A Multiple Scale Method Solution for the Nonlinear Vibration of Rectangular Plates

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In this paper, first, the equations of motion for a rectangular isotropic plate have been derived. This derivation is based on the Von Karmann theory and the effects of shear deformation have been considered. Introducing an Airy stress function, the equations of motion have been transformed to a nonlinear coupled equation. Using the Galerkin method, this equation has been separated into position and time functions. By means of the dimensional analysis, it is shown that the orders of magnitude for nonlinear terms are very small, with respect to linear terms. Then, for the first time, the invariant manifold theory has been applied to the plate problem and it is proved that the nonlinearities are stiffness and inertia types. Finally, the multiple scale method is applied to the equations of motion and closed-form relations for the nonlinear natural frequencies and mode shapes of the plate are derived. The obtained results are in good agreement in comparison with numerical methods. Using the obtained relation, the effects of initial displacement, thickness and dimensions of the plate on nonlinear natural frequencies and displacements have been investigated. These results are valid for a special range of the ratio of thickness to dimensions of the plate, which is a characteristic of the multiple scale method.

INTRODUCTION

Large amplitude vibrations of rectangular plates have been investigated rigorously by many authors. A good review has been done by Sathyamoorthy [1], and Benamar et al. [2] have developed one model for large displacements and nonlinear vibrations of plates with different boundary conditions. This model is based on the Hamilton principle and spectral analysis. In this method, one assumes that for a system with weakly nonlinear terms, the response is:

$$w(x, y, t) = \phi(x, y)q(t).$$

Assuming $q(t)$ as a harmonic function, the $\phi(x, y)$ will be found by the harmonic balance method.

Also, Bikri et al. [3] investigated the effects of geometrical nonlinearity in the free vibration case of thin isotropic laminated rectangular composite plates. They determined the fundamental nonlinear natural

frequency and related mode shapes of the plate using a numerical method.

Ribeiro and Petyt [4] used the hierarchical finite element method for the free vibration analysis and discovered the internal resonances of the system by this method. In this work, they considered the geometrical nonlinearity and demonstrated the plate mode shapes with the amplitude of the vibration.

Amabili [5] investigated, theoretically and experimentally, large amplitude (geometrically nonlinear) vibrations of rectangular plates subject to radial harmonic excitation in the spectral neighborhood of the lowest resonances with different boundary conditions.

Huang and Zheng [6] investigated the nonlinear vibration and dynamical response of simply supported shear deformable plates on elastic foundations. The plate was subjected to a transverse dynamic load, combined with initial in-plan static loads and resting on an elastic foundation.

In all the above works, the researchers have considered the effects of geometrical nonlinearity, which is a kind of nonlinearity in the stiffness. But, in a nonlinear system, there may be some other kinds of nonlinearity, such as nonlinear inertia and nonlinear damping.

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Shaw and Pierre [7] defined a new method, which is named the invariant manifold, for nonlinear continuous systems and detected the kind of nonlinearities in nonlinear equations. This method can be used for systems with weak nonlinearities. Based on this method, Nayfeh et al. [8] obtained the nonlinear frequencies and mode shapes for one-dimensional continuous systems, which have nonlinearities in stiffness and inertia. Mahmoodi et al. [9] analyzed the nonlinear free vibration of a continuous damped system, which contains nonlinearity terms in inertia, stiffness and damping.

In this paper, for the first time, the method of the invariant manifold is used for plate vibration analysis, which has nonlinearities in inertia and stiffness. The advantage of this method is that of obtaining closed-form relations for the nonlinear natural frequencies and nonlinear mode shapes. Also, using this method, the effects of system parameters on the natural frequencies and nonlinear mode shapes can be determined accurately. It is shown that by increasing the ratio of thickness to the dimensions of the plate, the nonlinear frequencies of the plate will increase.

The obtained results show good agreement, with respect to what has been obtained by other researchers, and, also, with respect to available numerical results.

DEFINITION OF THE PROBLEM

An isotropic elastic rectangular plate, with dimensions a and b and thickness h, is considered. Figure 1 shows the dimensions and displacement of the plate. The plate is under large deflection and the displacements are unknown.

