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Interpretation of Tensile Softening in Concrete, Using Fractal Geometry

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Concrete is a heterogeneous material with a wide variety of usage in structural design. Concrete under tension exhibits strain softening, i.e., a negative slope in the stress–deformation diagrams. Different softening curves have been proposed in the literature to interpret this phenomenon. In current research, a new softening curve for concrete has been proposed by using the newly introduced concept of fractal geometry. This new softening curve is denominated a 'Quasi-fractal' softening curve and consists of two parts, a linear portion at the beginning and an exponential portion in the rest of the curve. A comparison of a "Quasi-fractal" softening curve with a set of proposed experimental softening curves has been performed, which reveals good agreement.

INTRODUCTION

Since its first presentation by Mandelbrot [1], fractal geometry has found many applications in science and technology. The main merit of this new mathematical tool is in its ability to model natural irregularities. Owing to this, fractal geometry has found many applications, such as in electromagnetic, biology, fluid mechanics and many branches of solid and fracture mechanics.

The first step in using this new concept in fracture mechanics is the verification of the fact that fracture surfaces have fractal patterns. Several experiments and theory outcomes confirm that fracture surfaces in many engineering materials are fractal. The first investigation into this field concerned the fractal character of fracture surfaces in metals, which was carried out by Mandelbrot et al. [2]. Subsequently, some efforts were made to characterize the fracture surfaces of concrete. Among those researchers, one should mention Winslow [3], Saouma et al. [4], Brandt and Prokopski [5], Saouma and Barton [6], Carpinteri et al. [7] and Issa et al. [8]. In addition to the aforementioned experimental evidence of the hypothesis of the fractality of the fracture surfaces of concrete, theoretical proof also exists, such as in Carpinteri et al. [9].

Fracture in heterogeneous materials, e.g. concrete, is one of the most active branches in fracture mechanics. Resulting from inhomogeneities inside the structure of such materials, one of the best ways to analyze the behavior of these kinds of material is by the use of fractal geometry.

It is known that classes of material, such as concrete, rock, brick and ceramics, exhibit what is termed 'strain-softening' behavior. Thus, in a direct tensile test, there is a linear stress-strain relationship until, approximately, the ultimate strength, σ_u, is reached and further straining beyond this point results in stress relaxation, which depends on the strain-softening characteristics of the material. The ultimate behavior of such materials is characterized by the localization of a non-linear zone within a narrow band of the material, while the rest of the material outside this softening zone retains its linear behavior. In the cohesive model, this band is treated as the softening zone, where the material, though cracked, can still transfer stress. The tensile stress, σ, in the softening zone is described as the decreasing function of the relative displacement, w, of the opposite surfaces of the cohesive crack. The process zone starts forming after the tensile stress reaches its ultimate value, σ_u. The point where the displacement, w, reaches its critical value, w_c, beyond which no stress can be transferred, is called the real crack tip, while the point along the cohesive crack, at which the stress reaches σ_u, is called the cohesive crack tip. There are no stress singularities present in this model.

The "cohesive crack model" was described by Barenblatt [10,11] and Dugdale [12]. Many researchers have used cohesive cracks to describe the near-tip non-

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linear zone for cracks in most materials, e.g., metals, concrete, polymers, ceramics and geo-materials. In the late seventies, cohesive cracks were extended by proposing that the cohesive crack may be assumed to develop anywhere, even if no pre-existing macrocrack is actually present [13]. This extended cohesive crack is called a "fictitious crack model". In this model, the strain-softening behavior is expressed by an appropriate, $\sigma - w$, relationship.

Extensive investigations have been carried out to determine the form of the softening curve for concrete mixes. According to the numerous performed experiments, several forms have been proposed for the softening curve; among them, the bilinear curve, mentioned by Petersson [14], the exponential, mentioned by Cornelissen et al. [15,16], Gopalaratnam and Shah [17], Planas and Elices [18,19] and Slowik et al. [20] and the power-law, mentioned by Reinhardt [21,22]. These softening curves are useful in the analyses of the behavior of concrete structures, with or without notches.

This paper has the following structure: First, the generalized form of a Sierpinski carpet is presented for use as a flexible fractal set in the modeling of fractal surfaces. Second, the energy consumption, during the softening procedure in concrete-like materials, is interpreted, by the use of fractal geometry and, then, a new softening curve is presented, accordingly. Finally, a comparison of the results with several common experimental softening curves is successfully performed, which reveals good agreement.

