On the Existence of MFD Switch-on and Switch-off Shock Waves for Rectilinear Motion in Some Models of Plasma

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The shock waves structure problem in MFD for rectilinear motion in some models of plasma reduces to finding heteroclinic orbits for an ordinary differential equations system of six variables depending on five viscosity parameters, $\alpha, \eta, k, \sigma^{-1}$ and χ; one electrical parameter $\in \geq 0$ and a magnetic parameter $\delta > 0$. For $\in > 0$, this system admits four rest points say, $u_i(\in), 0 \leq i \leq 3$, which are all nondegenerate. For $\in = 0$, the system admits infinitely many rest points including the four rest points $\overline{u}_i = \lim_{\in \to 0} u_i(\in), 0 \leq i \leq 3$. Two of these rest points are nondegenerate and the other two are degenerate. The heteroclinic orbits between $u_0(\in)$ and $u_1(\in)$ correspond to the structure for the fast shocks, the heteroclinic orbits between $u_2(\in)$ and $u_3(\in)$ correspond to the structure for the slow shocks, while the heteroclinic orbits between $\overline{u}_0$ and $\overline{u}_1$; and $\overline{u}_2$ and $\overline{u}_3$ correspond to the switch-on and switch-off shocks, respectively. In [1] we have shown that the fast and the slow shocks admit structure. In this paper, by using some results from [1-4], we will show that for each pair of $(i,j), 0 \leq i < j \leq 3, (i,j) \neq (1,2)$ there is a complete orbit related to the rest points $\overline{u}_i$ and $\overline{u}_j$. Moreover, these orbits are obtained as the limiting case of the heteroclinic orbits for the fast and the slow shocks. In spite of these facts, the switch-on and the switch-off shocks do not admit structure and physically cannot occur.

INTRODUCTION

In this paper, we will continue our work begun in [1] on the structure of MFD shock waves of arbitrary strength for ionized gases governed by general equations of state. As discussed in [1], the structure problem reduces to finding heteroclinic orbits for a six-dimensional system of ordinary differential equations of the following form:

$$\begin{aligned}
\alpha V \dot{B}_2 &= y_2 , \\
\alpha V \dot{B}_3 &= y_3 , \\
\alpha V \dot{y}_2 &= \in + (V - \delta^2) B_2 - \chi \alpha^{-1} \delta y_3 - \frac{\sigma^{-1} y_2}{\alpha V} , \\
\alpha V \dot{y}_3 &= (V - \delta^2) B_3 + \chi \alpha^{-1} \delta y_2 - \frac{\sigma^{-1} y_3}{\alpha V} , \\
\eta \dot{V} &= p(V,T) + V + \frac{1}{2}(B_2^2 + B_3^2) - J , \\
k \dot{T} &= e(V,T) - \frac{1}{2}V^2 - \frac{1}{2}(V - \delta^2)(B_2^2 + B_3^2) \\
&\quad + JV + \frac{1}{2}(y_2^2 + y_3^2) - \in B_2 - C , \qquad (1)
\end{aligned}$$

where the symbols $\alpha, \eta, k, \sigma^{-1}$ and χ denote the "viscosity" parameters (which are always nonnegative), while $J > 0$, $\delta \geq 0$, $\in \geq 0$ and C are constants. The variables V and T correspond to volume and temperature and are naturally positive, while $p(V,T)$ and $e(V,T)$

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correspond to pressure and internal energy, respectively. For this system, the independent variable is t, which is a coordinate variable. The constant δ is the longitudinal component of the magnetic induction and the variables B_2 and B_3 are the components of this field along the axes perpendicular to the t-axis. Finally, y_2 and y_3 are auxiliary variables which are defined by the first two equations of the above systems.

It is known that the above system, under some hypotheses, admits (at most) four rest points, when $\delta > 0$ and $\in > 0$. We denote them by $u_i(\in), 0 \leq i \leq 3$. Moreover, $\lim_{\in \to 0} u_i(\in)$ exists. We denote it by $\overline{u}_i, 0 \leq i \leq 3$. For the case $\in = 0$, this system admits infinitely many rest points including $\overline{u}_i, 0 \leq i \leq 3$. For the case $\delta = 0$, the reader is referred to [3].

The following results are known [1]. For all values of the viscosity parameters and $\in > 0$, there is an orbit running from $u_0(\in)$ to $u_1(\in)$ (the heteroclinic orbits corresponding to the fast shocks) and likewise an orbit running from $u_2(\in)$ to $u_3(\in)$ (the heteroclinic orbits correspondence to the slow shocks). Moreover, the intermediate shocks, that is the shocks between $u_i(\in)$ and $u_j(\in)$, $i = 0, 1$, $j = 2, 3$, in general, do not admit structures.

This leaves open the question of the structure for the switch-on and the switch-off shocks, i.e., the existence of heteroclinic orbits between $\overline{u}_0$ and $\overline{u}_1$ (for the switch-on shocks), and heteroclinic orbits between $\overline{u}_2$ and $\overline{u}_3$ (for the switch-off shocks).

In order to handle this and the related problems, we shall study the case $\in = 0$ as a limiting case of $\in > 0$ whenever $\in \to 0$. This technique is different from the one used in [1]. The reason for using a different approach is that that topological technique cannot be applied here because:

1. After building the isolated invariant set, in [1], we have chosen a constant K_0 between $P(u_1(\in))$ and $P(u_2(\in))$ and split the isolated invariant set into two isolated invariant sets S_{01} and S_{23}, respectively, for the fast and the slow shocks. Since $\in = 0$ implies $P(\overline{u}_1) = P(\overline{u}_2)$, such a splitting is disallowed in this case. (see Lemma 4.1 in [1]).

2. To calculate the Conley indices $h(S_{01})$ and $h(S_{23})$, in [1], we let C increase; then, by cancellation of the rest points $u_0(\in), u_1(\in)$ and $u_2(\in)$, $u_3(\in)$, we could calculate these indices. But, when $\in = 0$ as C increases then $\overline{u}_1, \overline{u}_2$ and $\overline{u}_0$ and $\overline{u}_3$ simultaneously cancel.