The displacement relations, based on the first order shear deformation and the Von Karmann theory, are:
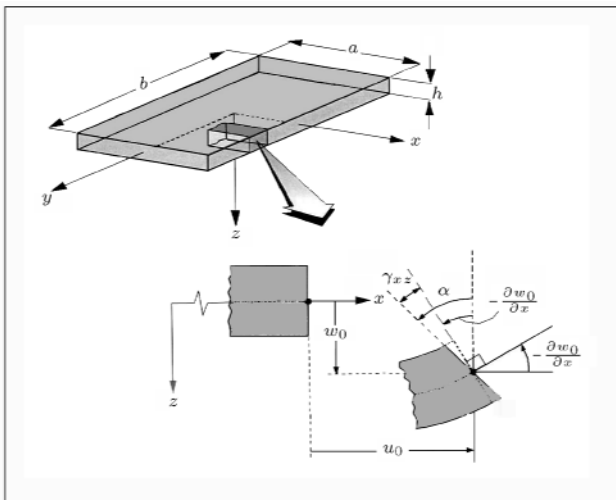
$$u(x, y, z, t) = u_0(x, y, t) + z\alpha(x, y, t),$$

$$v(x, y, z, t) = v_0(x, y, t) + z\beta(x, y, t),$$

$$w(x, y, z, t) = w_0(x, y, t), \qquad (1)$$

in which u, v and w are the displacements in the directions of x, y and z, respectively, and u_0, v_0 and w_0 are the displacements of the mid-plane. Also, α and β are the angles between the normal to mid-plane before and after deformation. The components of stresses are:

$$\sigma_x = \frac{N_x}{h} + \frac{12M_x}{h^3} = \frac{N_x}{h} + \frac{M_x}{I},$$

$$\sigma_y = \frac{N_y}{h} + \frac{12M_y}{h^3} = \frac{N_x}{h} + \frac{M_y}{I},$$

$$\sigma_{xy} = \frac{N_{xy}}{h} + \frac{12M_{xy}}{h^3} = \frac{N_{xy}}{h} + \frac{M_{xy}}{I},$$

$$\tau_{xz} = \frac{3}{2}\left(\frac{Q_x}{h}\right)\left(1 \quad 4\frac{z^2}{h^2}\right),$$

$$\tau_{yz} = \frac{3}{2}\left(\frac{Q_y}{h}\right)\left(1 \quad 4\frac{z^2}{h^2}\right), \qquad (2)$$

where N_x, N_y and N_{xy} are the membrane in-plane forces, M_x, M_y and M_{xy} are the bending moments and Q_x and Q_y are the internal shear forces. Also, I is the moment of inertia and E is the Young modulus. The internal forces and bending moments are shown in Figure 2.

Considering the first order shear deformation theory and using variational calculus, the force-displacement relations for an elastic isotropic plate will
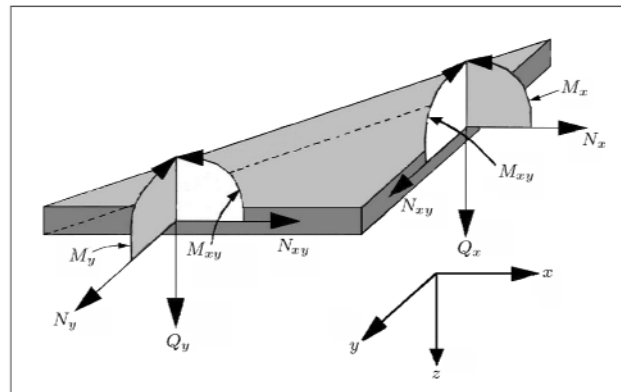

Figure 1. Rectangular isotropic plate and displacement after deflection.

Figure 2. Internal forces and bending moment in the plate.

become as follows:

$$\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \quad \frac{N_x}{Eh} + \nu\frac{N_y}{Eh} = 0,$$

$$\frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 \quad \frac{N_y}{Eh} + \nu\frac{N_x}{Eh} = 0,$$

$$\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\cdot\frac{\partial w_0}{\partial y} \quad 2\frac{(1+\nu)}{Eh}N_{xy} = 0,$$

$$\frac{\partial \alpha}{\partial x} \quad \frac{12M_x}{Eh^3} + \frac{12\nu M_y}{Eh^3} = 0,$$

$$\frac{\partial \beta}{\partial y} \quad \frac{12M_y}{Eh^3} + \frac{12\nu M_x}{Eh^3} = 0,$$

$$\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x} \quad \frac{24(1+\nu)}{Eh^3}M_{xy} = 0,$$

$$\alpha + \frac{\partial w}{\partial x} \quad \frac{12(1+\nu)}{5Eh}Q_x = 0,$$

$$\beta + \frac{\partial w}{\partial y} \quad \frac{12(1+\nu)}{5Eh}Q_y = 0. \tag{3}$$

