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A Methodology for Analyzing the Transient Reliability of Systems with Identical Components and Identical Repairmen

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In this paper, the Markov models, eigenvectors and eigenvalue concepts are used to propose a methodology for analyzing the transient reliability of a system with identical components and identical repairmen. The components of the systems under consideration can have two distinct configurations, namely; they can be arranged in series or in parallel. A third case is also considered, in which the system is up (good) if k-out-of-n components are good. For all three cases, a procedure is proposed for calculating the transient probability of the system availability and the duration of the system to reach the steady state.

INTRODUCTION

Reliability has been a major concern for system designers. Redundancy of components is usually required to design highly reliable systems. A common form of redundancy is k-out-of-n: a G system, in which at least k out of n components must be good for the system to be good [1]. Consider a system having n identical components. In parallel systems, the failure occurs when all of its n components fail. In series systems, the failure occurs if at least one of the components fails. Redundancy of components is usually incorporated in a system design for increasing system reliability.

Many systems consist of components having various failure modes. Several authors have considered a k-out-of-n system subject to two failure modes. Among those, Moustafa [2] has presented Markov models for analyzing the transient reliability of k-out-of-n: G systems subject to two failure modes. He proposed a procedure for obtaining closed form transient probabilities and the reliability for non-repairable systems. He, then, extended his work by providing a set of simultaneous linear differential equations for two different k-out-of-n repairable and non-repairable systems: G systems subject to M failure modes [3]. In his paper, numerical solutions for the reliability of the repairable systems were discussed and closed formulas for solutions of the reliability for the non-repairable systems were presented. Another research effort is the work of Pham, M. and Pham, H. [4], which has considered $[k, n \quad k+1]$-out of-n: F systems subject to two failure modes. Shao and Lamberson presented a model for k-out-of-n: a G system with load sharing [5].

Another attempt is the work conducted by Sarhan and Abouammoh [6], who applied the concept of a shock model to derive the reliability function for a k-out-of-n non-repairable system with non-independent and non-identical components. Later, El-Gohary and Sarhan [7] extended the work of Sarhan and Abouammoh, by proposing a Bayes estimator for a three non-independent and non-identical component series system under the condition of four sources of fetal shock. They support their estimation method by presenting a simulation study and show how one can utilize the theoretical results obtained in their paper.

In this paper, Markov models are presented for transient reliability and availability of series, parallel and k-out-of-n systems. The systems under consideration have n identical components and k identical repairmen. A methodology is proposed, based on Markov models, for obtaining the probability of a system to be available at time t, and for calculating the duration for the system to reach its steady state [8,9].

This paper is organized as follows: In the fol-

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lowing section, the nomenclature and definitions are presented. Then, the next section describes the methodology for analyzing the transient reliability and availability of the system, as well as the procedure for calculating the time elapse until the system reaches the steady state. After that, a numerical example and, finally, the concluding remarks are presented.

NOMENCLATURE, DEFINITIONS AND PRELIMINARIES

$N(t)$	number of components failed before time t,
$N'(t)$	number of repaired components before time t,
$X(t)$	number of failed components at time t; $X(t) = N(t) - N'(t)$,
$p_n(t)$	probability of having n failed components at time t; $p_n(t) = P(X(t) = n)$,
$A(t)$	probability of system to be up (good) at time t, regardless of its historical components failure and/or repair,
$A(\infty)$	long time system availability or system reliability,

$$P'(t) = \frac{dP(t)}{dt}, \qquad P'_n(t) = \frac{dP_n(t)}{dt}.$$

Definition 1

If Q be an $n \times n$ matrix, then, λ is an eigenvalue, such that $Q.X = \lambda.X$, where X is a non-zero vector and eigenvector.

Definition 2

Let $\{X(t) : t \geq 0\}$ be a continuous-time stochastic process with finite or countable state space R; usually R is $\{0, 1, 2, \cdots\}$, or a subset thereof.

It is said that $\{X(t)\}$ is a continuous-time Markov chain, if the transition probabilities have the following property: For every $t, s \geq 0$ and $j \in R$:

$$P(X(s+t) = j|X(u); u \leq s) = P(X(s+t)|X(s)),$$

and:

$$P_{ij}(t) = P(X(t+s) = j|X(s) = i) = P(X(t) = j|X(0) = i).$$

Considering $X(t)$ as the number of failed components at time t, one will have a Markov model, as shown in Figure 1.