In the next section, we will find the rest points $\overline{u}_i$, $0 \leq i \leq 3$ and its relationship to the rest points $u_i(\in)$, $0 \leq i \leq 3$. We will then find some bounds for the connecting orbits for the fast and the slow shocks independent of the viscosities and $\in > 0$, assuming that $\delta > 0$ is fixed and, afterward, will prove the existence of bounded complete orbits for the case $\in = 0$ and nonexistence of the switch-on and the switch-off shock waves.

HYPOTHESES AND REST POINTS

Let $S(V,T)$ be the entropy of the system. Following [1-9], we assume that the following hypotheses hold:

H_1: The function p, e and S are positive when $V, T > 0$.

H_2: For fixed $T > 0$, $p(V,T) \to 0$ as $V \to 0$.

H_3: Given any positive constants V_0 and K, there exists a $T_0 > 0$ such that if $0 < V \leq V_0$ and $T \geq T_0$, then $e(V,T) > K$.

H_4: On any interval $0 < V \leq V_0$, $S(V,T) \longrightarrow 0$ uniformly in V as $T \longrightarrow 0$.

H_5: Consider p as a function of V and S, then $p_V < 0$, $p_{VV} > 0$ and $p_S > 0$.

We will use these hypotheses directly, or we will take advantage of some results based on them.

As we mentioned before, the switch-on and the switch-off shock waves occur when $\in = 0$ in System 1, thus, in this case, our system of equations becomes:

$$\begin{aligned}
\alpha V\dot{B}_2 &= y_2 \ , \\
\alpha V\dot{B}_3 &= y_3 \ , \\
\alpha V\dot{y}_2 &= (V-\delta^2)B_2 - \chi\alpha^{-1}\delta y_3 - \frac{\sigma^{-1}y_2}{\alpha V} \ , \\
\alpha V\dot{y}_3 &= (V-\delta^2)B_3 + \chi\alpha^{-1}\delta y_2 - \frac{\sigma^{-1}y_3}{\alpha V} \ , \\
\eta\dot{V} &= p(V,T) + V + \frac{1}{2}(B_2^2+B_3^2) - J \ , \\
k\dot{T} &= e(V,T) - \frac{1}{2}V^2 - \frac{1}{2}(V-\delta^2)(B_2^2+B_3^2) \\
&\quad + JV + \frac{1}{2}(y_2^2+y_3^2) - C \ . \qquad (2)
\end{aligned}$$

Thus, at a rest point of this system, $y_2 = y_3 = 0$ and $(V-\delta^2)B_3 = (V-\delta^2)B_2 = 0$.

Case 1

If, at a rest point, $V \neq \delta^2$, then $B_2 = B_3 = 0$. Substituting these values of $y_2 = y_3 = B_2 = B_3 = 0$ into the equations for $\dot{V}$ and $\dot{T}$, we arrive at the following criterion for a rest point:

$$\begin{aligned}
\overline{F_1}(V,T) &= V - J + p(V,T) = 0 \ , \\
\overline{F_2}(V,T) &= e(V,T) - \frac{1}{2}V^2 + JV - C = 0 \ . \qquad (3)
\end{aligned}$$

The equations $\overline{F}_1(V,T) = 0$ and $\overline{F}_2(V,T) = 0$ determine the graphs of the functions $\overline{T}_1(V)$ and $\overline{T}_2(V)$ in the region $V, T > 0$, respectively (see Figure 1). Equations 3 are the same as Equations 2.6 in [4]. From Section 2 in [4], we have the following theorem.

Theorem 1

The equation $\frac{d\overline{T}_1}{dV} = 0$ has precisely one solution. At this point $\overline{T}_1(V)$ has a maximum value. For fixed $J > 0$, there is a constant C_1 such that for $C > C_1$, Equations 3 admit no solution. For $C < C_1$, they admit two solutions, say $(\overline{V}_i, \overline{T}_i)$, $i = 0, 3, \overline{V}_3 < \overline{V}_0$. The function $\overline{T}_2(V)$ is decreasing in the interval $(\overline{V}_3, \overline{V}_0)$. Moreover, in this interval, the curve $\overline{F}_2(V,T) = 0$ lies in the region $\overline{F_1}(V,T) < 0$.

Hence, in this case the following points are two of the rest points of System 2:

$$\overline{u}_i = (0,0,0,0,\overline{V}_i,\overline{T}_i), \quad i = 0,3 \ . \qquad (4)$$

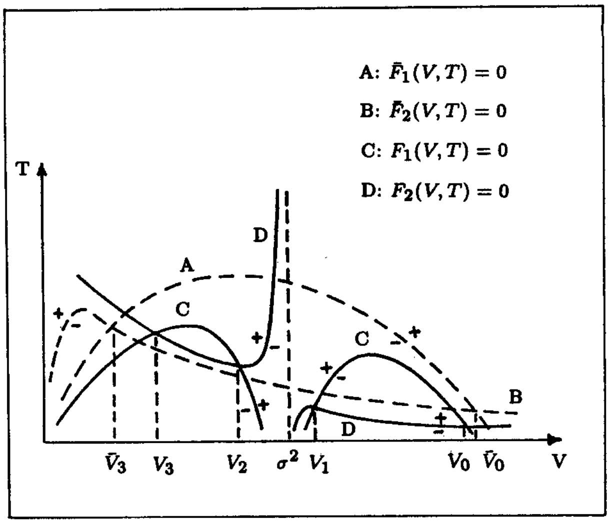

Figure 1. The + and − signs show the sign of the related function to the curve in the regions.

