Periodic Solutions of Certain n-th Order Non-Autonomous Differential Equations

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The nonlinear third order differential equation:

$$x''' + \psi(x')x'' + \varphi(x)x' + \theta(x,x',x'') = p(t) \quad , \quad p(t+\omega) \equiv p(t) \ ,$$

has been considered by Reissig [1]. He proved the existence of at least one non-trivial periodic solution of period ω. In this paper, we extend the above results to the case of n-th order equations. For this purpose Leray-Schauder principle, as suggested by G. Güssefeldt [2] is applied.

INTRODUCTION

Reissig [1] investigated the existence of periodic solutions of the equation:

$$x''' + \phi(x')x'' + K^2x' + fx) = p(t), \qquad p(t+\omega) = p(t) \ . \tag{1}$$

The object of this paper is to consider the more general type of equation:

$$x^{(n+1)} + \sum_{i=1}^{n-1} \phi_i(x^{(n-i)})x^{(n-i+1)} + [\phi_n(x) + K]x' + f(t,x) = p(t) \ , \tag{2}$$

where the functions $\phi_i(y), i = 1,2,\dots,n-1$ are continuous, $f(t,x)$ and $p(t)$ are continuous and ω-periodic with respect to t and $\phi_n(t)$ is such that $\Phi_n(y) = \int^y \phi_n(t)dt$ is continuous.

It will be shown that, under certain conditions, on functions involved, Equation 2 has at least one periodic solution of period ω. The method employed is similar to the method used in [1]. Now let:

$$\beta_k = \cos\frac{\pi(2k-1)}{n} \ , \ k = 1,2,\dots,n$$
$$\alpha = \text{Min}\ |\beta_k| \ , \ \text{if Min}\ |\beta_k| \neq 0$$
$$= 1 \ , \ \text{if Min}\ |\beta_k| = 0 \ .$$

Next we introduce the function:

$$G(t,s) = \begin{cases} \sum_{k=1}^{n} \frac{\exp K^{\frac{1}{n}}\lambda_k(\omega-s+t)}{n\lambda_k^{n-1}\exp K^{\frac{1}{n}}(\lambda_k\omega-1)} \ , \\ 0 \le t \le s \le \omega \\ \sum_{k=1}^{n} \frac{\exp K^{\frac{1}{n}}\lambda_k(t-s)}{n\lambda_k^{n-1}\exp K^{\frac{1}{n}}(\lambda_k\omega-1)} \ , \\ 0 \le s \le t \le \omega \end{cases} \tag{3}$$

where $\omega \in (0, \pi K^{-1/n})$ and $\lambda_k = \exp(i\frac{\pi}{n}(2k-1))$. The function $G(t,s)$ has the following properties:

(a) $G(t,s)$ is continuous and has continuous derivatives up to, and including, order $n-2$ on $[0,\omega] \times [0 \times \omega]$,

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(b) $\frac{\partial^{n-1}}{\partial t^{n-1}}G(t,t^-) - \frac{\partial^{n-1}}{\partial t^{n-1}}G(t,t^+) = -1$,

(c) $\frac{\partial^m}{\partial t^m}G(t,s)|_{t=0} = \frac{\partial^m}{\partial t^m}G(t,s)|_{t=\omega}$,
$m = 0, 1, \ldots, n-2$,

(d) $|\frac{\partial^m}{\partial t^m}G(t,s)| < \frac{\pi^l \exp(K^{l/n}\omega)}{\alpha\omega K^{1-m/n}}$,
$l = n-1 \pmod 2$. (4)

To show (a) one has to note:

$$\sum_{k=1}^{n} \lambda_k^{m-n+1} = 0, \ m = 0, 1, \ldots, n-2 .$$

To prove inequality (d), write:

$$|\sum \frac{\partial^m}{\partial t^m}G(t,s)| < A_1 + A_2 + A_3 ,$$

where:

$$A_1 = \frac{1}{n}\sum_{\beta_k>0}\alpha_k, \ A_2 = \frac{1}{n}\sum_{\beta_k<0}\alpha_k ,$$

$$\alpha_k = |\frac{\exp\lambda_k\omega}{\lambda_k^{n-m-1}(\exp\lambda_k\omega - 1)}| ,$$

and for $\beta_k = 0$:

$$A_3 = \frac{1}{n}K^{(m+1-n)/n} \left|\frac{\sin K^{1/n(t-s)} - \sin K^{1/n}(\omega - s + t)}{1 - \cos K^{1/n}\omega}\right| .$$