Computing kinetic and potential energy and using the Hamilton principle for conservative systems, the equations of motion are found, as follows:

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \quad Q_x = \frac{\rho h^3}{12}\alpha_{,tt}, \tag{4}$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \quad Q_y = \frac{\rho h^3}{12}\beta_{,tt}, \tag{5}$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \rho h u_{,tt}, \tag{6}$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = \rho h v_{,tt}, \tag{7}$$

$$\frac{\partial N_x}{\partial x}w_{,x} + N_x\frac{\partial^2 w}{\partial x^2} + \frac{\partial N_y}{\partial y}w_{,y} + N_y\frac{\partial^2 w}{\partial y^2}$$

$$+ \frac{\partial N_{xy}}{\partial x}w_{,y} + \frac{\partial N_{xy}}{\partial y}w_{,x} + 2N_{xy}\frac{\partial^2 w}{\partial x\partial y}$$

$$+ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = \rho h w_{,tt}. \tag{8}$$

Also, the boundary conditions are:

at $x = 0$ and $x = a$:

$u = 0$ or $N_x = 0,$

$v = 0$ or $N_{xy} = 0,$

$w = 0$ or $Q_x + N_x\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y} = 0,$

$\alpha = 0$ or $M_x = 0,$

$\beta = 0$ or $M_{xy} = 0,$ (9)

and at $y = 0$ and $y = b$:

$u = 0$ or $N_{xy} = 0,$

$v = 0$ or $N_y = 0,$

$w = 0$ or $Q_y + N_{xy}\frac{\partial w}{\partial x} + N_y\frac{\partial w}{\partial y} = 0,$

$\alpha = 0$ or $M_{xy} = 0,$

$\beta = 0$ or $M_y = 0.$ (10)

Equations 4 through 8 are the equations of motion for a rectangular isotropic elastic plate under large amplitude vibration, considering the shear deformation and rotary inertia phenomena.

Assuming, principly, transverse motion, an Airy stress function, ϕ, is introduced as:

$$N_x = \frac{\partial^2\phi}{\partial y^2}, \qquad N_y = \frac{\partial^2\phi}{\partial x^2}, \qquad N_{xy} = \frac{\partial^2\phi}{\partial x\partial y}. \tag{11}$$

Substituting the above function in the equations of motion, Equations 6 and 7 are satisfied automatically and the other equations become, as follows:

$$D\frac{\partial^2\alpha}{\partial x^2} + \frac{D(1 \quad \nu)}{2}\frac{\partial^2\alpha}{\partial y^2} + \frac{D(1+\nu)}{2}\frac{\partial^2\beta}{\partial x\partial y}$$

$$\frac{5Eh}{12(1+\nu)}\left(\alpha + \frac{\partial w}{\partial x}\right) \quad \frac{\rho h^3}{12}\alpha_{,tt} = 0, \tag{12}$$

$$D\frac{\partial^2\beta}{\partial y^2} + \frac{D(1 \quad \nu)}{2}\frac{\partial^2\beta}{\partial x^2} + \frac{D(1+\nu)}{2}\frac{\partial^2\alpha}{\partial x\partial y}$$

$$5\frac{Eh}{12(1+\nu)}\left(\beta + \frac{\partial w}{\partial y}\right) \quad \frac{\rho h^3}{12}\beta_{,tt} = 0, \tag{13}$$

$$\left[\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} \quad 2\frac{\partial^2\phi}{\partial x\partial y}\right.$$

$$+ \frac{5Eh}{12(1+v)}\left(\frac{\partial\alpha}{\partial x} + \frac{\partial\beta}{\partial y} + \nabla^2 w\right)$$

$$\left.\rho h w_{,tt} = 0\right], \tag{14}$$

where $D = \frac{Eh^3}{12(1-v^2)}$.

