Trajectories Connecting Critical Points

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In this paper, an existence criterion of trajectories connecting a moving pair of critical points of planar differential systems depending on a parameter is given.

INTRODUCTION

Many questions arising in the physical and biological sciences are concerned with the existence of trajectories joining a pair of critical points of systems of autonomous ordinary differential equations. In other words, is there a trajectory that, as $t \to +\infty$ and $t \to -\infty$, tends to a given pair of critical points of the equation? As pointed out by Gelfand [1], this is an interesting problem. In recent years, this problem has attracted the attention of many researchers. A considerable number of papers have been written in connection with this subject. For example, some results on the existence and uniqueness of connecting trajectories have been obtained by C.C. Conley and J.A. Smoller [2], J.G. Conlon [3], J.F. Selgrade [4], Jiang Jifa [5], and others. These results are applicable to cooperative systems [6]. Some results on their existence have been obtained by C.C. Conley and J.A. Smoller [7], C.C. Conley [8], P. Gordon [9], Z. Artstein and M. Slemrod [10], Yu Shu-Xiang [11,12], and others. The existence of trajectories connecting a pair of critical points of differential equations depending on a parameter has been discussed by A.G. Kulikovskii [13], J.A. Smoller and C.C. Conley [14], and others.

For an ordinary differential system, let its right-hand side function depend on a parameter μ and suppose that for $0 < \mu \leqslant \mu_1$, it has two critical points $P_1(\mu)$, $P_2(\mu)$ and that these critical points move continuously with μ, finally coalescing at $\mu = 0$, i.e., $P_1(0) = P_2(0)$. A problem is: Is there a value $\mu_0 (0 < \mu_0 \leqslant \mu_1)$ such that, for $0 < \mu \leqslant \mu_0$, there exists at least one trajectory connecting the critical points $P_1(\mu)$ and $P_2(\mu)$? This problem is of special physical significance [13,14].

In this paper, we consider the planar differential system depending on a parameter and give conditions on the differential system which assure the existence of a trajectory connecting two critical points when they move close enough together (see Theorem 2 below).

RESULTS

Consider the differential system defined in the region $G \subset R^2$:

$$\frac{dx}{dt} = X(x,y) ,$$
$$\frac{dy}{dt} = Y(x,y) . \qquad (1)$$

Suppose $X, Y \in C^1$. Let the vector field $V \equiv (X,Y)$ define a flow $f(p,t)$. Let $B \subset G$ be the closure of a bounded and connected open set

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with the boundary ∂B, and there are no critical points of Equation 1 in ∂B. We define a subset b^+ of ∂B by:

$$b^+ = \{p \in \partial B : \exists \varepsilon > 0 \text{ with } f(p,(-\varepsilon,0)) \cap B = \phi\} .$$

Hence, if $p \in b^+$, the trajectory through p leaves B for a short backwards time. Similarly, let:

$$b^- = \{p \in \partial B : \exists \varepsilon > 0 \text{ with } f(p,(0,\varepsilon)) \cap B = \phi\} ,$$

and

$$\tau = \{p \in \partial B : V \text{ is tangent to } B \text{ at } p\} .$$

We now introduce the following definitions [7,15].

Definition 1

We say that B is an isolating block for the flow defined by Equation 1 if $b^+ \cap b^- = \tau$ and $b^+ \cup b^- = \partial B$.

It follows from the above definition that, if B is an isolating block, then all the tangencies to B must be external.

Definition 2

If a simple closed curve C is the union of alternating nonclosed whole trajectories and critical points, and it is contained in the ω (or α)-limit set of some trajectory, then we say that C is a singular closed trajectory.

In an early paper [11], we proved the following theorem.

Theorem 1

If the system in Equation 1 admits an isolating block B such that the following two conditions are satisfied in B:

i. there are precisely two critical points, one of which is a repeller;

ii. there are no closed trajectories and singular closed trajectories;

then there must be a trajectory in B running from the repeller to the other critical point.

