] (5)

Let \( y = y(x) \) denote the positive semi-trajectory \( \gamma^+_M \) passing through the point \( M \). Thus, by Equation 4, \( y(x_2) < F(x_2) \). This implies that \( \gamma^+_M \) crosses the characteristic curve for \( 0 \leq x \leq x_2 \). Because, for \( 0 \leq x \leq x_2, y \leq y_0 \), we have:
\[ \frac{dy}{dx} \left|_{(s)} \right. \leq \frac{dy}{dx} \left|_{(s)} \right. < 0, \]
this shows that $\gamma^+$ must cross the characteristic curve for $0 \leq x \leq x_2$. It follows that $\gamma^+_M$ will cross the negative $y$-axis at $C(0,y_0)(y_0 < 0)$ by Lemma 1 and Condition 3 as $t$ increases. By symmetry, $\gamma$ is a closed curve. Therefore, we have proved that the result of Theorem 2 is correct. The proof is complete.

We can show the following theorem in a way similar to the proof of Theorem 2.

**Theorem 3**

Suppose that (1) $f(-x) = -f(x)$, $g(-x) = -g(x)$; (2) $g(x) > 0$ for $x > 0$; (3) there exist that $k > 0$, $x_1 > 0$ such that $F(x) \geq 0$ for $0 \leq x \leq x_1$ and $F(x) \geq -k > -\infty$ for all $x$;

$\lim_{x \to -\infty} \left[ \int_0^x \frac{g(u)}{c + F(u)} du - F(x) \right] = \infty(c > 0)$.

Then, all solutions of Equation 1 having initial conditions $x = 0, \dot{x} = y_0 < 0$, are periodic.

In the following, we give two examples to show the utility of our theorems.

**Example 1**

Take, in the system in Equation 2, $F(x) = \cos x - 1$, $g(x) = x$. It is easy to check that $F(x)$ and $g(x)$ satisfy the all conditions of Corollary 1. Thus, by Corollary 1 all the orbits of Equation 2 leaving the positive $y$-axis are closed. Therefore, those orbits are the periodic solutions of Equation 1.

**Example 2**

Take, in the system in Equation 2, $F(x) = \begin{cases} \cos x - 1, & 0 \leq x \leq 2\pi \\ -(x^2 - 4\pi^2)^2, & x \geq 2\pi \end{cases}$, $g(x) = xe^x$. It is obvious that Conditions 1, 2 and 3 of Theorem 2 are satisfied. We now check that Condition 4 is also satisfied. Because:

$$\int_0^x \frac{xe^{x^2}}{c + (x^2 - 4\pi^2)^2} dx - (x^2 - 4\pi^2)^2 = \frac{1}{2} c + (x^2 - 4\pi^2)^2 - \frac{1}{2} e^{4\pi^2} - (x^2 - 4\pi^2)^2$$

$$+ 2 \int_{2\pi}^x \frac{x(x^2 - 4\pi^2)e^{x^2}}{c + (x^2 - 4\pi^2)^2} dx$$

Condition 4 is satisfied. Therefore, the solutions of Equation 1 passing through any points on the positive $y$-axis are all periodic.

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**REFERENCES**