Case 2

If, at a rest point of System 2, $V = \delta^2$, then, at that rest point, we must have:

$$\begin{aligned}
&\frac{1}{2}(B_2^2+B_3^2) + p(\delta^2,T) + \delta^2 - J = 0 \ , \\
&e(\delta^2,T) - \frac{1}{2}\delta^4 + J\delta^2 - C = 0 \ . \qquad (5)
\end{aligned}$$

Then, $\overline{T}_2(\delta^2)$ is the unique solution of the second equation of Equations 5. If $\overline{V}_3 < \delta^2 < \overline{V}_0$, then, from the last statement in Theorem 1, we must have $\overline{F}_1(\delta^2, \overline{T}_2(\delta^2)) = \delta^2 - J + p(\delta^2, \overline{T}_2(\delta^2)) < 0$. Thus, from the first equation of Equations 5, we obtain:

$$\frac{1}{2}(B_2^2+B_3^2) + p(\delta^2,\overline{T}(\delta^2) + \delta^2 - J = 0 \ ,$$

or

$$\begin{aligned}
(B_2^2+B_3^2) &= 2(J - \delta^2 - p(\delta^2,\overline{T}(\delta^2))) \\
&= \beta^2 \ , \ \beta > 0. \qquad (6)
\end{aligned}$$

For the case $\in > 0$, as we shall see later, at a rest point, we have $B_3 = 0$. Thus, if we let $B_3 = 0$, then from Equation 6, we obtain $B_2 = \pm\beta$. In this way we find the following two rest points $\overline{u}_1$ and $\overline{u}_2$:

$$\overline{u}_i = ((-1)^i\beta, 0, 0, 0, \delta^2, \overline{T}_2(\delta^2)) \ , \ i = 1,2 \ . \qquad (7)$$

Therefore, we have the following theorem.

Theorem 2

For suitable values of $J > 0$, $C \in R$ and $\delta > 0$ the points $\overline{u}_i = (0,0,0,0,\overline{V}_i,\overline{T}_i)$, $i = 0,3$ and $\overline{u}_i = ((-1)^i\beta,0,0,0,\delta^2,\overline{T}_2(\delta^2))$, $i = 1,2$, are the four rest points for System 2. Moreover, each point of the set of points $\{u \in R^6 : (B_2,B_3,0,0,\delta^2,\overline{T}_2(\delta^2)), B_2^2+B_3^2=\beta^2, B_2,B_3 \in R\}$ is also a rest point of System 2.

Now we shall consider the rest points of System 1, whenever $\in > 0$. At a rest point, in this case, $y_2 = y_3 = B_3 = 0$ and $B_2 = -\frac{\in}{V-\delta^2}$. Substituting these values of y_i and B_i, $i = 2, 3$ into the equations for $\dot{V}$ and $\dot{T}$, we obtain the following equations at a rest point:

$$\begin{aligned} F_1(V,T) &= \frac{1}{2}\in^2 (V-\delta^2)^{-2} + V - J \\ &\quad + p(V,T) = 0\ , \\ F_2(V,T) &= \frac{1}{2}\in^2 (V-\delta^2)^{-1} - \frac{1}{2}V^2 \\ &\quad + JV - C + e(V,T) = 0\ . \end{aligned} \tag{8}$$

These equations are the same as Equations 2.3 in [4]. Thus, from [4] we have the following theorem.

Theorem 3

For fixed $J > 0$, $\delta > 0$ and $\in > 0$, there are two numbers $C_0 \geq C_1$ such that for $C > C_0$ the system of algebraic Equations 8 admits no solution at all. For $C < C_1$ it admits precisely four solutions. Two of these solutions are located in the region $V > \delta^2$ and the other two are located in the region $0 < V < \delta^2$. For $C = C_0$, $C_1 < C < C_0$ and $C = C_1$, it admits one, two and three solutions, respectively. At most, two of these are located in $V > \delta^2$ (or $0 < V < \delta^2$).

From now on we assume that Equations 8 admit four solutions. We denote these solutions by $(V_i(\in),T_i(\in))$, $0 \leq i \leq 3$ where $V_0(\in) > V_1(\in) > \delta^2 > V_2(\in) > V_3(\in)$. This means that System 1 admits the following four rest points:

$$\begin{aligned} u_i(\in) &= (-\in (V_i-\delta^2)^{-1},0,0,0,V_i,T_i), \\ &\quad 0 \leq i \leq 3\ , \end{aligned} \tag{9}$$

where V_i and T_i means $V_i(\in)$ and $T_i(\in)$, respectively.

Now we have the following lemma, its proof is the same as proof of Lemma 3.2 in [4].

Lemma 1

Suppose in System 1, $\delta > 0$, $J > 0$ and $C \in R$ are fixed. Given $\in_1 > 0$ there are positive numbers a_i and b_i, $i = 1,2$ such that if $u = (B_2,B_3,0,0,V,T)$ is a rest point of this system corresponding to $0 \leq \in \leq \in_1$, then $(V,T) \in [a_1,b_1] \times [a_2,b_2]$.

Finally, in this section, we have the following theorem. Its proof is the same as proofs of Theorem 2.4 and Corollary 2.1 in [4].

Theorem 4

Suppose System 2 admits the four rest points $\overline{u}_i$, $0 \leq i \leq 3$. Then there is an $\in_1 > 0$ such that for $0 < \in < \in_1$, System 1 admits the four rest points $u_i(\in)$, $0 \leq i \leq 3$. Moreover $T_i(\in) < T_{i+1}(\in)$, $i = 0,1,2$, and $\overline{u}_i = \lim_{\in \to 0} u_i(\in)$, $0 \leq i \leq 3$.

SOME BOUNDS ON HETEROCLINIC ORBITS

As we mentioned before, the existence of bounded complete orbits correspondence to $\in = 0$ will be proven as limits of the structure for the fast and the slow shock waves as $\in \longrightarrow 0$. In order to do this, we need to have the set of all heteroclinic orbits correspondence to the fast and the slow shocks bounded and bounded away from $V = 0$ and $T = 0$ independent of $\in$ and the viscosity parameters. This section is devoted to finding such bounds. Prior to this we have the following.