It is easy to see that Theorem 1 is a generalization of Conley's result in [8]. In particular, Theorem 1 does not require the condition that the system, Equation 1, is a gradient-like system. A generalization of Theorem 1 can be found in [12].

Consider now the differential system depending on a parameter:

$$\frac{dx}{dt} = X(x,y,\mu) ,$$
$$\frac{dy}{dt} = Y(x,y,\mu) . \qquad (2)$$

Suppose that $X(0,0,0) = 0$, $Y(0,0,0) = 0$ and that the system in Equation 2 is defined in the region $\Omega = G \times I \subset R^2 \times I (I \equiv [0,1])$, where G is a region in the xy-plane containing the origin $(x,y) = (0,0) = 0$ in its interior. Suppose that $X, Y \in C^1$ is in x, y and μ is in Ω.

In this paper, we shall prove the following theorem.

Theorem 2

Suppose that for $0 < \mu \leqslant \mu_1$, G contains precisely two critical points $P_1(\mu)$ and $P_2(\mu)$ of the system in Equation 2, one of which is a repeller, and that these critical points move continuously with μ, finally coalescing at $\mu = 0$, i.e., $P_1(0) = P_2(0) = (0,0) = 0$. Assuming that the following three conditions are satisfied:

i. $\dfrac{\partial Y}{\partial y} \neq 0$ at $(x,y,\mu) = (0,0,0)$;

ii. $\dfrac{\partial X}{\partial x} + \dfrac{\partial Y}{\partial y} \neq 0$ at $(x,y,\mu) = (0,0,0)$;

iii. there are $\alpha > 0$, $\beta > 0$ such that the function $X(x,y,0)$ does not change sign in the region $R = \{(x,y) \in G : -\alpha \leqslant x \leqslant \alpha$ and $-\beta \leqslant y \leqslant \beta\}$ and $X(x,y,0) \neq 0$ for $x \neq 0$ in R;

then there exists a value $\mu_0 (0 < \mu_0 \leqslant \mu_1)$ such that, for $0 < \mu \leqslant \mu_0$, the system in Equation 2 has a trajectory connecting the critical points $P_1(\mu)$ and $P_2(\mu)$.

Proof of Theorem 2

Since $Y(x,y,\mu) \in C^1$ in Ω, there is a region $G_1 \subset G$ containing the origin $0 = (0,0)$ in its interior and a value $\mu_2 (0 < \mu_2 \leqslant \mu_1)$ such that, in the region $\Omega_1 = G_1 \times [0, \mu_2]$, the function $Y(x,y,\mu)$ can be represented as follows:

$$\begin{aligned} Y(x,y,\mu) &= ax + by + c\mu + Q(x,y,\mu) \\ &= ax + by + Q_1(x,y,\mu) , \end{aligned} \tag{3}$$

where a, b, c are real constants, $Q(0,0,0) = 0$ and:

$$\frac{Q(x,y,\mu)}{\rho} \to 0 \text{ when}$$
$$\rho = \sqrt{x^2 + y^2 + \mu^2} \to 0 . \tag{4}$$

By the condition (i) of Theorem 2, it follows that $b \neq 0$. We first assume:

$$b > 0 . \tag{5}$$

By Equation 3 we have:

$$\begin{aligned} Y(0,y,0) &= by + Q(0,y,0) = by + \gamma . y \\ &= (b + \gamma) y , \end{aligned}$$

where $\gamma \to 0$ when $y \to 0$. Therefore, there exists a $\delta > 0$ such that $b + \gamma > 0$ for $|y| \leqslant \delta$. Hence, we get:

$$Y(0,y,0) = by + Q_1(0,y,0) > 0 \text{ for } 0 < y \leqslant \delta , \tag{6}$$

and

$$Y(0,y,0) = by + Q_1(0,y,0) < 0 \text{ for } -\delta \leqslant y < 0. \tag{7}$$

Let y_1 be a fixed value such that $0 < y_1 \leqslant \delta$. By Equations 6 and 7, we get:

$$Y(0,y_1,0) = by_1 + Q_1(0,y_1,0) > 0 , \tag{8}$$

and

$$Y(0,-y_1,0) = b(-y_1) + Q_1(0,-y_1,0) < 0 . \tag{9}$$

By virtue of the continuity of the function $Y(x,y,\mu)$ in μ, there exists a $\mu_3 (0 < \mu_3 \leqslant \mu_2)$ such that, for $0 \leqslant \mu \leqslant \mu_3$, the following two inequalities hold:

$$Y(0,y_1,\mu) = by_1 + Q_1(0,y_1,\mu) > 0 , \tag{10}$$

$$Y(0,-y_1,\mu) = b(-y_1) + Q_1(0,-y_1,\mu) < 0 . \tag{11}$$

By virtue of the continuity of the function $Y(x,y,\mu)$ in x, for y_1 and for any fixed $\mu (0 \leqslant \mu \leqslant \mu_3)$ there exists a $x(\mu) > 0$ such that, for $|x| \leqslant x(\mu)$, the following two inequalities hold:

$$Y(x,y_1,\mu) = ax + by_1 + Q_1(x,y_1,\mu) > 0 , \tag{12}$$

$$\begin{aligned} Y(x,-y_1,\mu) &= ax + b(-y_1) \\ &\quad + Q_1(x,-y_1,\mu) < 0 . \end{aligned} \tag{13}$$

Since the interval $[0, \mu_3]$ is compact, there exists a $x_1 > 0$ such that, for $|x| \leqslant x_1$ and for every $\mu \in [0, \mu_3]$, the inequalities in Equations 12 and 13 hold.

Altogether, we have proved that, for any fixed $y_1 (0 < y_1 \leqslant \delta)$, we can construct a rectangular region $R_1 = \{(x,y) \in G_1 : -x_1 \leqslant x \leqslant x_1 \text{ and } -y_1 \leqslant y \leqslant y_1\}$, its boundary being a rectangle $A_1B_1C_1D_1$, where $A_1 = (-x_1, y_1)$, $B_1 = (x_1, y_1)$, $C_1 = (x_1, -y_1)$ and $D_1 = (-x_1, -y_1)$. For every point (x,y) on the horizontal edge A_1B_1 and for every $\mu \in [0, \mu_3]$, the inequality $Y(x,y,\mu) > 0$ holds. For every point (x,y) on the horizontal edge C_1D_1 and for every $\mu \in [0, \mu_3]$, the inequality $Y(x,y,\mu) < 0$ holds. This means that for the system in Equation 2, we have $\frac{dy}{dt} > 0$ when $(x,y) \in A_1B_1$, $\mu \in [0, \mu_3]$ and $\frac{dy}{dt} < 0$ when $(x,y) \in C_1D_1$, $\mu \in [0, \mu_3]$. A rectangle which possesses such a property is called H-rectangle. Obviously, for any fixed $x_1' (0 < x_1' < x_1)$ we can construct a rectangular region $R_1' = \{(x,y) \in G_1 : -x_1' \leqslant x \leqslant x_1' \text{ and } -y_1 \leqslant y \leqslant y_1\}$, its boundary being a rectangle $A_1'B_1'C_1'D_1'$ where $A_1' = (-x_1', y_1)$, $B_1' = (x_1', y_1)$, $C_1' = (x_1', -y_1)$ and $D_1' = (-x_1', -y_1)$. By construction, we know that $R_1' \subset R_1$, and that rectangle $A_1'B_1'C_1'D_1'$ is also an H-rectangle, because, for the system in Equation 2, we have $\frac{dy}{dt} > 0$ when $(x,y) \in A_1'B_1'$, $\mu \in [0,\mu_3]$ and $\frac{dy}{dt} < 0$ when $(x,y) \in C_1'D_1'$, $\mu \in [0,\mu_3]$. Therefore, for an arbitrary region $G_2 \subset G$ containing the origin $0=(0,0)$ in its interior, we can construct an H-rectangle $ABCD$ such that the $ABCD$ is contained in G_2, provided we make y_1 small enough.

