Mixed Galerkin Finite Element Analysis of Nonaxisymmetrically Loaded Spherical Shells

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A thin spherical shell under nonaxistymmetric loading is considered. The shell governing equations are reduced to four second order differential equations in terms of three displacement components and the meridian moment, which are all functions of two principle angles ϕ and θ, in meridian and circumferential directions, respectively. Using the Fourier's series of expansion in the θ-direction, the shell equations are further reduced to four second order ordinary differential equations of the mth harmonic of dependent functions in terms of the variable ϕ. Two sets of linear and third order test functions are employed to formulate the finite element model of the shell based on Galerkin approximation. While the mixed formulation provides a more accurate implication of the boundary conditions, the results reveal minor differences between the two sets of approximations.

INTRODUCTION

The weighted residual methods, such as Galerkin, subdomain, collocation, and least square are alternative approaches for finite element modeling of engineering problems. The finite element formulations based on these methods are mathematically simple to apply and result in very accurate solutions compared to other numerical methods. The Galerkin method, among the weighted residual methods, is especially favorable because of its strong rate of convergence, particularly when applied to nonlinear problems. Complicated engineering problems are powerfully handled with the Galerkin method.

Traditionally, this method has been favorably used in fluid flow problems [1]. The importance of this method is magnified when variational principles are either not developed or are questionable. Eslami [2-5] has applied it to coupled thermoelasticity problems where the proper variational principles of the first law of thermodynamics for solid materials are controversial. However, shell structures are seldom treated by this method in the literature. Sharma [6] has applied the weighted residual method to axisymmetric shells, but his treatment is rather general. Eslami [7] and Eslami and Shakeri [8-11] have made extensive use of the Galerkin method in detailed analysis of cylindrical and spherical shells under static and dynamic forces, including coupled thermoelasticity of cylindrical shells. The general conclusion is that lower degrees approximation polynomials to model the dependent fields provide more accurate results in comparison with variational formulations. While a third order approximation for lateral deflection provides acceptable results for variational formulations

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of shell of revolution [12], the linear set offers a precise solution when the Galerkin method is used [7, 8].

When variational formulations are employed for finite element analysis of structural problems, the displacement versus equilibrium models set up the upper and lower bounds of the solution. The displacement compatible finite element model always underestimates the strain energy function and results in a more flexible element providing an upper boundary to the solution, while the stress model results in an upper bound for strain energy function and produces a stiffer element, thus setting up the lower boundary of the solution [13]. While this mathematical conclusion is drawn on the basis of variational analysis, the Galerkin method also exhibits identical properties [3]. Based on this justification, the mixed models should converge to a more exact solution compared to the analytical treatment. Furthermore, the mixed models provide better tools in handling both kinematical and forced boundary conditions.

This paper employs the Galerkin finite element method and, by means of mixed formulation, presents the static nonaxisymmetric solution of spherical shells under general external loadings. The field of dependent functions includes three displacement components and the meridian moment. The conclusion is that simple mathematics and accuracy are the basic criteria in selection of the Galerkin approach in finite element analysis of the shell structures.

DERIVATIONS

Consider a spherical shell under non-axisymmetric loadings. The general applied forces on the surface of the shell are in the directions of the meridian f_u, tangential f_v, and lateral f_w, and the applied moments in the meridian $f_{\beta\phi}$ and tangential $f_{\beta\theta}$, as follows:

$$\{f(\phi,\theta)\}^T = < f_u \, f_v \, f_w \, f_{\beta\phi} \, f_{\beta\theta} > . \tag{1}$$

It is assumed that the force matrix is a function of shell variables ϕ and θ, the angles measured along meridian and tangential directions. The displacement matrix caused by the applied forces is a function of ϕ and θ and is

$$\{D(\phi,\theta)\}^T = < u \, v \, w \, \beta_\phi \, \beta_\theta >, \tag{2}$$

where u, v, and w are the components of displacements in the meridian, tangential and lateral directions, respectively, β_ϕ and β_θ are the shell rotations about the meridian and tangential directions. In this general case, the strain matrix is

$$\{\varepsilon(\phi,\theta)\}^T = < \epsilon_\phi \, \epsilon_\theta \, \epsilon_{\phi\theta} \, k_\phi \, k_\theta \, k_{\phi\theta} >, \tag{3}$$

where ε, s are strains and k, s are curvatures in the indicated directions. The associated matrix of forces and moments per unit length of shell loaded in general ϕ and θ directions is

$$\{N(\phi,\theta)\} = < N_\phi \, N_\theta \, N_{\phi\theta} \, M_\phi \, M_\theta \, M_{\phi\theta} > . \tag{4}$$

Since all the shell dependent parameters are functions of the variables ϕ and θ, a Fourier series expansion in the θ direction will eliminate the variable θ from governing equations, leaving the variable ϕ alone. The resulting governing equations are the nth harmonic of Fourier expansion of the shell parameters in terms of ϕ.