FRACTAL GEOMETRY ASPECTS OF THE PROBLEM

Fractal geometry is a mathematical tool, which describes objects of irregular shape, as long as the requirement of self-similarity is satisfied. A fractal set is a geometrical pattern that deviates from its Euclidean dimension; if deviation is a positive value, the fractal set is "invasive" and, if it is negative, the fractal set is "lacunar" [1].

The measure of a fractal set is a function of the length of the used yardstick. The length (or area for 2D fractal sets) of a fractal set, according to the used yardstick in measuring, is determined from the following relation:

$$L_\varepsilon = L_p \varepsilon^{(d-D_f)}, \tag{1}$$

where L_p is the projected length (area) of the object, ε is the fractional measuring unit, which, for practical purposes, will be associated either with the magnification and resolution used in the microscope or to the relative length of a typical constructional segment of that line. The parameter, d, represents the Euclidean dimension of the object (d is equal to 1 for a line or 2 for a surface) and D_f is denominated as the fractal dimension. A general definition of the fractal dimension of an artificial fractal object is, as follows:

$$D_f = \frac{Ln\ N}{Ln\ m}, \tag{2}$$

where N is the number of elements in the basic picture ($n = 1$) and m is the applied reduction factor to the segments.

As mentioned in [3-8], the fracture surfaces of concrete are fractal sets. It has been proven that the area of the matrix in granular composites, such as concrete, is also a fractal set [9]. In the next section, it is important to have a flexible fractal set for modeling surfaces with different fractal dimensions. The used fractal set is a generalization on the Sierpinski carpet fractal set. A generalized Sierpinski carpet is similar to a Sierpinski carpet, with this difference, that a generalized to a Sierpinski set makes it useful when modeling surfaces with any fractal dimension. The structure of a generalized Sierpinski carpet consists of a rectangle that consists of p^2 similar rectangles, from which, at each iteration, the total number of q, $\frac{q}{p^2} \leq 1$, rectangles of the remaining area will be omitted. The fractal dimension of this fractal set, according to Equation 2, is equal to:

$$D_f = \frac{1n(p^2 - q)}{1n(p)}. \tag{3}$$

The remaining area of this fractal set, after n iterations, is calculated from the following relation:

$$A_n = A_p \left(\frac{1}{p^n}\right)^{2-D_f}. \tag{4}$$

The eliminated area, in iteration of number n, is equal to:

$$\Delta A_n = A_p \frac{q}{p^2} \left(1 - \frac{q}{p^2}\right)^{n-1}. \tag{5}$$

This new fractal set is denoted by $S_{p,q}^{[a,b]}$, where a and b are the side lengths of the rectangle. An example of a generalized Sierpinski carpet, $S_{5,5}^{[1,1]}$, in its first three iterations, is drawn in Figure 1.

The fractal dimension of $S_{5,5}^{[1,1]}$, according to Equation 3, is equal to $D_f = Ln(5^2 - 5)/Ln(5) = 1.861$. For the sake of simplicity, one can assume $q = 1$, therefore, parameter p is the only essential parameter throughout this investigation.

INTERPRETATION OF THE SOFTENING PHENOMENON IN CONCRETE UNDER TENSILE STRESSES BY USING FRACTAL GEOMETRY

In this section, it will be shown that the use of fractal patterns in the modeling of tensile fracture surfaces will

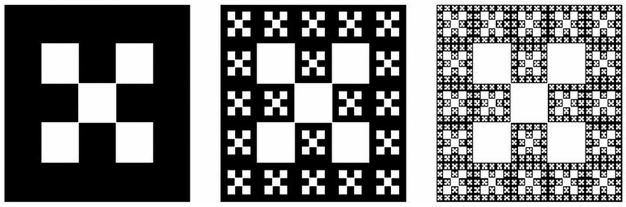

Figure 1. First three iterations of formation of generalized Sierpinski carpet set, $S_{5,5}^{[1,1]}$.

guide one to a new softening curve, which will be useful in cohesive crack models. Before starting the new approach, it is required to review some characteristics of the cohesive crack model.

The cohesive crack model, called a fictitious crack model by Hillerborg and co-workers [13], has been one of the most essential tools in the analysis of the fracture of concrete and cement-based materials since its first application to structural analysis in the mid seventies. The characteristics of the fictitious crack are contained in its stress-crack opening relationship (the softening curve shown in Figure 2), which, in the simplest approximation, is assumed to be unique, so, one can write:

$$\sigma = f(w). \tag{6}$$

Various forms have been proposed for the softening curve, some of which have been reviewed in the next section. All these curves have common essential features, as follows:

1. It is non-negative and non-increasing;
2. For zero crack openings, its value equals tensile strength;
3. It tends to zero for large crack openings (complete failure, zero strength);
4. It can be integrated over $(0, \infty)$.