Thus, it is easy to see if $\lambda = \text{Min}\{|\beta_k|; \beta_k \neq 0\}$, $k = 1, 2, \ldots, n$:

$$A_1 \leq \frac{1}{n}\sum_{\beta_k>0}\frac{\exp(K^{1/n}\omega)}{\lambda\omega K^{1-m/n}}$$

$$A_2 \leq \frac{1}{n}\sum_{\beta_k<0}\frac{1}{\lambda\omega K^{1-m/n}}$$

$$A_3 \leq \frac{\pi K^{m/n-1}}{n\omega} ,$$

where:

$$0 < \omega < \pi K^{-1/n} .$$

We remark here if the number n is odd, we always have $\beta_k \neq 0$ or $A_3 = 0$. In the latter case:

$$\frac{\partial^m}{\partial t^m}G(t,s) < \frac{\exp(K^{1/n}\omega)}{\lambda\omega K^{1-m/n}} .$$

Putting these two cases together we can write Equation 4 with $l = n-1 \pmod 2$.

THEOREM 1

Differential Equation 2 admits at least one ω-periodic solution if:

(i) $0 < \omega < \pi K^{-1/n}$,

(ii) $\int_0^\omega p(t)dt = 0$,

(iii) $|\Phi_i(t)| \leq M_i$, $\Phi_i(t) = \int_0^t \phi_i(t)dt$,
$i = 1, 2, \ldots, n-1$,

(iv) $\frac{|\Phi_n(x)|}{|x|} \to 0 \ (|x| \to \infty)$,

(v) $\frac{|f(t,x)|}{|x|} \to 0 \ (|x| \to \infty$, uniformly in $t)$,

(vi) $f(t,x)$ sign $x \geq 0 \ (|x| \geq h)$.

Proof

We consider the following differential equation containing a parameter μ, $0 \leq \mu \leq 1$:

$$x^{n+1} + Kx' + cx = \mu\{p(t) - f(t,x) + cx - \sum_{i=1}^{n}\phi_i(x^{(n-i)})x^{n-i+1}\} , \quad (5)$$

where c is an arbitrary real positive constant.

We note for $\mu = 1$, Equation 5 is identical to Equation 2 and, for $\mu = 0$, we obtain a linear homogeneous differential equation.

Due to the fact that, for non-zero K, the algebraic equation $S^{n+1} + KS^n + c = 0$ has no purely imaginary root, we conclude that Equation 5, for $\mu = 0$, has no non-zero ω-periodic solution.

Hence, Equation 5 admits at least one ω-periodic solution for $\mu \in [0,1]$, provided that for $\mu \in (0,1)$ the solutions of Equation 5, together with their derivatives, up to and including order $(n-1)$, are uniformly bounded [3-5]. Consequently the stated theorem can be proved with the aid of an a priori estimate on the solutions of Equation 5 and their derivatives.

Let $x(t) \equiv x(t+\omega)$ be a solution of Equation 5 and let $0 < \mu < 1$, the derivative

$y = x'$ satisfies the equation:

$$y^{(n)} + Ky = \mu\{p(t) - f(t, x(t)) - \sum_{i=1}^{n} \phi_i(x^{(n-i)}(t))x^{(n-i+1)}(t)\} - (1-\mu)cx(t) ,$$

which can be considered as a non-homogeneous linear equation:

$$y^{(n)} + Ky = g(t), \ g(t+\omega) = g(t) , \qquad (6)$$

where:

$$g(t) = \mu\{p(t) - f(t, x(t)) - \sum_{i=1}^{n} \phi_i (x^{(n-i)}(t))x^{(n-i+1)}(t)\} - (1-\mu)cx(t) .$$

It is clear that with the properties (a), (b), (c) and (d), the function $G(t,s)$ given by Equation 3, is the Green's function for the equation:

$$y^{(n)} + Ky = 0 ,$$

with periodic boundary conditions:

$$y^{(i)}(0) = y^{(i)}(\omega), \ i = 0, 1, \ldots, n-1 . \qquad (7)$$

Therefore, the solution of Equation 6 with the boundary conditions in Equation 7 is equivalent to the integral equation:

$$y(t) = \int_0^\omega G(t,s)g(s)ds ,$$

or

$$y^{(m)}(t) = \int_0^\omega \frac{\partial^m}{\partial t^m} G(t,s)g(s)ds , \quad m = 0, 1, \ldots, n-2 .$$