Given an autonomous system of ordinary differential equations in R^n:

$$\frac{dx}{dt} = f(x)\ , \tag{10}$$

we will denote by $x.t$ the value of the solution of this system at time t that is x initially. As f is assumed smooth, $x.t$ will be uniquely defined for each x on an open interval of t contains 0.

For $S \subset R^n, J \subset R$, we let $S.J = \{x.t : x \in S, t \in J\}$. The set S is called invariant if $S.R = S$. For $Y \subset R^n$, the ω-limit set of Y is defined to be the maximal invariant set in the closure of $Y.[0,\infty)$. Similarly, α-limit set of Y is defined to be the maximal invariant set in the closure of $Y.(-\infty, 0]$.

By an orbit we mean a solution of System 10 which is defined on an open interval. By a complete orbit we mean an orbit which is defined for all $t \in R$. We say that $\gamma(t)$ is an orbit running from x_0 to x_1 if $\gamma(t)$ is a complete orbit and $\lim_{t\to-\infty}\gamma(t) = x_0$ and $\lim_{t\to+\infty}\gamma(t) = x_1$. Then x_0 and x_1 must be rest points. Such an orbit is called a heteroclinic orbit.

System 10 is called gradient-like in the open set $U \subset R^n$, if there is a continuous real valued function h on U which is strictly increasing on nonconstant solutions of System 10. The function h is called a gradient-like function. We also say that this system is gradient-like with respect to h on U.

Note that the ω-limit set of the orbit $x.t$ means the set of limit points of sequences $x.t_n$ as t_n tends to plus infinity, and α -limit set of this orbit is the set of limit points of these sequences as t_n goes to minus infinity. It is known that, for a bounded complete orbit, each of these two sets is nonempty, compact, connected and invariant. In the case of a gradient-like system, the restriction of the gradient-like function to each of these sets is constant. Therefore, each of them consists of rest points [9,10].

Now we shall show that Systems 1 and 2 are both gradient-like. To do this, we defined the real valued functions P and $\overline{P}$ on $V > 0$ and $T > 0$ by:

$$\begin{aligned} P(u) = T^{-1}\{&\frac{1}{2}V^2 + \frac{1}{2}(V-\delta^2)(B_2^2+B_3^2) - JV \\ &- \frac{1}{2}(y_2^2+y_3^2) + \in B_2 - f(V,T) + C\}, \\ \overline{P}(u) = T^{-1}\{&\frac{1}{2}V^2 + \frac{1}{2}(V-\delta^2)(B_2^2+B_3^2) \\ &- JV - \frac{1}{2}(y_2^2+y_3^2) - f(V,T) + C\}, \end{aligned} \tag{11}$$

where $f(V,T)$ is the Helmholtz free energy function [11]. This function satisfies:

$$f_V = -p,\ f_T = -S,\ e = f + TS\ . \tag{12}$$

Theorem 5

System 1 is gradient-like with respect to $P(u)$ in the region $\{u \in R^6 : V, T > 0\}$ for all choices of the viscosities except for the case $\eta = k = \sigma^{-1} = 0$. System 2 is gradient-like with respect to $\overline{P}(u)$ in the above region for all choices of the viscosities except for the case $\alpha \neq 0$, $\sigma^{-1} = 0$.

For the proof of this theorem see Theorem 3.1 in [1].

Here we assume that δ, J, C and the viscosity parameters α, η, k and σ^{-1} are fixed and $\chi = 0$, moreover Theorems 4 and 5 hold.

Let $\in_1$ be the same as in Theorem 4, $\in_0$, $0 < \in_0 \leq \in_1$, a fixed number. For $0 < \in \leq \in_0$, let $S(\in)$ be defined as the set of all points which lie on bounded complete orbits of System 1 corresponding to the above parameter values and $S_0 = \bigcup_{0<\in\leq\in_0} S(\in)$.

The next four lemmas are modifications of Lemmas 3.2-3.5 in [1]. Those parts of their proofs which are similar will be referred to [1].

Lemma 2

On S_0, $V \leq J$.

Proof

If $\eta = 0$, then, from the fifth equation of System 1, we obtain $V < J$. If $\eta \neq 0$, then from $p(V,T) > 0$ we get $\dot{V} > 0$, whenever $V > J$. Thus, if V ever exceeds J on a complete orbit in $\{V, T > 0\}$, then the V coordinate on that orbit must go to $+\infty$ as t tends to $+\infty$.

Lemma 3

There are constants $\in_0 > 0$ and $a > 0$ such that on S_0, $T > a$ and $V > a$.

Proof

Suppose such an a and $\in_0$ do not exist. Then, similar to the proof of Lemma 3.4 in [2], we can conclude that there exists a sequence $\{\in_j\}$ and a sequence $\{u'_j\}$ in S_0, such that $u'_j \in S(\in_j)$, $\in_j \to 0$ and the T coordinate of $\{u'_j\}$ converges to zero, moreover on $u'_j.t$, the heteroclinic orbit of System 1, corresponding to the particular value of $\in = \in_j$, we have $\dot{T} = 0$ at u'_j. This means that the right hand side of the last equation in System 1 at u'_j is zero and $T'_j \to 0$

where T_j' is the sixth component of u_j'. If we denote the gradient-like function corresponding to $\in_j$ by $P_j(u)$, then from (3.4) in [1] we obtain $P_j(u_j') = S(u_j')$. Since $0 < V_j' \leq J$ (V_j' is the fifth component of u_j') and $T_j' \to 0$, by hypothesis $H_4, S(u_j') = S(V_j', T_j') \to 0$. Hence, $P_j(u_j') \to 0$.