The proper compatibility equation must be considered for the middle surface strains, which is stated, as follows:

$$\nabla^4 \phi = Eh \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right].$$ (15)

Eliminating α and β from Equations 12 through 14, will obtain:

$$\left[\frac{D^2(1-v)}{2} \nabla^4 - \frac{D\rho h^3(3-v)}{24} \nabla^2 \frac{\partial^2}{\partial t^2} - \frac{5DEh(3-v)}{24(1+v)} \nabla^2 \right.$$

$$+ \frac{25E^2h^2}{144(1+v)^2} + \frac{5\rho Eh^4}{72(1+v)} \frac{\partial^2}{\partial t^2} \bigg]$$

$$\times \left[\frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 \phi}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \right.$$

$$+ \frac{5Eh}{12(1+v)} \nabla^2 w - \rho h \ddot{w} \bigg]$$

$$+ \frac{5Eh}{12(1+v)} \left[\frac{5DEh}{24} \frac{(1-v)}{(1+v)} \nabla^4 \right.$$

$$- \frac{25E^2h^2}{144(1+v)^2} \nabla^2 - \frac{5\rho Eh^4}{144(1+v)} \frac{\partial^2}{\partial t^2} \bigg] w = 0.$$ (16)

The above equation is the equation of motion in a transverse direction, for a rectangular isotropic elastic plate.

Such an equation is derived for the Classical Plate Theory (CPT), without considering the effects of shear deformation and rotary inertia by Nayfeh [10], as follows:

$$\left[D\nabla^4 + \rho h \frac{\partial^2}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} \frac{\partial^2}{\partial y^2} \right.$$

$$\left. 2 \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial^2}{\partial x \partial y} \right] w = 0,$$ (17)

where ϕ is the Airy stress function. Comparing Equations 17 and 16, it can be seen that the existence of shear deformation and rotary inertia will complicate the equation of motion and will generate more nonlinear terms, which are explained later.

MULTIPLE SCALE METHOD

In order to calculate the natural frequencies and transverse mode shapes of the system, Equation 16 must be solved. For this purpose, there are many different methods, i.e., numerical methods, finite elements methods and analytical analysis, such as perturbation methods.

The method that will be used in this paper is the multiple scale method [9]. The most important advantage of this method is that, by identification of a non-dimensional small parameter, which has a physical interpretation and by using several time scales, one can obtain a complete physical understanding about the behavior of the system and the influence of different parameters and terms on the final response of the system.

To solve the above nonlinear equation, first, by using the Galerkin method, one discretizes the equation by writing the solution as:

$$w(x, y, t) = \sum_{i=1}^{N} \sum_{i=1}^{M} \psi_{ij}(x, y) f(t).$$ (18)

Then, by selecting $\psi_{ij}(x, y)$, which satisfies the boundary conditions, the multiple scale method is used to find the time function $f(t)$. For this purpose, without any lack of generality, the case of a square plate with all sides hinged will be considered in the first mode. Therefore;

$$\psi_{11}(x, y) = h \sin \left(\frac{\pi x}{a} \right) \sin \left(\frac{\pi y}{a} \right).$$ (19)

So:

$$w_{11}(x, y, t) = h f(t) \sin \left(\frac{\pi x}{a} \right) \sin \left(\frac{\pi y}{a} \right).$$ (20)

Also, the function, ϕ, will be selected, as follows:

$$\phi = f^2(t) \frac{Eh^3}{32} \left[\frac{2\pi^2}{a^2(1-v^2)} (x^2 + y^2) + \cos^2 \frac{\pi x}{a} \right.$$

$$\left. + \cos^2 \frac{\pi y}{a} \right],$$ (21)

to satisfy Equation 15. The boundary conditions of the problem are used, as follows:

at $x = 0$ and $x = a$:

$$w(x, y) = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0,$$ (22)

and at $y = 0$ and $y = a$:

$$w(x, y) = 0, \quad \frac{\partial^2 w}{\partial y^2} = 0.$$ (23)