Since:

$$\frac{\partial^m G}{\partial t^m} = (-1)^m \frac{\partial^m G}{\partial s^m}, \ m = 0, 1, \ldots, n-2 ,$$

substituting for $g(t)$ and using above equality, we obtain:

$$y^{(m)}(t) = \int_0^\omega \frac{\partial^m}{\partial t^m} G(t,s)\{\mu[p(s) - f(s, x(s))] - (1-\mu)cx(s)\}ds + \mu(-1)^m \int_0^\omega \frac{\partial^{m+1}}{\partial s^{m+1}} G(t,s) \sum_{i=1}^{n} \Phi_i(x^{(n-i)}(s))ds ,$$

$$m = 0, 1, 2, \ldots, n-2 .$$

For $m = n-1$, we have:

$$y^{(n-1)}(t) = \int_0^\omega \frac{\partial^{n-1}}{\partial t^{n-1}} G(t,s)\{\mu[p(s) - f(s, x(s))] - (1-\mu)cx(s)\}ds + \mu \sum_{i=1}^{n} \Phi_i(x^{(n-i)}(s)) + \mu K \int_0^\omega G(t,s) \sum_{i=1}^{n} \Phi_i(x^{(n-i)}(s))ds ,$$

and for $m = n$:

$$y^{(n)}(t) = -K \int_0^\omega G(t,s)\{\mu[p(s) - f(s, x(s))] - (1-\mu)cx(s)\}ds + K \int_0^\omega \frac{\partial}{\partial s} G(t,s) \sum_{i=1}^{n} \Phi_i(x^{(n-i)}(s))ds .$$

Now denoting:

$$R = \max|x(t)|; \ t \in [0, \omega] ,$$

$$\Phi_n = \Phi_n(R) = \max|\Phi_n(x)|; \ |x| \le R ,$$

$$F = F(R) = \max|f(t,x)|; \ t \in [0, \omega], |x| \le R ;$$

we derive the following estimates:

$$|y^{(m)}(t)| \le \rho[M + F(R) + cR + \Phi_n(R) + \sum_{i=1}^{n-1} M_i], \ m = 1, 2, \ldots, n-2 ,$$

$$|y^{(n-1)}(t)| \le \sigma[M + F(R) + cR + \Phi_n(R) + \sum_{i=1}^{n-1} M_i] ,$$

and:

$$|y^{(n)}(t)| \le \rho K[M + F(R) + cR + \Phi_n(R) + \sum_{i=1}^{n-1} M_i] ,$$

where:

$$\rho = \max\left\{\frac{\pi^l \exp(K^{1/n}\omega)}{\alpha K^{1-m/n}}\right\}, \ m = 1, 2, \ldots, n$$

$$\sigma = \max\{\rho, (1 + K\rho)\} .$$

Now, term by term integration of Equation 5 yields:

$$x^{(n)}(t) + Kx(t) + \mu \sum_{i=1}^{n} \Phi_i(x^{(n-i)}(t)) - \mu P(t)|_0^\omega + \int_0^\omega \{(1-\mu)cx(t) + \mu f(t, x(t))\}dt = 0 ,$$

or:

$$\int_0^\omega \{(1-\mu)cx(t) + \mu f(t,x(t))\}dt = 0 \ .$$

Since $1-\mu > 0$, we have:

$$[(1-\mu)cx(t) + \mu f(t,x(t))] \text{ sign } x > 0 \ ;$$
$$|x| \geq h \ .$$

These results show that the case $|x(t)| \geq h$ for $0 \leq t \leq \omega$ is excluded and we must have $|x(\tau)| < h$ for some $\tau \in (0,\omega)$.