Consider two harmonic expansions $|\Theta_1^n|$ and $|\Theta_2^n|$ as follows:

$$\lfloor \Theta_1^n \rceil = \lfloor \cos n\theta, \sin n\theta, \cos n\theta, \cos n\theta, \sin n\theta \rceil, \tag{5}$$

$$\lfloor \Theta_2^n \rceil = \lfloor \cos n\theta, \cos n\theta, \sin n\theta, \cos n\theta, \cos n\theta, \sin n\theta \rceil. \tag{6}$$

It is easily verified that the variable θ is eliminated from the governing equations of the shell and the nth harmonic of the parameters will become a function of ϕ, provided that the following Fourier series expansion is applied:

$$\{f\} = \sum_{n=0}^{\infty} \lfloor \Theta_1^n(\theta) \rceil \{f^n(\phi)\},$$
$$\{D\} = \sum_{n=0}^{\infty} \lfloor \Theta_1^n(\theta) \rceil \{D^n(\phi)\},$$
$$\{\epsilon\} = \sum_{n=0}^{\infty} \lfloor \Theta_2^n(\theta) \rceil \{\epsilon^n(\phi)\},$$
$$\{N\} = \sum_{n=0}^{\infty} \lfloor \Theta_2^n(\theta) \rceil \{N^n(\phi)\}. \quad (7)$$

The equilibrium equations of shell can be written in terms of the nth harmonic of the three components of displacement u, v, w and the meridian moment $M\phi$, as shown below [14]. The superscript n denoting the nth harmonic is omitted for simplicity:

$$\begin{aligned} & a_1u'' + a_2u' + a_3u + a_4v' + a_5v + a_6w' \\ & \quad + a_7w + a_8m' + a_9m = C_1, \\ & a_{10}u' + a_{11}u + a_{12}v'' + a_{13}v' + a_{14}v \\ & \quad + a_{15}w'' + a_{16}w' + a_{17}w + a_{18}m = C_2, \\ & a_{19}u' + a_{20}u + a_{21}v'' + a_{22}v' + a_{23}v \\ & \quad + a_{24}w'' + a_{25}w' + a_{26}w + a_{27}m'' \\ & \quad + a_{28}m' + a_{29}m = C_3, \\ & a_{30}u' + a_{31}u + a_{32}v + a_{33}w'' + a_{34}w' \\ & \quad + a_{35}w + a_{36}m = C_4, \end{aligned} \quad (8)$$

where $m = M\phi$ and $(')$ shows the derivative with respect to ϕ. The right hand side parameters represent the applied lateral forces and are $C_1 = -p_\phi$, $C_2 = -p_\theta$, $C_3 = -p_r$, and $C_4 = 0$. The coefficients a_l through a_{36} are given in the appendix and are functions of shell parameters, material constants, and the variable ϕ. Solution of this set of governing equations is based on application of the mixed formulation to the kinematical and force fields. The finite element modeling, based on mixed formulations, insures the continuity of both displacement and stress fields and provides better means for application of both kinematical and forced boundary conditions.

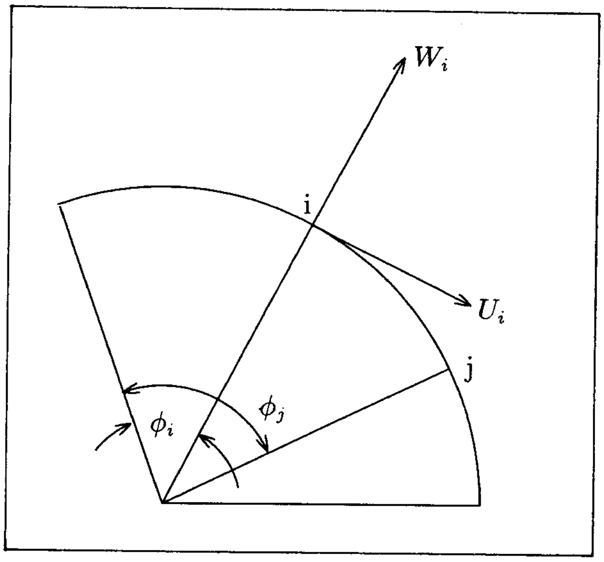

Figure 1. Spherical shell element.