By defining the non-dimensional parameters as:

$$\tau = t\sqrt{\frac{E}{\rho a^2}}, \qquad x^* = \frac{x}{a}, \qquad y^* = \frac{y}{a}, \qquad w^* = \frac{w}{h}, \tag{24}$$

and substituting Equations 20 and 21 into Equation 16, multiplying the left-hand side by $\sin(\frac{\pi x}{a})\sin(\frac{\pi y}{a})$ and integrating over the plate area, one obtains a non-dimensional equation, as follows:

$$a_1 f^3 + a_2 f + a_3 f'' + a_4 f^2 f'' + a_5 f(f')^2 = 0, \tag{25}$$

where:

$$a_1 = \frac{\pi^8 r^6}{2304} \frac{(3 \quad \nu)}{(1 \quad \nu^2)^2} + \frac{5\pi^6(3 \quad \nu)^2 r^4}{4608(1 \quad \nu^2)^2}$$

$$+ \frac{25\pi^4 r^2(3 \quad \nu)}{4608(1+\nu^2)(1 \quad \nu)},$$

$$a_2 = \frac{5\pi^6 r^4}{1728(1+\nu)^2(1 \quad \nu^2)} + \frac{25\pi^4 r^2}{1728(1+\nu)^2(1 \quad \nu^2)},$$

$$a_3 = \frac{\pi^4 r^4}{1728(1+\nu)^2(1 \quad \nu^2)} + \frac{5\pi^2 r^2(23 \quad 11\nu)}{3456(1+\nu)(1 \quad \nu^2)}$$

$$+ \frac{25}{576(1+\nu)^2},$$

$$a_4 = \frac{\pi^6 r^6(3 \quad \nu)^2}{1536(1 \quad \nu)(1 \quad \nu^2)} + \frac{5\pi^4 r^4(3 \quad \nu)}{768(1 \quad \nu^2)},$$

$$a_5 = \frac{\pi^6 r^6(3 \quad \nu)^2}{768(1 \quad \nu)(1 \quad \nu^2)} + \frac{5\pi^4 r^4(3 \quad \nu)}{384(1 \quad \nu^2)}, \tag{26}$$

and $r = \frac{h}{a}$.

Comparing Equation 25 with the equation which is derived by Nayfeh [8], it is found that the first term of this equation presents the nonlinearity in the stiffness and the two last terms are the nonlinear terms in inertia. So, this system has nonlinearity in stiffness and inertia.

Now, Equation 25 is rewritten as:

$$f'' + b_1 f + b_2 f^3 + b_3 f^2 f'' + b_4 f(f')^2 = 0, \tag{27}$$

where:

$$b_1 = \frac{a_2}{a_3} = O(1), \qquad b_2 = \frac{a_1}{a_3} = O(r^2),$$

$$b_3 = \frac{a_4}{a_3} = O(r^2), \qquad b_4 = \frac{a_5}{a_3} = O(r^2). \tag{28}$$

Equations 28 show that the order of magnitude of nonlinear terms, with respect to linear terms, is weak. So, the system has weak nonlinearities.

By definition, the small parameter, ε, is, as follows:

$$\sqrt{\varepsilon} = r = h/a. \tag{29}$$

Equation 27 will be written as:

$$f'' + \omega^2 f + \varepsilon G(f, \dot{f}, \ddot{f}) = 0, \tag{30}$$

and:

$$G(f, \dot{f}, \ddot{f}) = b_2^* f^3 + b_3^* f^2 f'' + b_4^* f(f')^2. \tag{31}$$

Therefore, due to the presence of ε, Equation 30 has weak nonlinear terms. Now, Equation 30 will be solved by the method of multiple scales. First, this equation is rewritten as:

$$f'' + \omega^2 f + \varepsilon(b_2^* f^3 + b_3^* f^2 f'' + b_4^* f(f')^2) = 0. \tag{32}$$

Two time scales are defined, as follows:

$$T_0 = t, \qquad T_1 = \varepsilon t. \tag{33}$$

Thus, the first and second derivatives of f are written as:

$$\frac{df}{dt} = \frac{df}{dT_0} + \varepsilon \frac{df}{dT_1},$$

$$\frac{d^2 f}{dt^2} = \frac{d^2 f}{dT_0^2} + 2\varepsilon \frac{d^2 f}{dT_0 dT_1} + \varepsilon^2 \frac{d^2 f}{dT_0 dT_1^2}. \tag{34}$$