Example 1

If one lets $n = 4$ and $k = 3$, the Markov model is represented by Figure 2.

Lemma 1

If one considers Q as the state transient rate matrix and $P(t)$ as the state transient probability in the exponential Markov chain with continuous time, then, one has [10]:

1. $P'(t) = P(t).Q$,
2. $P'_n(t) = P_n(0).Q$,
3. $P_n(t) = P_n(0).P(t)$.

In which Q and $P(t)$ are square matrices and $P_n(t)$ and $P_n(0)$ are row vectors.

Lemma 2

Let one consider a continuous time exponential Markov chain, in which $P'(t) = P(t).Q$, then, one has [11]:

$$P(t) = e^{Q.t}, \qquad P_n(t) = P_n(0).e^{Q.t}.$$

Lemma 3

Let one consider Q and d to be $n \times n$ matrices and V to be an invertible matrix, then, for every positive integer, k, one has [12]:

$$(V.d.V^{-1})^k = V.d^k.V^{-1}.$$

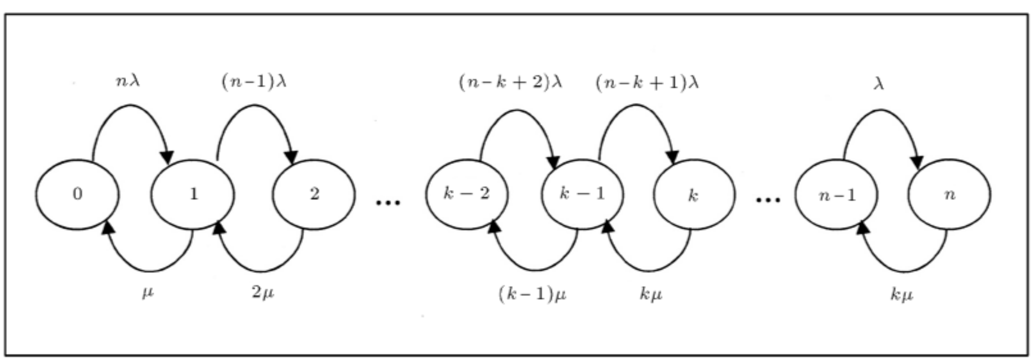

Figure 1. State transition diagram of the system with n components and k repairmen.

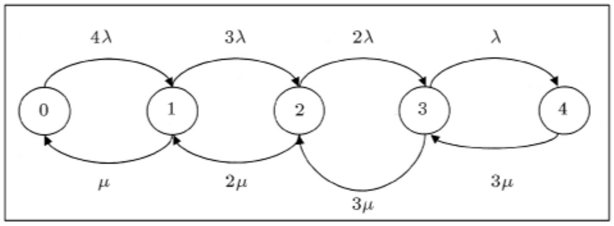

Figure 2. State transition diagram of the system with 4 components and 3 repairmen.

PROPOSED METHODOLOGY

To describe the proposed methodology for analyzing the system's transient reliability, consider a system having n identical components and k repairmen. The components can be arranged in parallel or series. The authors' methodology can analyze three distinct cases. In the case of parallel components, one considers that the system fails when all n components fail. In case of series components, one considers that the system fails when a component fails. The case is also considered in which k-out-of-n components fail. It is assumed that the time between two consecutive failures of a component is a random variable, having exponential distribution with parameter λ. It is also assumed that there are k identical repairmen providing services to the system. The service time of a component is also an exponentially distributed random variable with parameter μ. The goal is to provide a methodology for analyzing the transient availability of the system and the time until the system reaches its steady state.

The proposed methodology for obtaining system availability and transient probabilities are based on several theorems. These theorems are proved to provide the underlying theory of this methodology. First, these theorems are presented, as follows.