GALERKIN FINITE ELEMENT MODELING

The field of approximation, as given by Equation 8, includes the four independent functions u, v, w and m. However, the meridian moment m is related to the displacement components through the kinematical relations and the constitutive law as follows:

$$\begin{aligned} M\phi = & - w'' - \mu \cot\phi\, w' + \mu w / \sin 2\phi \\ & + u' + \mu \cot\phi\, u + \mu v / \sin\phi. \end{aligned} \quad (9)$$

Selection of independent shape functions for u, v and w provides a definite expression for M_ϕ from Equation 9. However, to test the convergence of the Galerkin finite element analysis, two independent sets of shape functions are examined. The first set constitutes four distinct linear shape functions approximating u, v, w and m, regardless of the relationship between the displacements and moment as given by Equation 9. Considering a spherical shell element as shown by Figure 1, the four nodal degrees of freedom at nodes i and j are (u_i, v_i, w_i, m_i) and (u_j, v_j, w_j, m_j) and a simplex shape function in terms of the variable

ϕ approximates the field as:

$$\begin{aligned} u &= < N_1 > \{u\}, \\ v &= < N_1 > \{v\}, \\ w &= < N_1 > \{w\}, \\ m &= < N_1 > \{m\}. \end{aligned} \tag{10}$$

In this type of formulation the relationship of m and u, v and w is ignored.

On the other hand, the second set of shape functions consists of linear approximations for u and v and a third order polynomial for w. The meridian moment is then obtained from w as $m = -w''$, which is the first term of Equation 9. The third order approximation for w yields a linear shape function for m as follows:

$$\begin{aligned} u &= < N_1 > \{u\}, \\ v &= < N_1 > \{v\}, \\ w &= < N_3 > \{w\}, \\ m &= < N_1 > \{m\}. \end{aligned} \tag{11}$$

In terms of the variable ϕ, the members of matrix $< N_1 >$ are $N_i = (1-\phi)/1$ and $N_j = \phi/1$, where $1 = \phi_j - \phi_i$ and $\phi = \phi - \phi_i$.

On the basis of these two sets of shape functions, the finite element model of shell is derived using the Galerkin method. Applying the formal Galerkin method, the equilibrium equations of shell as given by Equation 8 are orthogonalized on the selected shape functions. In this process, the first equation is minimized with respect to the shape function governing the meridian displacement, that is, u, the second equation with respect to v, the third equation with respect to w and the last equation with respect to m. As the second derivative of dependent functions appear in equilibrium equations and the shape functions are linear, the integration by part of the second derivatives is necessary. Through weak formulations, the natural boundary conditions appear on the nodal boundaries, which cancel out each other between any two adjacent elements, except for the first and last elements. The resulting natural boundary conditions on real boundary of the solution domain should be fitted with the given boundary conditions. To clarify the statement, the following term, which appears in the Galerkin approximation, will be examined:

$$\int_0^1 N_m u'' d\phi = N_m u' |_0^1 - \int_0^1 N'_m u' d\phi. \tag{12}$$

While the second term on the right hand side is transformed into the stiffness matrix, the first term should be evaluated at the nodal boundaries. Due to continuity, the nodal values of this term are eliminated between any two adjacent elements except for its value on the boundary of the solution domain.