This integral is the work of the fracture per unit surface of the complete crack, which is equal to the area under the softening curve. Figure 2 is denominated by

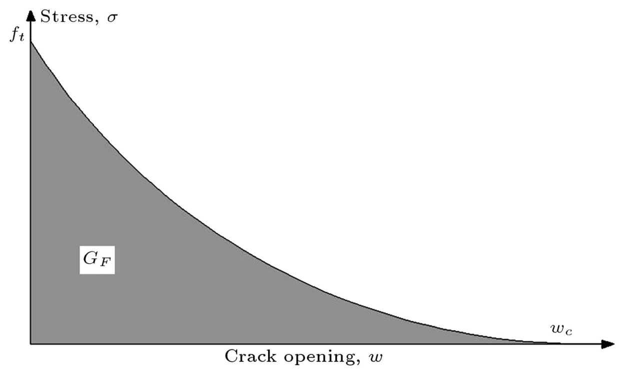

Figure 2. Schematic form of softening curve.

G_f, which is as follows:

$$G_f = \int_0^\infty \sigma dw. \tag{7}$$

It is intended to interpret the softening behavior of concrete-like materials by using the aforementioned principles and fractal geometry. As mentioned in the literature, it is possible to model the cross section of such materials by the use of fractal patterns [9]. In this fractal pattern-based model, the omitted areas are exhibitors of the aggregates and the remaining parts are exhibitors of the adhesive matrix. By using this model, the softening behavior of concrete-like materials under tension is interpretable. According to Equation 7, the total required breakage energy, per unit of the cross section area, is equal to G_f.

The bond zones around aggregates are the weakest link in the structure of concrete-like materials and tensile failure begins from shaping cracks around particles in the bond zones. The process of softening in quasi-brittle materials could be depicted in this way that, at the peak load, f_t, a set of macrocracks start to grow through the bar cross section. A continuation of applying elongation to the specimen causes smeared cracking behavior inside the cross section, which leads to a gradually decreasing cross section for stress transfer and, thus, to a gradually decreasing external stress (Figure 3). At the beginning, smeared cracks start at the weak phase, around larger aggregates and, by continuation of applying strain to the specimen, the cracking extends around smaller particles, consecutively. This process will continue until the increase of the total area of the macrocrack becomes less than a certain value, which is denominated by the "last resistant ligament".

On the other hand, it is known that the final tensile fracture surface in concrete is a fractal set. Now, it is possible to use a fractal set for modeling

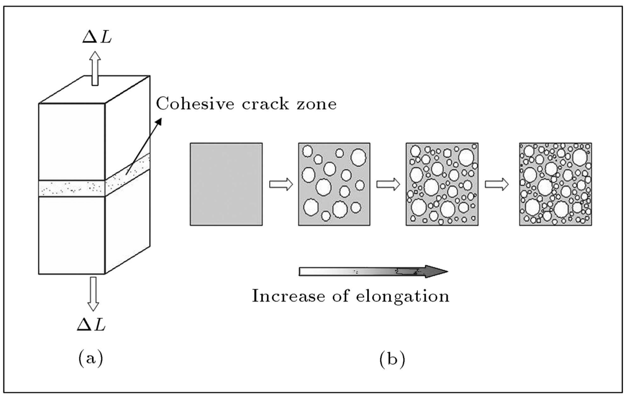

Figure 3. a) Formation of cohesive crack in a specimen under uniaxial tension test; b)Decrease of resistant cross section after peak load, due to increase of subjected elongation.

the total area of the macrocracks. In this fractal set, the eliminated surfaces, at each iteration, represent an increase in the total macrocrack area. The generalized Sierpinski carpet is a useful tool, so, from Equation 5, the increase in the area of macrocracks in the nth iteration is equal to:

$$\Delta a_n = A_p \frac{q}{p^2}\left(\frac{p^2-q}{p^2}\right)^{n-1}, \tag{8}$$

where A_p is the nominal area of the cross section. It is obvious that:

$$\sum_{n=1}^{\infty} \Delta a_n = A_p. \tag{9}$$

If it is assumed that the consumed energy at each level is proportional to an increase in the total area of the macrocrack, the amount of consumed energy at each step is equal to:

$$\Delta U_n = G_f.\frac{\Delta a_n}{A_p}. \tag{10}$$

On the other hand, the required energy for forming new cracks in the cross section is equal to the area under the stress-elongation curve, in an interval with the length of Δw, which reads, as follows:

$$\Delta U_n = \sigma_n \Delta w. \tag{11}$$

The condition of crack growth dictates that the value of Equation 10 should be equal to Equation 11 at each point, which yields:

$$G_f \frac{\Delta a_n}{A_p} = \sigma_n \Delta w. \tag{12}$$

According to previous discussions, the resistance of the specimen against elongation continues until the area added to the macrocrack, in a critical iteration 'n_c', becomes smaller than a certain value, which is denominated the "last resistant ligament". The value of n_c may be calculated from the following relation:

$$\frac{\Delta a_{n_c}}{A_p} < \frac{1}{m} \Rightarrow \left(\frac{1}{p^2}\right)\left(\frac{p^2-1}{p^2}\right)^{n_c-1} < \frac{1}{m}. \tag{13}$$

For example, if m is equal to 1000, n_c is equal to the iteration number at which the eliminated area from the fractal set becomes less than 0.001 of the cross sectional area. For the sake of simplification, q is equated with 1.

Now, by having the critical elongation of the specimen, w_c, it is possible to calculate the length of the intervals from the following relation:

$$\Delta w = \frac{w_c}{n_c}. \tag{14}$$

Before applying Equation 14 to Equation 12, it is noticed that, by using fractal patterns, a discrete distribution of the stress values will be gained. Some modifications on this model are required to make it applicable. The required modifications are, as follows:

1. The value of the softening function in ($w = 0$) is equal to the tensile strength of the specimen and the softening curve is linear at the interval of $[0, \Delta w]$, point A to point B, in Figure 4;
2. An energy modification factor, α, is defined to consider the consumption of energy at the interval of $[0, \Delta w/2]$, shown in Figure 4. This modification factor makes the area under the softening curve equal to G_F. The energy modification factor can be calculated from the following relation:

$$(1-\alpha)G_F = \left[f_t - \left(\frac{f_t - \frac{\alpha}{p^2\Delta w}}{4}\right)\right] \times \frac{\Delta w}{2}. \tag{15}$$

Now, it is possible to calculate the value of the stress at every point of the stress-elongation curve, by using the following relation:

$$\sigma = \begin{cases} \frac{n_c}{w_c}\left[G_f\left[\frac{p^2(8G_F-3f_t\Delta w)}{8p^2G_F+1}\right]\frac{1}{p^2}\left(1-\frac{1}{p^2}\right)^{\frac{n_c}{w_c}w-1}\right] & \frac{w_c}{n_c} \le w \le w_c \\ f_t\left(1-\frac{w}{\delta w}\right) + \left[\frac{p^2(8G_F-3f_t\Delta w)}{8p^2G_F+1}\right]\frac{G_F w}{p^2\Delta w^2} & w < \frac{w_c}{n_c}. \end{cases} \tag{16}$$

This softening curve is called "Quasi-fractal", because, not only is it a function of the fractal dimension of the fracture surface, it also depends on another parameter, m. A 'Quasi-fractal' softening curve consists of a linear part and an exponential part, which makes it very

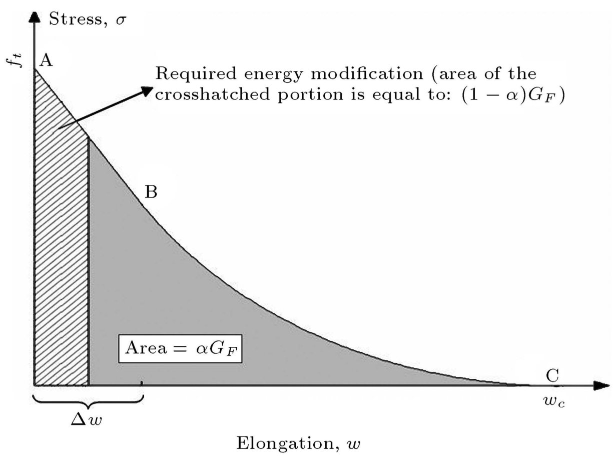

Figure 4. Schematic diagram of 'Quasi-fractal' softening curve.

adaptive to any kind of common softening curve. This will be shown in the next section.