Considering the above relations and expanding the function, f, as follows:

$$f = f_0 + \varepsilon f_1 + \cdots, \tag{35}$$

and, by substituting it in Equation 32 and equating the coefficients of ε^0 and ε^1 with zero, one obtains the following:

$$0(\varepsilon^0) = 0(1): \qquad \frac{d^2 f_0}{dT_0^2} + \omega^2 f_0 = 0, \tag{36}$$

$$0(\varepsilon^1): \quad \frac{d^2 f_1}{dT_0^2} + \omega^2 f_1 = \quad 2\frac{d^2 f_0}{dT_0 dT_1}$$

$$b_2^* f_0^3 \quad b_3^* f_0^2 \frac{d^2 f_0}{dT_0^2} \quad b_4^* f_0 \left(\frac{df_1}{dT_0}\right)^2, \tag{37}$$

a general solution for Equation 36 can be written as:

$$f_0(T_0, T_1) = A(T_1)e^{i\omega T_0} + \overline{A}(T_1)e^{\quad i\omega T_0}, \tag{38}$$

$A(T_1)$ is an undetermined function of T_1 and will be determined later. $\overline{A}(T_1)$ is the complex conjugate of

$A(T_1)$. Substituting f_0 from Equation 38 in Equation 37 will result in:

$$\frac{d^2 f_1}{dT_0^2} + \omega^2 f_1 = \quad 2iA'\omega e^{i\omega T_0} + 2i\overline{A'}e^{-i\omega T_0}$$

$$b_2^*(Ae^{i\omega T_0} + \overline{A}e^{-i\omega T_0})^3$$

$$b_3^*(Ae^{i\omega T_0} + \overline{A}e^{-i\omega T_0})^2(-A\omega^2 e^{i\omega T_0} - \overline{A}\omega^2 e^{-i\omega T.})$$

$$b_4^*(Ae^{i\omega T_0} + \overline{A}e^{-i\omega T.})(Ai\omega e^{i\omega T_0} - \overline{A}i\omega e^{-i\omega T_0})^2$$

$$+ cc, \tag{39}$$

where prime denotes a derivative, with respect to T_1 and cc means complex conjugates. Secular terms, which are the terms that produce nonperiodic solutions, must be eliminated from Equation 39. So, the solvability condition for Equation 39 is, as follows:

$$2A'i\omega + [3b_2^* + \omega^2(b_4^* - 3b_3^*)]A^2\overline{A} = 0. \tag{40}$$

Now, $\overline{A}(T_1)$ is stated in the polar form, as:

$$A(T_1) = \frac{1}{2}\gamma(T_1)e^{i\beta(T_1)}, \tag{41}$$

where γ is the amplitude and β is the phase. Substituting Equation 41 in Equation 40 and separating the real and imaginary components equal to zero, the following are obtained:

$$\gamma' = 0, \tag{42}$$

$$\beta' = \frac{3}{8}b_2\omega\gamma^2(b_4 - 3b_3). \tag{43}$$

Equation 42 means that the amplitude of the mode shape is constant and it is not a function of any time scales. This is because the system does not possess any kind of damping. Also, from Equation 43, it is obtained that:

$$\beta = \frac{3}{8}b_2^*\omega\gamma^2(b_4^* - 3b_3^*)T_1 + \beta_0, \tag{44}$$

where β_0 is constant. Therefore, substituting Equations 42 and 44 in Equation 38, it is obtained that:

$$f_0(T_0, T_1) = \frac{1}{2}\gamma e^{i[\omega T_0 + \frac{3}{8}b_2^*\omega\gamma^2(b_4^* - 3b_3^*)T_1 + \beta_0]} + cc. \tag{45}$$

So, by eliminating the secular terms, Equation 39 will become:

$$\frac{d^2 f_1}{dT_{0.}^2} + \omega^2 f_1 = [-b_2^* + \omega^2(b_3^* + b_4^*)]A^3 e^{3i\omega T_0} + cc. \tag{46}$$

The solution of this equation is:

$$f_1(T_0, T_1) =$$

$$\frac{\gamma^3}{64\omega^2}[b_2^* - \omega^2(b_3^* + b_4^*)]e^{i[\frac{9}{8}b_2\omega\gamma^2(b_4 - 3b_3)T_1 + 3\beta_0] + 3i\omega T_0}. \tag{47}$$