Now let $Y = \{(V,T) : a_1 \leq V \leq b_1, a_2 \leq T \leq b_2\}$, where a_i and b_i, $i = 1,2$ are given in Lemma 1. Then $A = \inf_{(V,T)\in Y} S(V,T) > 0$. By Theorem 3.2 in [1], $u_j'.t$ tends to a rest point, say u_j'', as t tends to $-\infty$. By Lemma 1 $(V_j'', T_j'') \in Y$, where V_j'' and T_j'' are the fifth and the sixth components of u_j''. Since P_j is increasing along the orbits, $P_j(u_j') > P_j(u_j'') = S(u_j'') \geq A > 0$. This contradicts $P_j(u_j') \to 0$. Hence such $a > 0$ and $\in_0$ exist.

Lemma 4

There is a constant M such that on $S_0, |B_2|, |B_3|, |y_2|$ and $|y_3|$ are less than M.

Proof

We will consider the following cases.

Case 1

$\alpha = 0$. From the first two equations of System 1, we obtain $y_1 = y_2 = 0$. Then the last four equations of System 1 reduce to the following system of ordinary differential equations (assuming $\chi = 0$):

$$\begin{aligned}
\sigma^{-1}\dot{B}_2 &= \in + (V - \delta^2)B_2 \ , \\
\sigma^{-1}\dot{B}_3 &= (V - \delta^2)B_3 \ , \\
\eta\dot{V} &= p(V,T) + V + \frac{1}{2}\,(B_2^2 + B_3^2) - J \ , \\
k\dot{T} &= e(V,T) - \frac{1}{2}\,V^2 - \frac{1}{2}\,(V - \delta^2)(B_2^2 + B_3^2) \\
&\quad + JV - \in B_2 - C \ . \qquad (13)
\end{aligned}$$

If $\sigma^{-1} = 0$, then $B_3 = 0$ and $B_2 = -\frac{\in}{V-\delta^2}$. Substituting in the third and fourth equations we obtain:

$$\begin{aligned}
\eta\dot{V} &= p(V,T) + V - J + \frac{1}{2}\in^2 (V - \delta^2)^{-2} \ , \\
k\dot{T} &= e(V,T) - \frac{1}{2}V^2 + JV - C \\
&\quad + \frac{1}{2}\in^2 (V - \delta^2)^{-1}.
\end{aligned}$$

This system of equations is the same as (4.3.4) in [2]. Thus from [2] such an M, independent of $0 < \in \leq \in_0$, in this case exists. If $\sigma^{-1} \neq 0$ and $a \leq V \leq J$, then $|B_2|$ and $|B_3|$ decrease at most exponentially. Thus, there is a positive constant b, such that if an orbit of Equations 13 initiating at a point with $|B_2| + |B_3| > b$, then either V sometimes exceeds J or:

$$B_2^2(t) + B_3^2(t) \geq 2(J + 1 + \eta) \ ,$$

independent of $0 < \in \leq \in_0$, on an interval of time of length J, say for $t \in [t_0, t_0 + J]$. If $\eta \neq 0$, then from the third equation of Equations 13, we obtain $\dot{V} > 1$ on $[t_0, t_0 + J]$. Thus, $V(t_0 + J) > J$. Hence, such an orbit is not in S_0. If $\eta = 0$, then from the third equation of Equations 13 we get $B_2^2(t) + B_3^2(t) \leq 2J$ on any orbit of Equations 13. Hence, such an M must exist, in this case, independent of $0 < \in \leq \in_0$.

Case 2

$\alpha \neq 0$. Using $\chi = 0$, the first five equations of System 1 become:

$$\begin{aligned}
\alpha V\dot{B}_2 &= y_2 \ , \\
\alpha V\dot{B}_3 &= y_3 \ , \\
\alpha V\dot{y}_2 &= \in + (V - \delta^2)B_2 - \frac{\sigma^{-1}}{\alpha}\frac{y_2}{V} \ , \\
\alpha V\dot{y}_3 &= (V - \delta^2)B_3 - \frac{\sigma^{-1}}{\alpha}\frac{y_3}{V} \ , \\
\eta\dot{V} &= \frac{1}{2}(B_2^2 + B_3^2) + V - J + p(V,T) \ .
\end{aligned}$$

If $a \leq V \leq J$, from the first four equations, we see that $|B_i|$ and $|y_i|, i = 2,3$ decrease at most exponentially. Thus, given any positive constants K and t_0, two constants $M > M_1 > K$ can be found such that any orbit initiating at a point $(B_{20}, B_{30}, y_{20}, y_{30}, V_0, T_0)$ with $y_{i0}^2 \geq M_1$ which is defined for $t \in [0, J]$, either $V \notin [a, J]$ for some $t \in [0, J]$ or $y_i^2 > K$ for all $t \in [0, J]$, $i = 2,3$, independent of $0 < \in \leq \in_0$. Moreover, any orbit initiating at a point $(B_2', B_3', y_2', y_3', V', T')$ with $(B_i'^2 + y_i'^2) \geq M^2$ which is defined for $t \in [0, t_0]$, either $V \notin [a, J]$ for some $t \in [0, t_0]$ or $(B_i^2 + y_i^2) > 2M_1$ for all $t \in [0, t_0]$, $i = 2,3$, independent of $0 < \in \leq \in_0$.

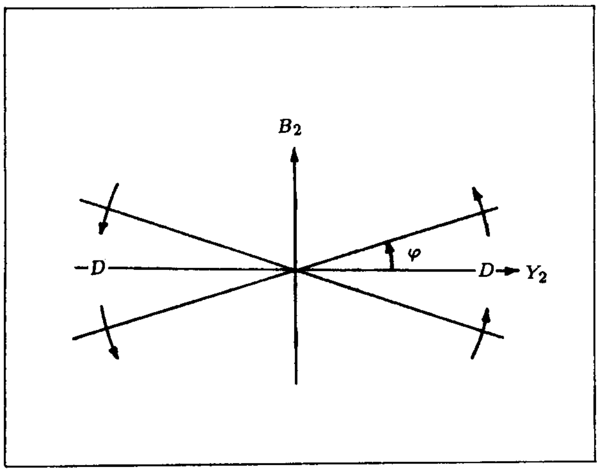

Figure 2. Arrows on the boundary of D show the direction of the flow on this set.