By condition (ii) of Theorem 2 and by the continuity of the functions X and Y, it follows that there are three sufficiently small real positive constants α_1, β_1 and μ_4 such that when $(x,y) \in R_2 = \{(x,y) \in G_1 : -\alpha_1 \leqslant x \leqslant \alpha_1$ and $-\beta_1 \leqslant y \leqslant \beta_1\}$ and $\mu \in [0,\mu_4]$, the following expression holds:

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \neq 0 \ . \tag{14}$$

Further, noting α and β in the Condition iii, we can assume here $\alpha_1 < \alpha$ and $\beta_1 < \beta$ (if necessary, we can lessen α_1 and β_1; in this case, Equation 14 still holds for $(x,y) \in R_2$ and $\mu \in [0,\mu_4]$). Thus, we have $R_2 \subset R$. It follows from Condition iii of Theorem 2 that the sign of function $X(x,y,0)$ does not change in R_2 and $X(x,y,0) \neq 0$ for $x \neq 0$ in R_2.

We now construct an H-rectangle $A_2B_2C_2D_2$ such that it is contained in R_2. Let $R_3 = \{(x,y) \in R_2 : -x_2 \leqslant x \leqslant x_2$ and $-y_2 \leqslant y \leqslant y_2\}$ be a rectangular region enclosed by the rectangle $A_2B_2C_2D_2$, where $0 < x_2 < \alpha_1$ and $0 < y_2 < \beta_1$, $A_2 = (-x_2, y_2), B_2 = (x_2, y_2), C_2 = (x_2, -y_2)$ and $D_2 = (-x_2, -y_2)$. The fact that $A_2B_2C_2D_2$ is a H-rectangle implies that there exists a value $\mu_5 > 0$ such that, for the system in Equation 2, we have $\frac{dy}{dt} > 0$ when $(x,y) \in A_2B_2$, $\mu \in [0,\mu_5]$ and $\frac{dy}{dt} < 0$ when $(x,y) \in C_2D_2$, $\mu \in [0,\mu_5]$.

Since $R_3 \subset R_2$, the sign of function $X(x,y,0)$ does not change in R_3 and $X(x,y,0) \neq 0$ for $x \neq 0$ in R_3. Suppose first that:

$$X(x,y,0) \geqslant 0 \text{ in } R_3 \ . \tag{15}$$

Hence, for every point (x,y) on the edge B_2C_2 of the rectangle $A_2B_2C_2D_2$ we have $X(x,y,0) = X(x_2,y,0) > 0$. By the continuity of the function $X(x,y,\mu)$ in μ, it follows that for any fixed point (x_2,y) on the edge B_2C_2 there exists a $\mu_6(y) > 0$ such that $X(x_2,y,\mu) > 0$ for $\mu \in [0,\mu_6(y)]$. Since the set composed of all points on the line segment B_2C_2 is compact, there exists a $\mu_6 > 0$ such that for every point (x,y) on the edge B_2C_2 and for every $\mu \in [0,\mu_6]$ we have $X(x,y,\mu) > 0$. Similarly, using exactly the same type of argument, it follows that there exists a $\mu_7 > 0$ such that $X(x,y,\mu) > 0$ for every point (x,y) on the edge A_2D_2 and for every $\mu \in [0,\mu_7]$. If we let $\mu_8 = \min\{\mu_6, \mu_7\}$, then we claim that for every point (x,y) on the edge B_2C_2 and A_2D_2, and for every $\mu \in [0,\mu_8]$, the following expression holds:

$$X(x,y,\mu) > 0 \ . \tag{16}$$

Finally, by the hypothesis of Theorem 2, as $\mu \to 0, P_1(\mu)$ and $P_2(\mu)$ tend to the origin $0=(0,0)$, thus there exists a $\mu_9 > 0$ such that for every $\mu \in [0,\mu_9]$, the critical points $P_1(\mu)$ and $P_2(\mu)$ are contained in the interior of R_3.