Substituting Equations 45 and 47 into Equation 35 and converting it to a triangular form, the time function will be:

$$f = \gamma\cos\theta + \varepsilon\frac{\gamma^3}{64\omega^2}[b_2^* - \omega^2(b_3^* + b_4^*)]\cos(3\theta) + \cdots, \tag{48}$$

where θ and ω_N are introduced as:

$$\theta = \omega_N t + \beta_0, \tag{49}$$

and:

$$\omega_N = \omega + \frac{3b_2^*\omega^2(b_4^* - 3b_3^*)}{8\omega}\varepsilon\gamma^2 + \cdots, \tag{50}$$

ω and ω_N are the linear and nonlinear natural frequencies of the first mode, respectively, and γ is the amplitude of vibration.

Equation 50 means that the frequency of nonlinear oscillation is dependent on the parameters of the system, the small parameter (relative thickness), ε, and the amplitude of oscillation, which are characteristics of nonlinear systems.

One of the advantages of the multiple scale method, with respect to other methods, is that, by using this method, the effects of system parameters on the system responses can be recognized accurately.

For example, the dependence of the first nonlinear frequency to amplitude (initial displacement of the mid-point of the plate) is illustrated in Figure 3.
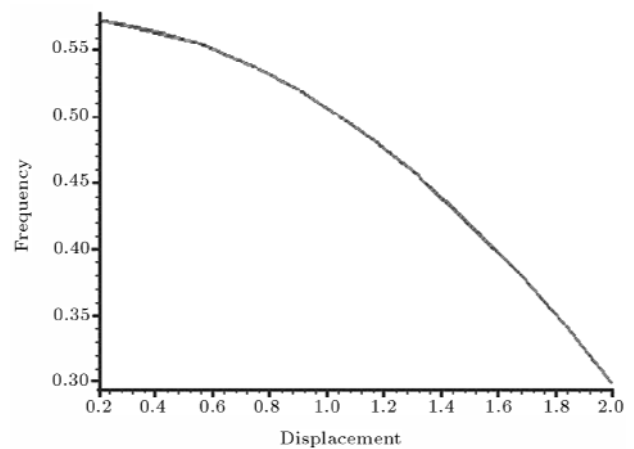
Figure 3. Variation of first nonlinear frequency with respect to initial displacement of mid-point.
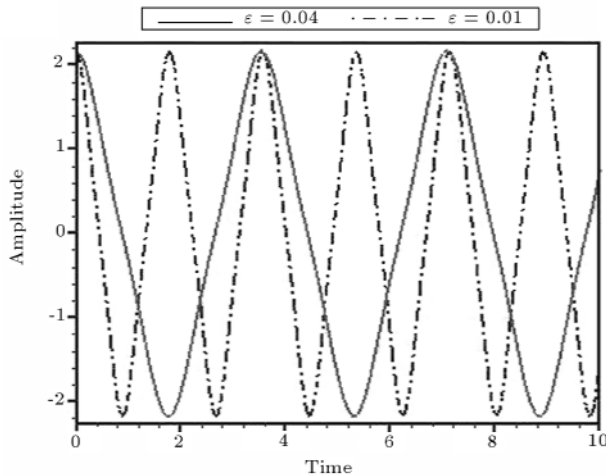
This figure shows that, for an elastic plate, by increasing the initial displacement, the nonlinear natural frequencies are decreased quadratically.

Figure 4 shows the effect of small parameters on the amplitude and first frequency of the plate.
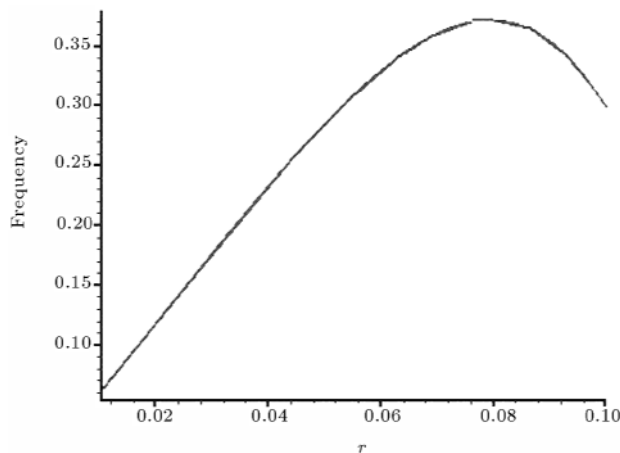
It is shown that, because there is no damping in this system, the amplitude is constant and does not vary with respect to time. However, the period of harmonic oscillations is decreased by increasing this small parameter.