Now let $\theta = \tan^{-1}\frac{B_2}{y_2}$ with $\theta = 0$ (mod 2π) on the positive $y_2 - axis$. A straightforward calculation gives:

$$\frac{d\theta}{dt} = \frac{1}{\alpha V}\left[\cos^2\theta - (V-\delta^2)\sin^2\theta + \frac{\sigma^{-1}}{\alpha V}\sin\theta\cos\theta - \frac{\in\sin\theta}{(B_2^2+y_2^2)^{\frac{1}{2}}}\right].$$

Choose $0 < \varphi < \frac{\pi}{4}$ such that for $a \le V \le J$ and $|\theta| \le \varphi$ or $|\theta - \pi| \le \varphi$, we have $\frac{d\theta}{dt} > \frac{1}{2\alpha J}$, for all $0 < \in \le \in_0$. Now define:

$$D = \{u : B_2^2 \le c(B_2^2 + y_2^2),\ a \le V \le J\}\ ,$$

where $c = \tan^2\varphi(1+\tan^2\varphi)^{-1} < \frac{1}{2}$.

From $\frac{d\theta}{dt} > \frac{1}{2\alpha J}$ we see that any orbit segment on which $V \in [a, J]$ either stay out of D or else spends time (i.e., an interval for t) in D which is at most $4\alpha\varphi J$. Note that such an orbit, having crossed out of any component of D, cannot re-enter the same component of D without passing through the other component of D (see Figure 2).

Now choose $cK > 2(\alpha J + \eta)$ and $t_0 = (2 + 4\varphi)\alpha J$. Then, by the above argument, there are two constants $M > M_1 > K$ such that any orbit initiating at a point $y_2^2 \ge M_1$, either $V \notin [a, J]$ for some $t \in [0, J]$ or $y_2^2 > K$ for all $t \in [0, J]$. Moreover, any orbit initiating at a point $B_2'^2 + y_2'^2 > M^2$, either $V \notin [a, J]$ for some $t \in [0, t_0]$ or $B_2^2 + y_2^2 > 2M_1 > 2K$ for all $t \in [0, t_0]$.

We shall now show that $B_2^2(t) + y_2^2(t) < M^2$ on S_0. Supposing this statement to be untrue, then there is an orbit in S_0 which starts at $B_2^2(0) + y_2^2(0) \ge M^2$ and along that orbit $B_2^2(t) + y_2^2(t) > 2M_1 > 2K > \frac{4}{c}(\alpha J + \eta)$ for $t \in [0, t_0]$. Note that in S_0, $V \notin [a, J]$ cannot occur. If the orbit segment is ever in D, it gets out in time $4\alpha\varphi J$. Thus, if the orbit does not lie completely out of D for the time interval $[0, J]$, then it goes through a boundary point of D at the time $t_1 \in [0, \alpha J + 4\alpha\varphi J]$. Since $B_2^2(t_1) = c[B_2^2(t_1) + y_2^2(t_1)]$ and $0 < c < 1/2$, we must have $y_2^2(t_1) \ge M_1$. Thus, by the above argument, in the time interval $[t_1, t_1 + J]$ along the orbit, $y_2^2(t) > K > 0$. Since the orbit cannot re-enter D without crossing $y_2 \equiv 0$, the orbit must lie outside of D for the time interval $[t_1, t_1 + J]$. Since $[t_1, t_1 + J] \subset [0, t_0]$, $B_2^2(t) \ge c[B_2^2(t) + y_2^2(t)] \ge 2(\alpha J + \eta)$ for a time interval J. Therefore:

$$\eta\frac{dV}{dt} = \frac{1}{2}[B_2^2(t) + B_3^2(t)] + V(t) - J + p(V(t), T(t)) \ge \frac{1}{2}B_2^2(t) - J \ge \eta\ ,$$

or $\frac{dV}{dt} \ge 1$ for the time interval $[t_2, t_2 + J]$ where $t_2 = 0$ or t_1. Thus:

$$\int_{t_2}^{t_2+J}\frac{dV}{dt}dt \ge J \ or\ V(t_2 + J) \ge J.$$

Such an orbit cannot lie in S_0. Hence $B_2^2(t) + y_2^2(t) \le M^2$ on S_0. With a similar argument for B_3 and y_3, we obtain $B_3^2(t) + y_3^2(t) \le M^2$ on S_0.

Lemma 5

There is a constant T_0 such that if $u \in S_0$, then $T \le T_0$, where T is the sixth component of u.

Proof

By considering the sixth equation of System 1 the proof is similar to the proof of Lemma 3.6 in [3].

Now, suppose that the numbers $J > 0$, $\delta > 0$, $C \in R$, $\in_1 > 0$ and $(\alpha, \eta, k, \sigma^{-1}, \chi) \ge 0$ are such that Theorems 4 and 5 hold. Let $0 \le \in_0 \le \in_1$. For $0 < \in \le \in_0$, let $\overline{S}(\in)$ be defined as the set of all points which lie on bounded complete orbits of System 1 correspondence to the above parameter values and $\overline{S}_0 = \bigcup_{0<\in\leq\in_0} S(\in)$. Note that for the set S_0 we had $\chi = 0$, but for the set $\overline{S}_0$ we have $\chi \geq 0$.

Theorem 6

Let $\overline{S}_0$ be as above. Then there are positive numbers a' and M' such that $\overline{S}_0 \subset \{u \in R^6 : |u| < M', V > a', T > a'\}$.