It is more convenient to convert the softening curve to a dimensionless form. With this normalization procedure, the total area under the softening curve and the appropriate tensile strength will become equal to one. The dimensionless form of the 'Quasi-fractal' softening curve is, as follows:

$$\hat{\sigma}=\begin{cases}\frac{n_c}{\hat{w}_c}\left[\left[\frac{p^2(8-3\Delta\hat{w})}{8p^2+1}\right]\frac{1}{p^2}\left(1-\frac{1}{p^2}\right)^{\frac{n_c}{\hat{w}_c}\hat{w}-1}\right] & \frac{\hat{w}_c}{n_c}\leq\hat{w}\leq\hat{w}_c\\ \left(1-\frac{\hat{w}}{\Delta\hat{w}}\right)+\left[\frac{p^2(8-3\Delta\hat{w})}{8p^2+1}\right]\frac{\hat{w}}{p^2\Delta\hat{w}^2} & \hat{w}<\frac{\hat{w}_c}{n_c}\end{cases} \quad (17)$$

where the dimensionless parameters in Equation 17 are defined by the following relations:

$$\hat{\sigma}=\frac{\sigma}{f_t}, \qquad \hat{w}=\frac{w}{w_{ch}}, \qquad w_{ch}=\frac{G_f}{f_t}. \quad (18)$$

In the next section, the presented dimensionless form of the 'Quasi-fractal' softening curve will be compared to a set of experimental softening curves.

COMPARISON OF 'QUASI-FRACTAL' SOFTENING CURVE WITH COMMON SOFTENING CURVES

Introduction of Some Famous Experimental Softening Curves of Concrete

As stated above, several forms of the softening curve have been proposed for concrete. Five experimental softening curves have been chosen for comparison with a 'Quasi-fractal' softening curve. All the presented softening curves are in their dimensionless form, in order to have a single measurement criterion.

The first of these is Petersson's bilinear model [14]. He proposed the first experimental softening curve for concrete. This curve is a bilinear curve with a kink point at (0.8,1/3) and a dimensionless critical opening, i.e., $\hat{w}_c=\frac{w_c}{w_{ch}}$, equal to 3.6.

The second model is a power law softening, which has been proposed by Reinhardt [22]. This model has the following relation:

$$\sigma=\sigma_u\left[1-\left(\frac{w}{w_c}\right)^n\right], \qquad 0<n<1. \quad (19)$$

After conducting a series of experiments on concrete, Reinhardt [22] suggested values of 0.29-0.40 for n and 0.12-0.20 mm for w_c. Some typical values for concrete, suggested by Reinhardt [22], for n, w_c and σ_u, are 0.31, 0.175 mm and 3.2 N/mm^2, respectively. The corresponding value of G_f is 133 N/m. By using the suggested values of Reinhardt [22], the dimensionless form of Equation 19 is, as follows:

$$\hat{\sigma}=\left[1-\left(\frac{\hat{w}}{4.226}\right)^{0.31}\right], \qquad 0<w<4.226. \quad (20)$$

The third model is the "CHR" curve. Cornelissen et al. [15,16] proposed this model, which has the following relation:

$$\hat{\sigma}=(1+0.199\hat{w}^3)e^{-1.35\hat{w}}-0.00533\hat{w},$$

$$0<\hat{w}\leq 5.14. \quad (21)$$

Hordijk [23] analyzed the experimental results from 12 different sources and concluded that this equation can yield a reasonable approximation for them all.

On the other hand, some experimental data indicate that the tail of the softening curve can be extremely long, with w_c as large as $12G_f/f_t$. The fourth model, called ELT, has been proposed for these kinds of softening behavior by Planas and Elices [24], which has the following relation:

$$\hat{\sigma}=0.0750-0.00652\hat{w}+0.9250e^{-1.614\hat{w}},$$

$$\hat{w}\leq 11.5. \quad (22)$$

The fifth model is resultant of edge splitting experiments, which have recently been carried out by Slowik et al. [20]. They used a new optimization method to fit a softening curve to the results of edge splitting experiments. The proposed softening curve for concrete, by this new method, has the following mathematical form:

$$\sigma=c_1\left[\left\{1+\left(c_3\frac{w}{c_2}\right)^3\right\}.e^{-c_4\frac{w}{c_2}}-\frac{w}{c_2}(1+c_3^3).e^{-c_4}\right]. \quad (23)$$

They suggested a range for each of the parameters in the above equation. By using the proposed 'master parameter set' and converting the outcome equation into a dimensionless form, the following relation was obtained:

$$\hat{\sigma}=(1+0.1908\hat{w}^3).e^{-1.343\hat{w}}-0.0049\hat{w},$$

$$\hat{w}<5.211. \quad (24)$$

An interesting point concerning the above equation, which will be called "SVBV" for the length of this paper, is in its deep similarity with the CHR softening curve (Equation 21).