Applying the mean value theorem to an arbitrary interval $[\tau, t] \subset [\tau, \tau+\omega]$, we find:

$$|x(t) - x(\tau)| = (t-\tau)y(\tau + \theta(t-\tau))$$
$$\leq \omega\rho\left[M + F(R) + cR + \Phi_n(R) + \sum_{i=1}^{n-1} M_i\right] ,$$
$$0 \leq \theta \leq 1$$

or:

$$|x(t)| < h + \omega\rho[M + F(R) + cR + \Phi_n(R) + \sum_{i=1}^{n-1} M_i] \ .$$

Hence:

$$\max|x(t)| = R < h + \omega\rho[M + F(R) + cR + \Phi_n(R) + \sum_{i=1}^{n-1} M_i], \ 0 \leq t \leq \omega \ .$$

Choosing $0 < c < \frac{1}{\omega\rho}$, we obtain:

$$1 < \frac{h + \omega\rho(M + \sum_{i=1}^{n-1} M_i)}{1-\omega\rho c}\frac{1}{R} + \frac{\omega\rho}{1-\omega\rho c}\left(\frac{F(R)}{R} + \frac{\Phi_n(R)}{R}\right) \ . \qquad (8)$$

An immediate consequence of the assumptions (iv) and (v) of Theorem 1 is:

$$\frac{F(R)}{R} \to 0 \text{ and } \frac{\Phi_n(R)}{R} \to 0 \ (R \to \infty) \ .$$

Therefore, we conclude from Equation 8:

$$R = \max_{t \in [0,\omega]}|x(t)| \leq R_0 \text{ (independent of } \mu) ,$$
$$F(R) = \max_{t\in[0,\omega], |x|\leq R}|f(t,x)| \leqq F_0 = \max_{t\in[0,\omega], |x|\leq R_0}|f(t,x)| ,$$
$$\Phi_n(R) = \max_{|x|\leq R}|\Phi_n(x)| \leq \Phi_{n_0} = \max_{|x|\leq R_0}|\Phi_n(x)| \ .$$

The resulting a priori estimates:

$$|x(t)| \leq R_0 \ ,$$
$$|y^{(m)}(t)| \leq \rho\left[M + F_0 + cR_0 + \Phi_{n_0} + \sum_{i=1}^{n-1} M_i\right] ,$$
$$|y^{(n-1)}(t)| \leq \sigma\left[M + F_0 + cR_0 + \Phi_{n_0} + \sum_{i=1}^{n-1} M_i\right] ,$$
$$|y^{(n)}(t)| \leq \rho K\left[M + F_0 + cR_0 + \Phi_{n_0} + \sum_{i=1}^{n-1} M_i\right] ,$$

ensure the existence of a periodic solution of Equation 2.

Remark

The results of Theorem 1 remain valid if, instead of the assumption (vi), we use:

$$(vi)' f(t,x) \text{ sign } x \leq 0 \ .$$

To see this, it is sufficient to introduce a new independent variable $t^* = -t$ and obtain a differential equation of previous type.

Theorem 2

The differential equation:

$$x^{(n+1)} + \sum_{i=1}^{n-2} \phi_i(x^{(n-i)}(t))x^{(n-i+1)} + (\phi_{n-1}(x') + K)x'' + \phi_n(x)x' + f(t,x) = p(t) \ , \qquad (9)$$

where $f(t,x) = f(t+\omega, x$ and $p(t) = p(t+\omega)$ admits at least one ω-periodic solution if:

i) $0 < \omega < \pi K^{-1/n-1}$,

ii) $\int_0^\omega p(t)dt = 0$,

iii) $|\Phi_i(x)| \leq M_i$; $(\Phi_i(x) = \int_0^x \phi(\tau)d\tau)$, $i = 1, 2, \ldots, n-1$,

iv) $\frac{|\Phi_n(x)|}{|x|} \to 0$; $(\Phi_n(x) = \int_0^x \phi_n(\tau)d\tau)$, $|x| \to \infty$,

v) $\frac{|f(t,x)|}{|x|} \to 0$; ($|x| \to \infty$ uniformly in t) ,

vi) $f(t,x)$ sign $x \geq 0$; $|x| \geq h$.

Again the proof is based on an a priori estimate of the ω-periodic solutions of the system:

$$\begin{aligned} &x' = y,\ y' = z\ , \\ &z^{n+1} + Kz = \mu\{p(t) - f(t,x) - \sum_{i=1}^{n} \phi_i(x^{(n-i)}) \\ &\qquad x^{(n-i+1)}\} - (1-\mu)cx\ , \qquad (10) \end{aligned}$$

where $0 < \mu < 1$ and $c > 0$ (properly chosen).