The finite element equilibrium equation obtained upon application of the Galerkin method based on the linear set of shape functions, Equation 10, and weak formulations of the governing Equation 8, result in the following:

$$\begin{aligned} &[A]\{u\} + [B]\{v\} + [C]\{w\} + [D]\{m\} = \{T_1\} + \{R\}, \\ &[E]\{u\} + [F]\{v\} + [G]\{w\} + [H]\{m\} = \{T_2\} + \{X\}, \\ &[I]\{u\} + [J]\{v\} + [K]\{w\} + [L]\{m\} = \{T_3\} + \{Y\}, \\ &[N]\{u\} + [O]\{v\} + [P]\{w\} + [Q]\{m\} = \{Z\}, \end{aligned} \tag{13}$$

where the matrices $\{T_1\}$, $\{T_2\}$ and $\{T_3\}$ are the resulting external forces acting on the shell and the matrices $\{R\}$, $\{X\}$, $\{Y\}$ and $\{Z\}$ are the force matrices resulting from weak formulations and boundary conditions. The matrices $[A]$ through $[Q]$ for the base element e are each a 2×2 matrix, where their members for the base element in local coordinate are defined as:

$$\begin{aligned} [\bar{A}] &= \int_0^1 [-(N_1 a_1)' N'_m + a_2 N_1 N'_m \\ &\quad + a_3 N_1 N_m\} d\phi \quad \begin{matrix} l = i, j \\ m = i, j \end{matrix} \\ [\bar{B}] &= \int_0^1 (a_4 N_1 N'_m + a_5 N_1 N_m) d\phi \\ [\bar{C}] &= \int_0^1 (a_6 N_1 N'_m + a_7 N_1 N_m) d\phi \\ [\bar{D}] &= \int_0^1 (a_8 N_1 N'_m + a_9 N_1 N_m) d\phi \\ [\bar{E}] &= \int_0^1 (a_{10} N_1 N'_m + a_{11} N_1 N_m) d\phi \end{aligned}$$

$$[\bar{F}] = \int_0^1 [-(a_{12}N_1)'N_m' + a_{13}N_1N_m' + a_{14}N_1N_m]d\phi$$

$$[\bar{G}] = \int_0^1 [-(a_{15}N_1)'N_m' + a_{16}N_1N_m' + a_{17}N_1N_m]d\phi$$

$$[\bar{H}] = \int_0^1 (a_{18}N_1N_m)d\phi$$

$$[\bar{I}] = \int_0^1 (a_{19}N_1N_m' + a_{20}N_1N_m)d\phi$$

$$[\bar{J}] = \int_0^1 [-(a_{21}N_1)'N_m' + a_{22}N_1N_m' + a_{23}N_1N_m]d\phi$$

$$[\bar{K}] = \int_0^1 [-(a_{24}N_1)'N_m' + a_{25}N_1N_m' + a_{26}N_1N_m]d\phi$$

$$[\bar{L}] = \int_0^1 [-(a_{27}N_1)'N_m' + a_{28}N_1N_m' + a_{29}N_1N_m]d\phi$$

$$[\bar{N}] = \int_0^1 (a_{30}N_1N_m' + a_{31}N_1N_m)d\phi$$

$$[\bar{O}] = \int_0^1 a_{32}N_1N_m d\phi$$

$$[\bar{P}] = \int_0^1 [-(a_{33}N_1)'N_m' + a_{34}N_1N_m' + a_{35}N_1N_m]d\phi$$

$$[\bar{Q}] = \int_0^1 a_{36}N_1N_m d\phi. \qquad (14)$$

The force matrices resulting from application of external forces in local coordinates are:

$$\{\bar{T}_1\} = \int_0^1 C_1N_1 d\phi \quad l = i, j$$

$$\{\bar{T}_2\} = \int_0^1 C_2N_1 d\phi$$

$$\{\bar{T}_3\} = \int_0^1 C_3N_1 d\phi, \qquad (15)$$

and the force matrices resulting from weak formulations and application of boundary conditions in local coordinates are:

$$\{\bar{R}\} = \{a_{1u,\phi}\}_{\phi=0\&L}$$

$$\{\bar{X}\} = \{a_{12v,\phi} + a_{15w,\phi}\}_{\phi=0\&L}$$

$$\{\bar{Y}\} = \{a_{21v,\phi} + a_{24w,\phi} + a_{27m,\phi}\}_{\phi=0\&L}$$

$$\{\bar{Z}\} = \{a_{33w,\phi}\}_{\phi=0\&L}, \qquad (16)$$

where $\phi = 0$ and $\phi = L$ are located at the problem solution domain. The force matrices resulting from boundary conditions must be specified on the boundary of the solution domain where kinematical conditions are specified. However, some of the functions specified in matrices, Equation 16, are not directly specified on the boundary, or they do not have kinematical meaning. In the event of this situation, the ordinary differentiations of the dependent functions with respect to ϕ are expanded by forward finite difference at node 1 and by backward finite difference at node N, where the range on nodes is from 1 to N. While the values of the dependent functions on the boundary are kept in the matrix of boundary conditions, the coefficients of neighboring values of dependent functions are transformed into the stiffness matrix and added to the appropriate term of the existing member of stiffness matrix. For example, assuming that the values of u, v, w and m are known at node 1 on the boundary, the matrix of boundary conditions is expanded by forward finite difference between nodes 1 and 2 and the coefficients of u_2, v_2, w_2 and m_2 are transformed into the stiffness matrix. The resulting modified stiffness matrix for element 1 is:

$$\begin{aligned}
K'_{11} &= K_{11} + a_1/l & K'_{36} &= K_{36} - a_{21}/l \\
K'_{15} &= K_{15} - a_1/l & K'_{33} &= K_{33} - a_{24}/l \\
K'_{22} &= K_{22} - a_{21}/l & K'_{37} &= K_{37} - a_{24}/l \\
K'_{26} &= K_{26} - a_{26}/l & K'_{34} &= K_{34} - a_{27}/l \\
K'_{23} &= K_{23} - a_{15}/l & K'_{38} &= K_{38} - a_{27}/l \\
K'_{27} &= K_{27} - a_{15}/l & K'_{43} &= K_{43} - a_{33}/l \\
K'_{32} &= K_{32} - a_{21}/l & K'_{47} &= K_{47} - a_{33}/l
\end{aligned} \qquad (17)$$

where $l = \phi_2 - \phi_1$, and the coefficient a's are evaluated at node 1.

It is to be noted that the general finite element equilibrium Equation 13 is written in the global coordinate system, where the proper rotation matrix is applied for transformation from the local to the global coordinate system. Calling the rotation matrix $[\Phi]$, for the base element e of nodes i and j reduces this to

$$[\Phi] = \begin{vmatrix} [\Phi_i] & [0] \\ [0] & [\Phi_i] \end{vmatrix}, \tag{18}$$

where

$$[\Phi_i] = \begin{vmatrix} \sin\phi_i & 0 & \cos_i\phi_i & 0 \\ 0 & 1 & 0 & 0 \\ -\cos\phi_i & 0 & \sin\phi_i & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}. \tag{19}$$

The rotation matrix practically applies to U and W.

RESULTS

Consider a spherical arc under uniform external pressure. The half angle of the cone is 39°, inside radius 563 in, shell thickness 23.6 in, the modules of elasticity is $30*10^6$ psi, and $\mu = 0.2$. The pressure is 284 psi. Due to the uniform load, the resulting stresses are symmetric with respect to the axis of cap and shear stresses must vanish. The total membrane and bending stresses on the outer surface in meridian and circumferential directions are shown in Figures 2 and 3. The solid line curves are the exact analytical solution given by Timoshenko [15], and the finite element solution is shown by (x). Close agreement is observed between the two solutions.

Now consider a hemispherical shell of the same material with an inside radius of 50 in and a thickness of 1 in exposed to wind loading. The boundary of the shell at great circle, $\phi = 90°$, is assumed to be clamped. The circumferential and meridian stresses at $\phi = 0$ are plotted versus ϕ in Figure 4. These results may be compared with the membrane solution of the spherical shell under wind loading given by Timoshenko. It can be verified that for $\phi = 0$, the shear force $N_{\theta\phi} = 0$ and N_ϕ and N_θ are zero at $\phi = 0$. At $\phi = 90$, $N_\phi = 0$ but $N_\theta \neq 0$.