All of the curves, shown in Figure 5, have been sketched simultaneously in a coordinate system. It will be shown, in the next part of this section, that all of the presented curves have a "Quasi-fractal" counterpart.

Comparison of 'Quasi-Fractal' Softening Curve with Existing Softening Curves

A 'Quasi-fractal' softening curve, as the direct outcome of an energy consumption theory during the softening

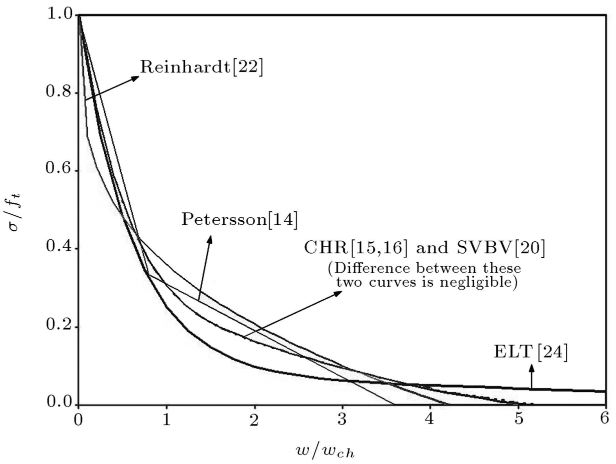

Figure 5. Different proposed softening curves for concrete.

procedure, is a function of two variables, p and m. For comparison of a 'Quasi-fractal' softening curve with other softening curves, a computer program was written to calculate the best adaptive pairs of p and m. After finding the best ranges for the previously mentioned parameters, because of the indistinct values of p and m, calculations were performed by a set of assumed values of m, which were 50, 100, 200 and 400.

The best value of parameter p is obtained from a simple algorithm. This algorithm is based on the criterion that, among a wide range for the parameter, the selected value has the least difference with the objective softening model. The outcomes for the above algorithm have been collected in Table 1.

After finding the p values for selected values of m by the above method, one more criterion is required for determining the best pair of parameters, m and p. This criterion is in concordance with the center of the area of the curves. The importance of this criterion is in the application of the softening curve. In fact, the location of the center of the area of a softening curve is the expositor of the influence point of the cohesive forces in an existing cohesive zone ahead of a crack, or, mathematically, the exerted moment by the cohesive zone forces can be calculated, simply by the following relation:

$$\int_0^{w_c} \sigma(w) \times w dw = G_f.\overline{w}, \tag{25}$$

where, in the above equation, $\overline{w}$, which, in its dimensionless notation is $\hat{\overline{w}}$, is the coordinate of the center of the area of the softening curve on the crack opening axis. Therefore, $\overline{w}$ is useful in the prediction of the behavior of notched concrete members or structures.

The results of the calculation of the center of the area for different values of p and m, from Table 1, have been gathered in Table 2. The presented values in Table 2 help the process of choosing an appropriate value of m.

The selected value, from Table 2, for the parameter, m, of an adaptive 'Quasi-fractal' softening curve, which is obtained by a comparison of the 'Quasi-fractal' model with other models, is equal to 100 for Petersson, CHR and SVBV curves and 400 for Reinhardt. For the 'Quasi-fractal' counterpart of an ELT curve, because of the negligible difference between the $\hat{\overline{w}}$ values for $m = 50$ and $m = 100$ and the good accordance of other curves with $m = 100$, the selected value for m is equal to 100. Each pair of selected softening curves and its 'Quasi-fractal' counterpart have been drawn through Figures 6 to 10.

From all the above, it is evident that 'Quasi-fractal' softening curves are in very good agreement

Table 1. The best values of parameter p for selected value of m of acquired adaptive 'Quasi-fractal' softening curve in comparison to other softening curves.

Comparison Model	$m = 50$	$m = 100$	$m = 200$	$m = 400$
Petersson [14]	1.74	2.19	2.82	3.72
Reinhardt [22]	2.04	2.62	3.52	4.79
CHR [15,16]	1.79	1.87	2.19	2.81
ELT [24]	2.04	1.98	1.95	1.93
SVBV [20]	1.78	1.87	2.11	2.75

Table 2. Calculated values of $\hat{\overline{w}}$ for the 'Quasi-fractal' softening curves of Table 1.