Theorem 1

Let one consider Q as an $n \times n$ square matrix, which has n non-repeating eigenvalues, then, one has:

$$e^{Q.t} = V.e^{d.t}.V^{\;1},$$

where t represents time, V is a matrix of eigenvectors of Q, V $^{\;1}$ is the inverse of V and d is a diagonal matrix of eigenvalues of Q, defined, as follows:

$$d = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Proof

$P(t) = e^{Q.t} = \sum_{k=0}^{\infty} \frac{(Q.t)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!}(Q)^k$. Since Q has non-repeating eigenvalues, then, one has:

$$Q = V.d.V^{\;1},$$

and, by Lemma 3, one has:

$$Q^k = V.d^k.V^{\;1}.$$

Therefore, one has:

$$P(t) = \sum \frac{t^k}{k!}(V.d^k.V^{\;1}) = \sum V.\frac{(d.t)^k}{k!}.V^{\;1}$$

$$= V.\left(\sum \frac{(d.t)^k}{k!}\right).V^{\;1} = V.e^{d.t}.V^{\;1}.$$

Theorem 2

Consider $P(t) = e^{Q.t}$ in which Q is the transition matrix. In matrix Q, one of the eigenvalues is zero and the remaining eigenvalues are complex numbers with negative real parts.

Proof

Since in every row of the transition matrix, the summation of row elements is zero, it can be deduced that one of the eigenvalues of matrix Q is zero. By Theorem 1, one has:

$$P(t) = V.e^{d.t}.V^{\;1} = (p_{ij}(t)),$$

$$p_{ij}(t) = \pi_j + \sum_{k=1}^{n} \alpha_{ijk}.e^{\lambda_k.t},$$

in which λ_k is the kth eigenvalue, α_{ijk}'s are constant values and π_j is the limiting probability. Using the contradiction, if one assumes that one of the eigenvalues of Q is a complex number with a positive real part, then, one has $\lim_{t\to\infty} e^{\lambda_k t} = \infty$. Therefore:

$$\lim_{t\to\infty} p_{ij}(t) = \infty,$$

which contradicts the fact that $\lim_{t\to+\infty} p_{ij}(t) = \pi_j$ and, therefore, the eigenvalues of Q are complex numbers with negative real parts.

Theorem 3

Consider $P(t) = e^{Q.t}$, in which Q is the transition matrix. The time until the system reaches the steady state $(P(t) =)$ can be calculated by the following formula:

$$t = \frac{\ln \varepsilon}{S_r},$$

in which ε is a very small positive number (i.e., $0\langle \varepsilon \leq 0.0001$), S_r is the largest real part of the eigenvalues, excluding the zero elements of matrix Q and $\prod$ is a square matrix, representing the limiting probabilities. The elements of matrix $P(t)$ and $\prod$ are shown, as follows:

$$P(t)=\begin{bmatrix} p_{00}(t) & p_{01}(t) & \cdots & p_{0n}(t) \\ p_{10}(t) & p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n0}(t) & p_{n1}(t) & \cdots & p_{nn}(t) \end{bmatrix}$$

$$=\begin{bmatrix} \pi_0 & \pi_1 & \cdots & \pi_n \\ \pi_0 & \pi_1 & \cdots & \pi_n \\ \vdots & \vdots & \ddots & \vdots \\ \pi_0 & \pi_1 & \cdots & \pi_n \end{bmatrix}.$$

Proof

$$p_{kj}(t)=\pi_j+\sum_{m=1}^{n}\alpha_{kjm}.e^{\lambda_m.t}$$

$$=\pi_j+\sum_{m=1}^{n}\alpha_{kjm}.e^{(S_m.+C_m.i).t}$$

By Theorem 2, all S_m are negative and $i = \sqrt{\ 1}$ (π_j, α_{kjm}, S_m and C_m are constant numbers). Now, suppose S_r is greater then S_m, then, for large values of t, one has:

$$p_{kj}(t)=\pi_j+\varepsilon',$$

where ε' is a very small positive number. Therefore, one has:

$$p_{kj}(t)\approx\pi_j,\qquad \varepsilon=e^{S_r.t},\qquad S_r.t=\ln\varepsilon,\qquad t=\frac{\ln\varepsilon}{S_r}.$$

Note that computational calculations are done according to 4 digits precision, therefore, $\varepsilon < 0.0001$ and, if $\varepsilon < 0.0001$, then, the system reaches the steady state.

Based on the proof of these theorems, an algorithm is now proposed for calculating the availability of the system.

Algorithm

1. Determine the transition matrix Q,
2. Determine the eigenvalues and eigenvectors of matrix Q,
3. Determine $P(t)=V.e^{d.t}.V^{\ 1}$,
4. Determine $P_n(t)=P_n(0).P(t)$,
5. Determine the availability of the system, according to the type of the system, as follows:
   - For a parallel system: $A(t)=1\quad p_n(t)$,
   - For a series system: $A(t)=p_0(t)$,
   - For k-out-of-n system: $A(t)=\sum_{i=0}^{n\ \ k}p_i(t)$.