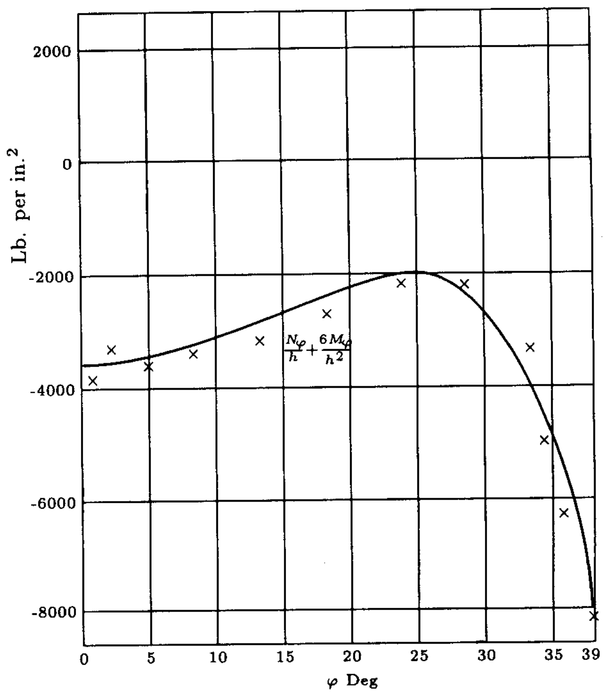

Figure 2. Spherical arc under uniform pressure; comparison of Timoshenko and finite element solution.

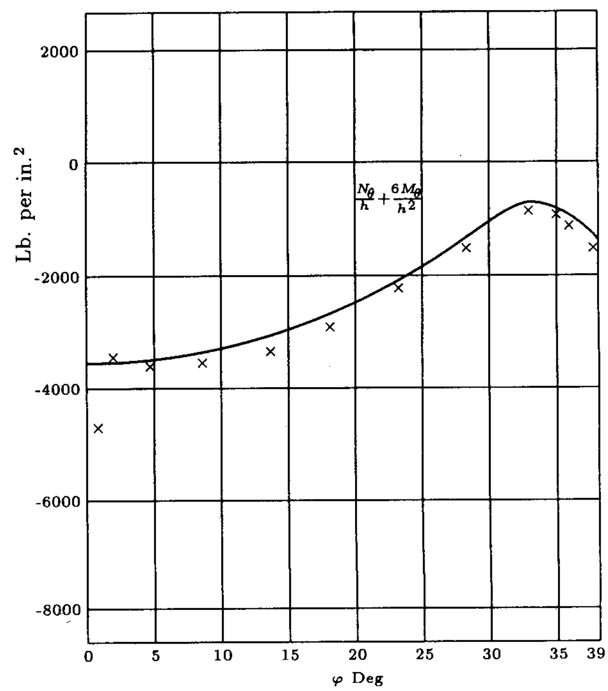

Figure 3. Spherical arc under uniform pressure; comparison of Timoshenko and finite element solution.

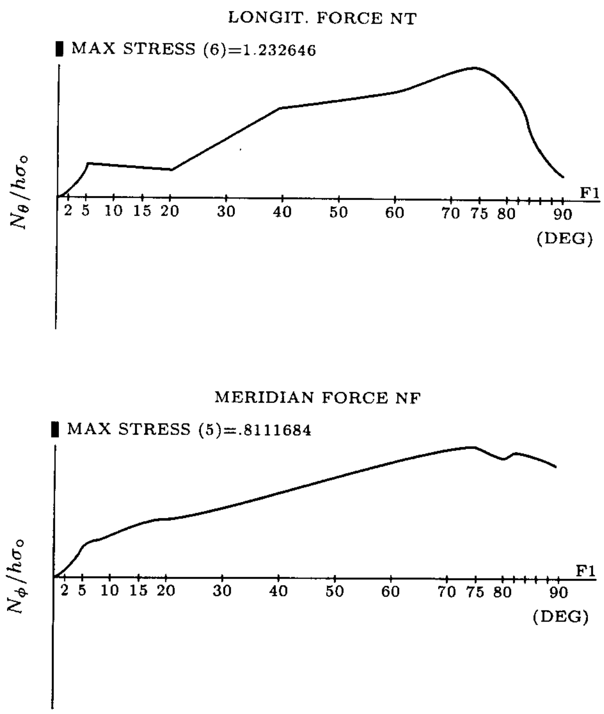

Figure 4. Circumferential and meridian stresses for a hemispherical shell exposed to wind loading.