Proof

Let a, M and T_0 be the same as in the above lemmas. Define $N_0 = \{u \in R^6 : |u| < M + J + T_0,\ V > \frac{a}{2},\ T > \frac{a}{2}\}$. Then N_0 is an isolating neighborhood for parameters $\alpha, \eta, k, \sigma^{-1}, J, \delta, C, 0 < \in \leq \in_0$ and $\chi = 0$. Since χ is a regular parameter, N_0 can be widened to a block valid for all flows parameterized by $\chi \geq 0$ which contains all of the bounded complete orbits of System 1 for $0 < \in \leq \in_0$ and $0 \leq \chi \leq \chi_0$ for any given $\chi_0 > 0$. Thus, such an $a' > 0$ and $M' > 0$ must exist.

THE SWITCH-ON AND THE SWITCH-OFF SHOCKS

In this section we will study the existence of the switch-on and the switch-off shock waves. As we mentioned before, the switch-on and the switch-off shocks occur when $\in = 0$. Thus, for these shocks, we should consider System 2, its rest points $\overline{u}_i$, $0 \leq i \leq 3$ and its other rest points which are given in Theorem 2. Given $\alpha, \eta, k, \sigma^{-1} > 0$ and $\chi \geq 0$, in this section we shall prove the existence of some bounded complete orbits of System 2 . Moreover, we will see that for each $(i,j) \in \{(m,n) : 0 \leq m < n \leq 3\} \setminus \{(1,2)\}$, there is a heteroclinic orbit which is running from $\overline{u}_i$ to $\overline{u}_j$ lying in the subspace $\{u \in R^6 : B_3 = y_3 = 0\}$. The technique we use here is to obtain these bounded complete orbits as limiting of the heteroclinic orbits for the fast and slow shocks as $\in \longrightarrow 0$. In the first step we have the following lemma.

Lemma 6

Let $D \subset R^n$ be a bounded open set, $\in_0 > 0$, $I = (0, \in_0)$ and $f : \overline{D} \times \overline{I} \longrightarrow R^n$ be continuous. Moreover, assume that the following conditions hold:

C_1: The function f is uniformly Lipschitz on $D \times I$.

C_2: There is a uniformly Lipschitz vector field g on $\overline{D}$ such that $\lim_{\in\to 0} f(u,\in) = g(u)$ uniformly on D.

C_3: For each $\in\in I$, there is a heteroclinic orbit of the system:

$$\frac{du}{dt} = f(u, \in)\ , \tag{14}$$

say $\gamma(u,\in)$, which is running from the rest point $u_0(\in)$ to the rest point $u_1(\in)$ lying in D.

C_4: $\lim_{\in\to 0} u_i(\in) = u_i$, $i = 0,1$ and $u_0 \neq u_1$.

C_5: The system of equations:

$$\frac{du}{dt} = g(u)\ , \tag{15}$$

is gradient-like with respect to a function P on a neighborhood of $\overline{D}$ with $P(u_0) < P(u_1)$. Moreover for each $\lambda \in (0,1)$ the hypersurface $P(u) = \lambda P(u_0) + (1-\lambda) P(u_1)$ contains no rest points of System 15. Then there is a complete orbit of System 15 lying in D, its ω-limit set and α-limit set consist of rest points. The first set contains u_1 and the second set contains u_0.

Proof

Choose a sequence of points $\in_k \in I$ such that $\in_k \longrightarrow 0$. Denote the corresponding heteroclinic orbit to $\in_k$ by $\gamma_k(t)$. The set $Q = \{u \in \overline{D}: P(u) = \frac{1}{2}P(u_0) + \frac{1}{2}P(u_1)\}$ intersects $\gamma_k(t)$, for large values of k say at the point $u'_k = \gamma_k(t_k)$. Since Q is compact, $\{u'_k\}$ must have a convergent subsequence. We may assume that $u'_k \longrightarrow u'_0$ and $t_k = 0$ for all k. Let $a > 0$ be given. Then Condition C_1 implies that there is a constant M such that for $0 \leq t \leq a$:

$$\begin{aligned}|\dot\gamma_k(t) - \dot\gamma_m(t)| &\leq |f(\gamma_k(t), \in_k) - f(\gamma_m(t), \in_m)| \\ &\leq M[|\gamma_k(t) - \gamma_m(t)| + |\in_k - \in_m|],\end{aligned}$$

where $. = \frac{d}{dt}$. On the other hand, for $0 \le t \le a$ we can write:

$$|\gamma_k(t) - \gamma_m(t)| \le |\gamma_k(0) - \gamma_m(0)| + \int_0^t |f(\gamma_k(s), \epsilon_k) - f(\gamma_m(s), \epsilon_m)| ds \le |\gamma_k(0) - \gamma_m(0)| + aM| \epsilon_k - \epsilon_m | + M\int_0^t |\gamma_k(s) - \gamma_m(s)| ds .$$

Therefore, by Gronwall's inequality we must have:

$$|\gamma_k(t) - \gamma_m(t)| \le [|\gamma_k(0) - \gamma_m(0)| + aM| \epsilon_k - \epsilon_m |]e^{aM}.$$

Hence $\{\gamma_k(t)\}$ and $\{\dot{\gamma_k}(t)\}$ are both uniformly Cauchy on $[0, a]$. Let $\gamma(t) = \lim_{k\to\infty} \gamma_k(t)$. Then $\dot{\gamma}(t)$ exists and $\dot{\gamma}(t) = \lim_{k\to\infty} \dot{\gamma}_k(t)$. Now for $0 \le t \le a$ we can write:

$$|\dot{\gamma}(t) - g(\gamma(t))| \le |\dot{\gamma}(t) - \dot{\gamma}_k(t)| + |f(\gamma_k(t), \epsilon_k) - g(\gamma_k(t))| + |g(\gamma_k(t)) - g(\gamma(t))|.$$

Thus if $k \to \infty$, Condition C_2 yields $|\dot{\gamma}(t) - g(\gamma(t))| \le 0$. Hence, $\gamma(t)$ is a solution of System 15 in the interval $[0, a)$. Similarly $\gamma(t)$ is a solution of System 15 in the interval $(-a, 0]$. Since a is arbitrary, $\gamma(t)$ is a solution of System 15 on R containing u_0'. Since u_0' is a regular point of System 15, $\gamma(t)$ is a nonconstant complete orbit of System 15. Since this complete orbit is bounded, its α-limit set and ω-limit set both are nonempty and connected. Then, from Condition C_5 we see that α-limit set and ω-limit of $\gamma(t)$ consist of rest points of System 15 and u_0 is in α-limit set and u_1 is in ω-limit set of this orbit.