If we let $\mu_0 = \min\{\mu_9, \mu_8, \mu_5, \mu_4\}$, then we claim that for $0 < \mu \leqslant \mu_0$, the system in Equation 2 has a trajectory connecting the critical points $P_1(\mu)$ and $P_2(\mu)$. In fact, as stated above, for the system in Equation 2 we have $\frac{dy}{dt} > 0$ when $(x,y) \in A_2B_2$, $\mu \in [0,\mu_0]$ and $\frac{dy}{dt} < 0$ when $(x,y) \in C_2D_2$, $\mu \in [0,\mu_0]$. Further, it follows from Equation 16 that for the system in Equation 2 we have $\frac{dx}{dt} > 0$ when $(x,y) \in B_2C_2$ and $(x,y) \in A_2D_2$, $\mu \in [0,\mu_0]$. Therefore, it is easy to see that R_3 is an isolating block for the flow defined by Equation 2. It precisely contains two critical points, $P_1(\mu)$ and $P_2(\mu)$, one of which is a repeller. Moreover, it follows from Equation 14 that there are no closed trajectories and singular closed trajectories in R_3 [16]. Thus, Theorem 1 implies that there must be a trajectory in R_3 connecting the critical points $P_1(\mu)$ and $P_2(\mu)$.

If we consider the case when $b < 0$ instead of Equation 5, the proof is similar. In this case, for the system in Equation 2, we have $\frac{dy}{dt} < 0$ when $(x,y) \in A_2B_2$, and $\frac{dy}{dt} > 0$ when $(x,y) \in C_2D_2$. Thus, R_3 still is an isolating block for the system in Equation 2. Similarly, if we consider the case when $X(x,y,0) \leqslant 0$ instead of Equation 15, then for the system in Equation 2 we have $\frac{dx}{dt} < 0$ when $(x,y) \in B_2C_2$ and A_2D_2. R_3 still is an isolating block for the system in Equation 2. Thus, the above proof works. Therefore, Theorem 2 is proved.

AN EXAMPLE

Consider the system of ordinary differential equations in the plane:

$$\begin{aligned} \frac{dx}{dt} &= y^2 + x^{2m} - \mu \equiv X(x,y,\mu) , \\ \frac{dy}{dt} &= y - \mu \equiv Y(x,y,\mu) , \end{aligned} \tag{17}$$

where $\mu \in \left(0, \frac{1}{2}\right]$ is a parameter, and m is a positive integer.

It is easy to show the following properties of the system in Equation 17.

1. The system in Equation 17 has precisely two critical points $P_1(\mu) = (-[\mu(1-\mu)]^{\frac{1}{2m}}, \mu)$ and $P_2(\mu) = ([\mu(1-\mu)]^{\frac{1}{2m}}, \mu)$ and $P_2(\mu)$ is a repeller. As $\mu \to 0$ we have $P_1(\mu) \to 0 = (0,0)$ and $P_2(\mu) \to 0 = (0,0)$.
2. $\dfrac{\partial Y}{\partial y} = 1$.
3. $\dfrac{\partial X}{\partial x} + \dfrac{\partial Y}{\partial y} = 2mx^{2m-1} + 1 = 1$ at $(x,y,\mu) = (0,0,0)$.
4. $X(x,y,0) = y^2 + x^{2m} \geqslant 0$ and $X(x,y,0) \neq 0$ for $x \neq 0$ in the plane.

Therefore, Equation 17 satisfies all conditions of Theorem 2. Hence, Theorem 2 implies that there exists a value $\mu_0 \in (0, \frac{1}{2}]$ such that, for $0 < \mu \leqslant \mu_0$, the system in Equation 17 has a trajectory connecting the critical points $P_1(\mu)$ and $P_2(\mu)$ (In fact, in this example we can take $\mu_0 = \frac{1}{2}$).

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