Comparison Model	$m = 50$	$m = 100$	$m = 200$	$m = 400$
Petersson ($\hat{\overline{w}} = 0.987$) [14]	0.944	0.946	0.937	0.924
Reinhardt ($\hat{\overline{w}} = 1.198$) [22]	1.115	1.149	1.177	1.193
CHR ($\hat{\overline{w}} = 1.178$) [15,16]	1.208	1.173	1.153	1.154
ELT ($\hat{\overline{w}} = 2.009$) [24]	2.07	2.075	1.91	1.787
SVBV ($\hat{\overline{w}} = 1.185$) [20]	1.057	1.186	1.146	1.157

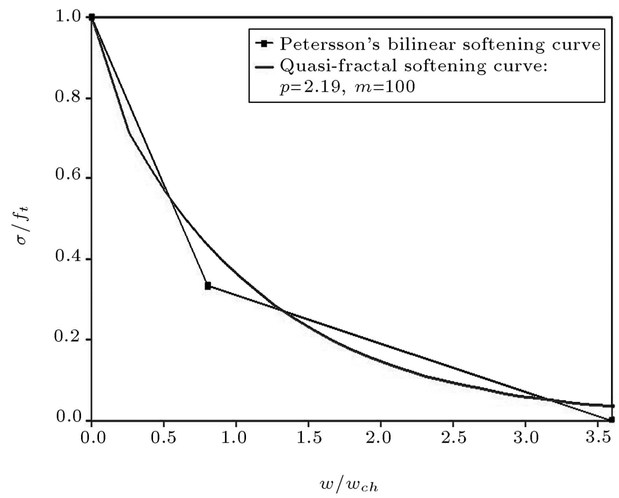

Figure 6. Petersson's bilinear softening curve vs. its 'Quasi-fractal' counterpart.

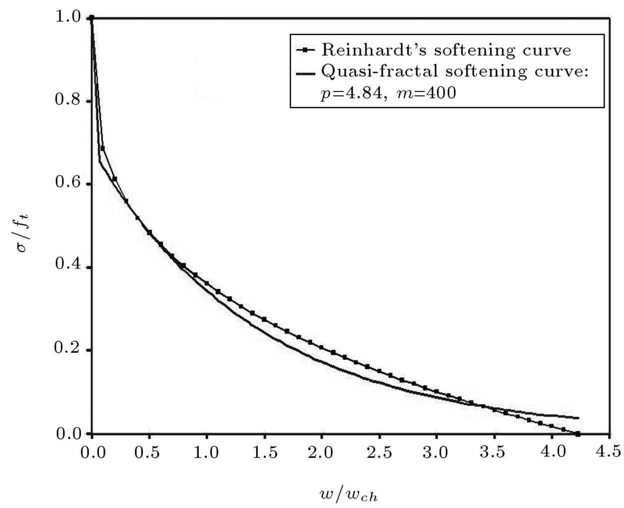

Figure 7. Reinhardt's softening curve vs. its 'Quasi-fractal' counterpart.

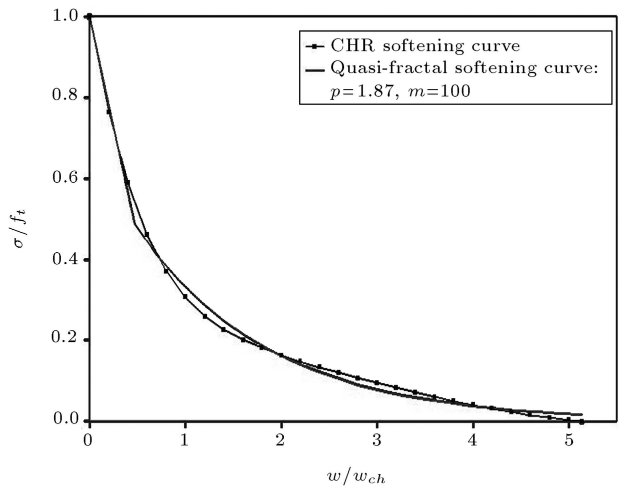

Figure 8. CHR softening curve vs. its 'Quasi-fractal' counterpart.