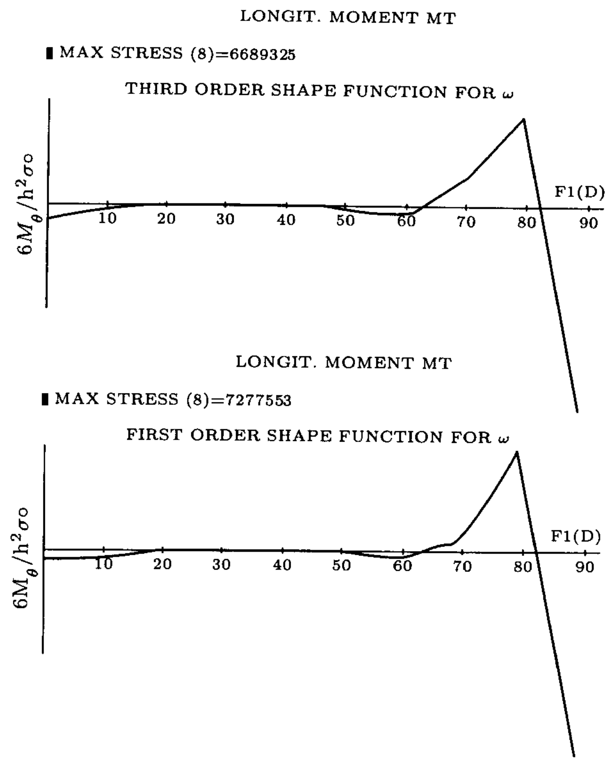

Figure 5. Distribution of circumferential bending stresses versus ϕ at $\theta = 0$.

Comparing these results with Figure 4, the condition at $\phi = 0$ is identical to the membrane solution, however at $\phi = 90$, $N_\phi \mp 0$. This difference with the membrane solution is justified if one notices that the Timoshenko solution at this boundary is equivalent to a free edge, while the boundary condition at $\phi = 90$ is assumed to be clamped, and obviously $N_\phi = 0$, due to the mechanical balance of the applied force and reactions. Figures 5 and 6 show the distribution of the meridian and circumferential bending stresses versus ϕ at $\theta = 0$. The results are shown for two series of shape functions, Equations 9 and 10. Close agreement is observed between the two sets of shape functions. The distribution of moments show that they are around zero from $\phi = 0$ to about 70°, justifying the membrane solution in this region. Between $\phi = 70$ to 90, the bending stresses change due to clamped conditions at $\phi = 90°$. This result can be checked by noticing that $\epsilon_\theta = 0$ at $\phi = 90$. From the stress-strain relations at this boundary $N_\theta = \mu N_\phi$.

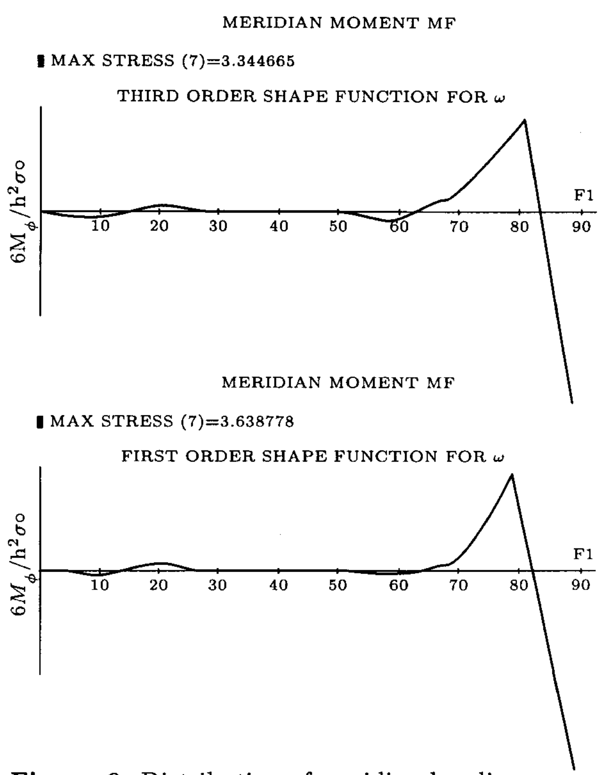

Figure 6. Distribution of meridian bending stresses versus ϕ at $\theta = 0$.

Also, due to clamped conditions at $\phi = 90$, $k_\theta = 0$ or $M_\theta = \mu M_\phi$. Noticing that $\mu = 0.2$, Figures 5 and 6 prove this theoretical result at $\phi = 90°$.