Next we let $x(t) \equiv x(t+\omega)$ be a solution of Equation 9 and consider the nonhomogeneous differential equation:

$$\begin{aligned} &z^{(n-1)} + Kz = g(t),\ g(t+\omega) = g(t)\ , \\ &g(t) = \mu\{p(t) - f(t,x(t)) \\ &\quad - \sum_{i=1}^{n} \phi_i(x^{(n-i)}(t))x^{(n-i+1)}(t)\} \\ &\quad - (1-\mu)cx(t)\ . \qquad (11) \end{aligned}$$

Or, equivalently, we consider the integral equation:

$$\begin{aligned} z^{(m)}(t) &= \int_0^{\omega} \frac{\partial^m}{\partial t^m} G(t,s)g(s)ds, \\ &m = 0,1,\dots,n-1\ , \qquad (12) \end{aligned}$$

where:

$$G(t,s) = \begin{cases} \sum_{k=1}^{n-1} \frac{\exp K^{\frac{1}{n}} \lambda_k(\omega - s + 1)}{n\lambda_k^{n-1} \exp K^{\frac{1}{n}}(\lambda_k\omega - 1)}\ , \\ 0 \leq t \leq s \leq \omega \\ \sum \frac{\exp K^{\frac{1}{n}} \lambda_k(t-s)}{n\lambda_k^{n-1} \exp K^{\frac{1}{n}}(\lambda_k\omega - 1)}\ , \\ 0 \leq s \leq t \leq \omega \end{cases} \qquad (13)$$

and:

$$\lambda_k = K^{1/n-1} \exp\left[\left(\frac{i\pi}{n}\right)(2k-1)\right]\ .$$

As in the previous case, we find the following estimates:

$$\begin{aligned} |z^{(m)}(t)| &\leq \rho[M + F(R) + cR + \Phi_n(R) \\ &\quad + \sum_{i=1}^{n-1} M_i]\ ,\ m = 0,1,\dots,n-3\ , \\ |z^{(n-2)}(t)| &\leq \sigma[M + F(R) + cR + \Phi_n(R) \\ &\quad + \sum_{i=1}^{n-1} M_i]\ , \\ |z^{(n-1)}(t)| &\leq \rho K[M + F(R) + cR + \Phi_n(R) \\ &\quad + \sum_{i=1}^{n-1} M_i]\ , \end{aligned}$$

where σ and ρ are determined by the bounds on Green's function in Equation 13 and its derivatives and, therefore, they depend on K and ω. $M, R, F(R)$ and $\Phi_n(R)$ are defined as before.

Following the same argument as in the case of Theorem 1, we finally obtain the required boundedness results.

As an application of previous theorems, we look at the following two examples.

Example 1

Consider the equation:

$$x'' + \left(\frac{1}{3} + e^{-x}\right)x' + |\sin t|\frac{x}{1+x^2} = \sin t. \qquad (14)$$

Here $p(t) = \sin(t)$, $f(t,x) = |\sin t|\frac{x}{1+x^2}$, $\phi_n(x) = e^{-x}$ and $\Phi_n(x) = 1 - e^{-x}$, $(n = 1)$. We have:

$$\lim_{|x|\to\infty} \frac{|\Phi_n(x)|}{|x|} = 0\ ,\ f(t,x) \text{ sign } x > 0$$

$$\lim_{|x|\to\infty} \frac{|f(t,x)|}{|x|} = 0\ ,$$

that is, all the assumptions of Theorem 1 are satisfied. Hence, there exists at least one 2π-periodic solution of Equation 14.

Example 2

Consider the equation:

$$\begin{aligned} &x''' + \left(\frac{1}{3} + \cos x'\right)x'' + e^{-x}x' \\ &\quad + \sin^2 t \frac{x}{1+|x|} = \cos t\ . \qquad (15) \end{aligned}$$

Here, $n = 2$, $\phi_{n-1}(x') = \cos x'$, $\phi_n(x) = e^{-x}$, $\Phi_n(x) = 1 - e^{-x}$, $f(t,x) = \sin^2 t \frac{x}{1+|x|}$, $p(t) = \cos t$. Hence:

$$\lim_{|x|\to\infty} \frac{|\Phi_n(x)|}{|x|} = 0,\ f(t,x)\ \text{sign}\ x > 0 (h = 0)$$

$$\lim_{|x|\to\infty} \frac{|f(t,x)|}{|x|} = 0\ ,$$

and therefore all the assumptions of Theorem 2 are satisfied. This proves the existence of, at least, one 2π-periodic solution of Equation 15.

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