Figure 5 shows the variation of the first nonlinear frequency of the plate, with respect to the ratio of thickness to dimension, i.e., $r = \frac{h}{a}$.

Thus, the frequency is increased, with respect to the ratio of thickness to dimension and, after a critical value of this ratio, the nonlinear frequency is decreased. For some values above the critical value of this ratio, the frequency will become negative. This means that the method which is described here is valid when the ratio of thickness to dimension is small and, therefore, the multiple scale method can be used.

NUMERICAL SOLUTION

To compare the above solution, with respect to other solutions, Equation 25 has been solved by the Runge-Kutta numerical method. This solution has been obtained using MAPLE 9 software and the following values for numerical parameters as $\nu = 0.3$, $h = 2$, $a = 50$. The numerical method and multiple scale method results have been compared in Figure 6.
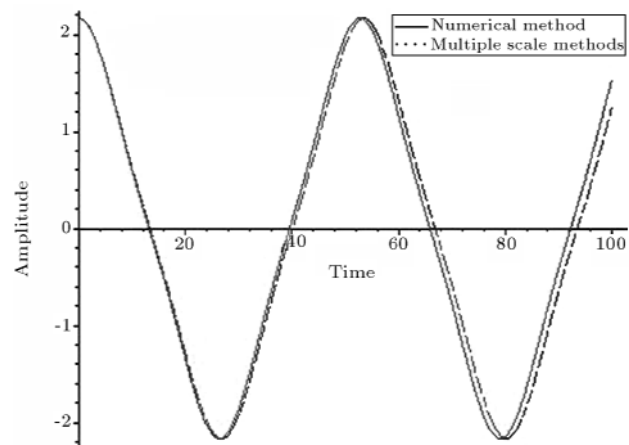
One can see from Figure 6 that the numerical results are in good agreement with the multiple scale method. However, the multiple scale method solution has the advantage that it provides a closed-form solution with a good physical insight, whereas the numerical methods do not provide a closed form solution and lack this type of physical insight.

CONCLUSION

In this paper, first, the equations of motion for an isotropic elastic plate were derived. In this derivation, the effects of shear deformation and rotary inertia were considered. Then, by using an Airy stress function, these equations were converted to one coupled equation and a compatibility equation. Using the Galerkin method, a nonlinear differential equation, with respect to time, was obtained. This equation has nonlinearities in stiffness and inertia.

Then, by using the method of multiple scales, this equation was solved and the nonlinear natural frequencies and response of the system were obtained. It is shown that the analytical results are in good agreement with the numerical results.

The advantage of the present solution is that the effects of nonlinearities can be determined accurately.

Figure 4. Effect of small parameter on the displacement of the mid-point ($x = a/2$, $y = a/2$).

Figure 5. Variation of first nonlinear frequency with respect to the square ratio of thickness to dimension.

Figure 6. Comparison of the numerical and multiple scale method, $\nu = 0.3$, $h = 2$, $a = 50$.

Dimensional analysis shows that the order of magnitude of nonlinear terms, with respect to linear terms, is weak. Also, the obtained results and the figures show that, by increasing the initial displacement of the mid-point, the first nonlinear frequency is decreased quadratically. Also, it is shown that, by increasing the ratio of thickness to the dimension of the plate, the nonlinear frequency of the plate will increase, but, this result is valid for a special range of this ratio, which is a characteristic of the multiple scale method.

NOMENCLATURE

a, b	dimensions of plate
E	Young's modulus
f	time function
h	thickness of plate
I	moment of inertia
N_x, N_y, N_{xy}	in-plane internal forces
M_x, M_y, M_{xy}	internal bending moments
Q_x, Q_y	internal shear forces
T_0, T_1	time scales
u, v	in-plane displacement
w	lateral displacement

Greek

α, β	angle of rotation
ε	non-dimensional small parameter
ϕ	Airy stress function
ν	Poisson's ratio
ρ	density
ω	linear natural frequency
ω_N	nonlinear natural frequency
τ	non-dimensional time parameter
ψ	position function

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