Now consider System 2. According to Theorem 2, for suitable values of $J > 0$, $C \in R$ and $\delta > 0$, this system admits infinitely many rest points. Moreover, by Theorem 4, for these values of J, C, δ, and $\epsilon > 0$ and small, System 1 admits the four rest points $u_i(\epsilon)$. By considering, this fact we have the following theorem.

Theorem 7

Suppose in System 2, α, η, k and σ^{-1} are fixed and positive. Suppose for fixed $J > 0$, $\delta > 0$ and $C \in R$ this system admits infinitely many rest points, given in Theorem 2. Then there are two complete orbits of this system such that their α-limit sets are $\overline{u}_0$ and their ω-limit sets are contained in the set $E = \{\overline{u} \in R^6 : \overline{u} = (B_2, B_3, 0, 0, \delta^2, \overline{T}(\delta^2)),\ B_2,\ B_3 \in R,\ B_2^2 + B_3^2 = \beta^2\}$. The rest point $\overline{u}_1$ is in one of these α-limit sets and $\overline{u}_2$ is in another one. Also there are two other complete orbits of this system such that $\overline{u}_3$ is their ω-limit set and their α-limit sets are contained in the set E. Moreover $\overline{u}_1$ is in one of these α-limit sets and $\overline{u}_2$ is in another one. Besides, there is a complete orbit which is running from $\overline{u}_0$ to $\overline{u}_3$.

Proof

We will show that System 1 as System 14 together with System 2 as System 15, the set $\{u \in R^6\colon\ \overline{P}(u) < \frac{1}{2}[(\overline{P}(\overline{u}_2) + \overline{P}(\overline{u}_3)],\ |u| < M', V > a',\ T > a'\}$ as D, the rest points $u_0(\epsilon), u_1(\epsilon), \overline{u}_0$ and $\overline{u}_1$ as the rest points $u_0(\epsilon)$, $u_1(\epsilon)$, u_0 and u_1 satisfy Conditions $C_1 - C_5$ of Lemma 6.

Condition C_1 obviously holds. By Theorem 4.1 in [1] Condition C_3 holds too. Condition C_4 is satisfied by Theorem 4. By Theorem 5, System 2 is gradient-like with respect to $\overline{P}(u)$, which is given by Equations 11, moreover $\overline{P}(\overline{u}_0) < \overline{P}(\overline{u}_1)$ and the hypersurface $\overline{P}(u) = \frac{1}{2}[\overline{P}(\overline{u}_0) + \overline{P}(\overline{u}_1)]$ contains no rest points of System 2. This means that Condition C_5 holds too.

In order to see that Condition C_2 holds, we denote the vector fields of Systems 1 and 2 by $G_1(u)$ and $G_2(u)$ respectively. Then:

$$|G_1(u) - G_2(u)| \le \epsilon\,(1 + |B_2|) \le \epsilon\,(1 + M') ,$$

where M' is the same as above. Thus Condition C_2 holds too. Therefore, by Lemma 6 there is a complete orbit of System 2 which is running from $\overline{u}_0$, and $\overline{u}_1$ is in its ω-limit set and this limit set is contained in the set E which is defined in the above. This orbit is lying in the set $\{u \in R^6\colon |u| \le M'$ and $V \ge a',\ T \ge a'\}$.

Now, if $u(t) = (B_2(t), B_3(t), y_2(t), y_3(t), V(t), T(t))$ is a solution of System 2 then, from symmetry, we see that $(-B_2(t), -B_3(t), -y_2(t), -y_3(t), V(t), T(t))$ must be another solution of this system. Thus, by considering Equation 7, there must be another orbit running from $\overline{u}_0$, and $\overline{u}_2$ is in its ω-limit set and this limit set is contained in the set E. By using a similar argument, we can show that there are two different complete orbits, their ω-limit set is the rest point $\overline{u}_3$ and their α-limit sets are contained in the set E. One of these α-limit sets contains $\overline{u}_1$ and another one contains $\overline{u}_2$.

Finally consider the rest points $\overline{u}_0$ and $\overline{u}_3$. Note that the subspace $\{u \in R^6\colon B_2 = B_3 = y_2 = y_3 = 0,\ V > 0,\ T > 0\}$ is invariant under System 2, and in this subspace the system becomes the gas dynamics equations. It is known that there is a unique orbit running from one rest point to another one for the gas dynamics equations [2,9]. Thus, there is an orbit running from $\overline{u}_0$ to $\overline{u}_3$.

Remark

If, in System 2, we replace B_2, B_3, y_2 and y_3 by $B, 0, y$ and 0, respectively, we obtain the system (2.5) in [4]. Thus by Theorem 6.1 in [4] for each $(i,j) \in \{(n,m)\colon 0 \le n < m \le 3,\ n,\ m = 0,1,2,3,\ (n,m) \ne (1,2)\}$ there is a complete orbit which is running from $\overline{u}_i$ to $\overline{u}_j$. Moreover there is no connecting orbit between $\overline{u}_1$ and $\overline{u}_2$. These orbits lie in the subspace $\{u \in R^6 : B_3 = y_3 = 0\}$.

Since each neighborhood of $\overline{u}_1(\overline{u}_2)$ contains infinitely many other rest points of System 2 and the rest points $\overline{u}_0$ and $\overline{u}_3$ are nondegenerated, the switch-on and the switch-off shocks do not admit structure. Since small change on $\overline{u}_1(\overline{u}_2)$ makes the related heteroclinic orbits disappear, these shocks physically are called nonevolutionary [12]. This means that these shocks, physically, cannot occur.

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