Note that the above complexity of the algorithm is $o(n^2)$.

A NUMERICAL EXAMPLE

Let one consider a system having five identical components. There are two identical repairmen for repairing this system. It is assumed that the time to the failure of the repaired component is a random variable with an exponential distribution function with a mean of 0.2 hours. The repair time is also considered to be a random variable, distributed exponentially, with a mean of 0.1 of an hour. It is required to calculate the system availability at any given time and the time until the system reaches the steady state. To show the flexibility of the proposed methodology, the following system configurations are considered:

1. Components are arranged in series;
2. Components are arranged in parallel;
3. The system is working if at least 2-out-of-5 componants are good.

SOLUTION

The graphical Markov model can be represented in Figure 3.

The transition rate matrix is first determined, as follows:

$$Q=\begin{bmatrix} 25 & 25 & 0 & 0 & 0 & 0 \\ 10 & 30 & 20 & 0 & 0 & 0 \\ 0 & 20 & 35 & 15 & 0 & 0 \\ 0 & 0 & 20 & 30 & 10 & 0 \\ 0 & 0 & 0 & 20 & 25 & 5 \\ 0 & 0 & 0 & 0 & 20 & 20 \end{bmatrix},$$

$$P_n(0)=\ 1\quad 0\quad 0\quad 0\quad 0\quad 0),$$

$$P_n(t)=P_n(0).V.e^{d.t}.V^{\ 1}$$

$$=\ p_0(t)\quad p_1(t)\quad p_2(t)\quad p_3(t)\quad p_4(t)\quad p_5(t)),$$

$$p_0(t)=0.0502e^{\ 62.1t}+0.159e^{\ 43.7t}+0.292e^{\ 30.8t}$$

$$+0.236e^{\ 19.6t}+0.153e^{\ 8.85t}+0.11,$$

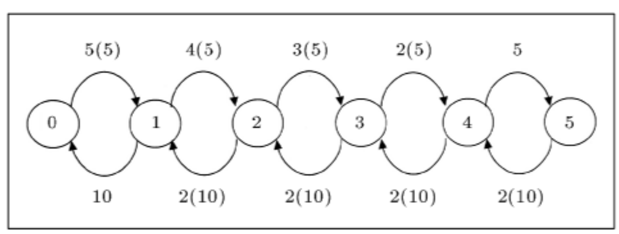

Figure 3. State transition diagram of the system with 2 repairmen.

$$p_1(t) = \quad 0.187e^{\quad 62.1t} \quad 0.296e^{\quad 43.7t} \quad 0.168e^{\quad 30.8t}$$
$$+ 0.128e^{\quad 19.6t} + 0.246e^{\quad 8.85t} + 0.277,$$

$$p_2(t) = 0.237e^{\quad 62.1t} + 0.00387e^{\quad 43.7t} \quad 0.358e^{\quad 30.8t}$$
$$0.229e^{\quad 19.6t} + 0.0698e^{\quad 8.85t} + 0.277,$$

$$p_3(t) = \quad 0.135e^{\quad 62.1t} + 0.294e^{\quad 43.7t} + 0.0928e^{\quad 30.8t}$$
$$0.304e^{\quad 19.6t} \quad 0.155e^{\quad 8.85t} + 0.208,$$

$$p_4(t) = 0.0388e^{\quad 62.1t} \quad 0.204e^{\quad 43.7t} + 0.265e^{\quad 30.8t}$$
$$+ 0.0130e^{\quad 19.6t} \quad 0.217e^{\quad 8.85t} + 0.104,$$

$$p_5(t) = \quad 0.00460e^{\quad 62.1t} + 0.0431e^{\quad 43.7t} \quad 0.123e^{\quad 30.8t}$$
$$+ 0.156e^{\quad 19.6t} \quad 0.0971e^{\quad 8.85t} + 0.026.$$

Now, one can calculate the system availability for different configurations of the system, as follows:

1. Components are arranged in parallel:

$$A(t) = 1 \quad p_5(t), \qquad A(\infty) = 0.974.$$

2. Components are arranged in series:

$$A(t) = p_0(t), \qquad A(\infty) = 0.11.$$

3. The system is working if at least 2-out-of-5 are good:

$$A(t) = p_3(t) + p_2(t) + p_1(t) + p_0(t)$$
$$= 1 \quad p_4(t) \quad p_5(t),$$
$$A(\infty) = 0.87.$$

One can calculate the time until the system reaches the steady state. Table 1 represents the probability of the system to be up (good) at time t, for different values of t and different system configurations.