NOMENCLATURE

E	modules of elasticity
h	shell thickness
f_i	distributed lateral forces
F_i	concentrated lateral forces
k_i	curvatures
N_i	forces per unit length of shell
M_i	moments per unit length of shell
R	shell radius
u, v, w	displacement components
σ_{ij}	stress tensor
ϵ_{ij}	strain tensor
μ	Poisson's ratio
$[\Phi]$	rotation matrix
ϕ	meridian angle
Θ	circumferential angle

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APPENDIX

The coefficients of the system in Equation 8 are in terms of the shell variable ϕ, and the matrial constants as follows:

$$
\begin{aligned}
a_1 &= k \\
a_2 &= k \cot\phi \\
a_3 &= -k\mu - (k + \lambda^2/2)\cot^2\phi \\
&\quad - (1 + \lambda^2/12)n^2 \csc^2\phi/12(1+\mu) \\
a_4 &= [1/(1-\mu^2) + \lambda^2/12]n \csc\phi/2(1+\mu) \\
a_5 &= -[k(3-\mu)/2 \\
&\quad + (1.5+\mu)\lambda^2/12(1+\mu)]n \csc\phi \cot\phi \\
a_6 &= k(1+\mu) + \lambda^2 \cot^2\phi/12 \\
&\quad + \lambda^2 n^2 \csc^2\phi/12(1+\mu) \\
a_7 &= -(2+\mu)\lambda^2 n^2 \cot\phi \csc^2\phi/12(1+\mu) \\
a_8 &= \lambda^2 \\
a_9 &= (1-\mu)\lambda^2 \cot\phi \\
a_{10} &= -a_4 \\
a_{11} &= a_5 \\
a_{12} &= (1+\lambda^2/12)/2(1+\mu) \\
a_{13} &= a_{12} \cot\phi \\
a_{14} &= (1+\lambda^2/12)\cot^2\phi/2(1+\mu) \\
&\quad + (1+\lambda^2/12)/2(1+\mu) \\
&\quad - (k + \lambda^2/12)n^2 \csc^2\phi
\end{aligned}
$$

$$
\begin{aligned}
a_{15} &= \lambda^2 n \csc\phi/12(1+\mu) \\
a_{16} &= \lambda^2 n \csc\phi \cot\phi/12 \\
a_{17} &= [-1/(1+\mu^2) + \lambda^2/12]n \csc\phi/(1+\mu) \\
&\quad - \lambda^2 n^3 \csc^3\phi/12 \\
a_{18} &= -\mu\lambda^2 n \csc\phi \\
a_{19} &= -a_6 \\
a_{20} &= [1/(1-\mu) - \lambda^2/12]\cot\phi \\
&\quad + \lambda^2(1-n^2)\cot\phi \csc^2\phi/12 \\
a_{21} &= a_{15} \\
a_{22} &= -\lambda^2 n \csc\phi \cot\phi/12 \\
a_{23} &= -[1/(1-\mu^2) - \lambda^2/12]n \csc\phi/(1+\mu) \\
&\quad + \lambda^2 n(1-n^2)\csc^3\phi/12 \\
a_{24} &= 2\lambda^2 n^2 \csc^2\phi/12(1+\mu) + \lambda^2 \cot^2\phi/12 \\
a_{25} &= -\lambda^2(2\cot\phi + \cot^3\phi)/12 \\
&\quad - \lambda^2 n^2 \cot\phi \csc^2\phi/6(1+\mu) \\
a_{26} &= -2k(1+\mu) + \lambda^2 n^2[(1+\mu)(1-n^2) + 2] \\
&\quad \csc^2\phi/12(1+\mu) + \lambda^2 n^2 \csc^2\phi \cot^2\phi/12 \\
a_{27} &= \lambda^2 \\
a_{28} &= (2-\mu)\lambda^2 \cot\phi \\
a_{29} &= -(1-\mu)\lambda^2 - \mu\lambda^2 n^2 \csc^2\phi \\
a_{30} &= D \\
a_{31} &= D_\mu \cot\phi \\
a_{32} &= D_{\mu n} \csc\phi \\
a_{33} &= -D \\
a_{34} &= D_\mu \cot\phi \\
a_{35} &= D\mu n^2 \csc^2\phi \\
a_{36} &= -1,
\end{aligned}
$$

where λ is the Lame constant and

$$
\begin{aligned}
k &= 1/(1-\mu^2) \\
D &= 1/12(1-\mu^2).
\end{aligned}
$$