By the following closed form formula, one can also calculate the elapsed time until the system reaches the steady state:

$$t = \frac{\ln \varepsilon}{S_r} = \frac{\ln 0.0001}{8.85} = 1.04.$$

As can be seen from both Table 1 and the above closed form formula, the system reaches the steady state after one time unit.

The limiting probability can also be calculated, as follows:

$$\pi_0 = 0.11,\ \pi_1 = 0.277,\ \pi_2 = 0.277,\ \pi_3 = 0.208,$$
$$\pi_4 = 0.104,\ \pi_5 = 0.026.$$

Table 1. Elapsed time until system reaches the steady state.

t	$A(t)$ for Series	$A(t)$ for Parallel	$A(t)$ for 2-out-of-5
0.05	0.38	0.9996	0.9941
0.10	0.222	0.9972	0.971
0.15	0.166	0.9926	0.9429
0.20	0.141	0.9877	0.9198
0.25	0.129	0.9835	0.9029
0.40	0.115	0.9767	0.879
0.50	0.112	0.9751	0.8741
0.70	0.11	0.9742	0.8702
1.00	0.11	0.974	0.87
1.20	0.11	0.974	0.87
1.50	0.11	0.974	0.87

CONCLUSION

In this paper, a methodology for analyzing systems transient reliability and availability with identical components and identical repairmen is proposed. The Markov model, eigenvector and eigenvalue concepts are employed to develop the methodology for the transient reliability of such systems. The proposed methodology is versatile in the sense that it can be applied to a variety of systems, i.e. series, parallel and k-out-of-n systems. The proposed methodology can also be employed for determining the time until the system reaches the steady state.

By developing this method, the following new research areas have been provided, which are under study:

1. Analyzing the transient reliability and availability of a system with non-identical components and repairmen;
2. Analyzing the transient reliability and availability of a system with standby components and repairable components;
3. Analyzing the transient reliability and availability of all reliability exponential models.

REFERENCES

1. Wasserman, G.S. "Reliability verification, testing, and analysis in engineering design", *Dekker Mechanical Engineering*, **153**, Marcel Dekker Inc., New York, USA (2002).
2. Moustafa, M.S. "Transient analysis of reliability with and without repair for K-out-of-N: G systems with two failure modes", *Reliability Engineering and System Safety*, **53**(1), pp 31-35 (1996).
3. Moustafa, M.S. "Transient analysis of reliability with and without repair for K-out-of-N: G systems with

M failure modes", *Reliability Engineering and System Safety*, **59**, pp 317-320 (1998).

4. Pham, H. and Pham, M. "Optimal designs of $\{k, n \quad k+ l\}$-out-of-n: F systems (subject to 2 failure modes)", *IEEE Trans, Reliability*, **40**(5), pp 559-562 (1991).
5. Shao, J. and Lamberson, L.R. "Modeling a shared-load k-out-of-n: G system", *IEEE Trans. Reliability*, **40**(2), pp 202-208 (1991).
6. Sarhan, A. and Abouammoh, A.M. "Reliability of k-out-of-n non-repairable systems with non-independent components subjected to common shocks", *Microelectronic Reliability*, **41**, pp 617-621 (2001).
7. El-Gohary, A.I. and Sarhan, A.M. "Estimation of the parameters in three non-independent component series systems subjected to sources of shock, *Applied Mathematics and Computation*, **160**, pp 29-40 (2005).
8. Gross, D. and Harris, C.M., *Fundamentals of Queuing Theory*, Wiley, New York, USA (1985).
9. Stewart, W.J., *Introduction to the Numerical Solution of Markov Chains*, Princeton University Press, Princeton., N.J., USA (1994).
10. Hoyland, A. and Rausend, M., *System Reliability Theory Models and Statistical Methods*, John Wiley, 3rd Ed., pp 220-230 (1994).
11. Papoulis, A., *Probability, Random Variables, and Stochastic Processes*, McGraw Hill, 3rd Ed., pp 640-644 (1991).
12. Singh, S. and Zameeruddin, Q., *Modern Algebra*, Printed at Ravindira Printing Press, Delhi, 2nd Ed., p 58 (1990).