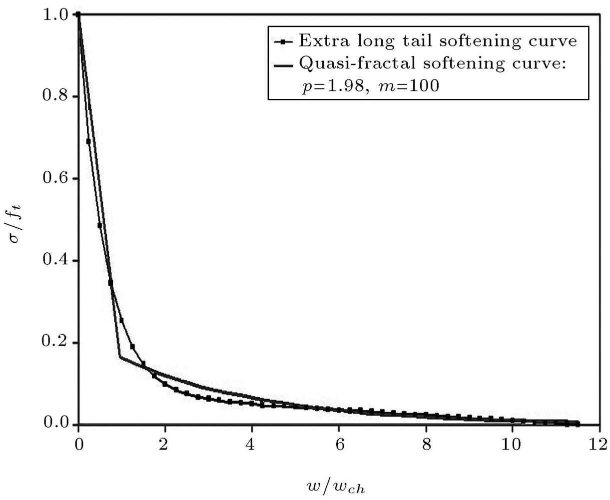

Figure 9. ELT softening curve vs. its 'Quasi-fractal' counterpart.

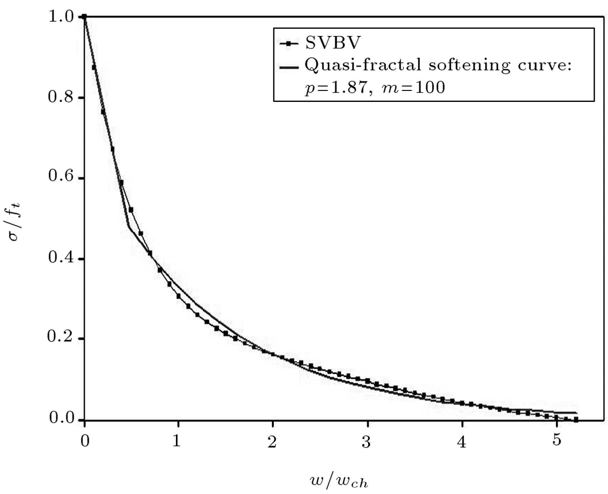

Figure 10. SVBV softening curve vs. its 'Quasi-fractal' counterpart.

with all presented softening curves. In other words, every softening curve has a counterpart in a 'Quasi-fractal' form. An interesting point about the acquired result is that the 'Quasi-fractal' counterpart of the advanced softening curves, i.e., CHR, ELT and SVBV, is in very close concordance with them. All of the presented softening curves are resultant of experiments, so, it can be stated that the 'Quasi-fractal' softening curve is close to the true materials softening behavior. In fact, the main merit of the 'Quasi-fractal' softening curve is that it is based on a theory for a softening procedure, which, itself, is based on the microstructure of the concrete. On this basis, because of the negligible difference in the value of parameter m in all of the studied curves, the difference between the existing softening curves arises from their different fractal dimensions, which is a direct result of the sieve curve of the aggregates. In addition, the impact of parameter m could be studied in future research.

SUMMARY AND CONCLUSION

In this paper, a new softening curve for concrete has been proposed. This model is based on a presented theory for a softening procedure that is a direct outcome of the microstructure of concrete. Studies into the so-called 'Quasi-fractal' softening curve have shown that it adapts itself well to existing softening curves. These adaptations to common softening curves may indicate that this new model is a good approximation of the actual softening procedure of concrete-like materials. This linear-exponential softening curve adapts well to other kinds of softening behavior, because many of the gained softening curves have a similar form.

REFERENCES

1. Mandelbrot, B.B., *Fractal Geometry of Nature*, New York, Freeman (1983).
2. Mandelbrot, B.B., Passoja, D.E. and Paullay, A.J. "Fractal character of fracture surfaces in metals", *Nature*, **308**, pp 721-722 (1984).
3. Winslow, D.N. "The fractal nature of the surface of cement paste", *Cement Concrete Res.*, **15**, pp 817-24 (1985).
4. Saouma, V.E., Barton, C.C. and Ganaledin, N.A. "Fractal characterization of fracture surfaces in concrete", *Engineering Fracture Mechanics*, **35**, pp 47-53 (1990).
5. Brandt, A.M. and Prokopski, G. "On the fractal dimension of fracture surfaces of concrete elements", *J. Mater Sci.*, **28**, pp 4762-6 (1993).
6. Saouma, V.E. and Barton, C.C. "Fractals, fracture, and size effect in concrete", *Journal of Engineering Fracture Mechanics*, **120**(4), pp 835-854 (1994).
7. Carpinteri, A., Chiaia, B. and Invernizzi, S. "Three-dimensional fractal analysis of concrete fracture at the meso-level", *Theoretical and Applied Fracture Mechanics*, **31**, pp 163-172 (1999).
8. Issa, M.A., Issa, M.A., Islam, M.S. and Chudnorsky, A. "Fractal dimension - A measure of fracture roughness and toughness of concrete", *Engineering Fracture Mechanics*, **70**, pp 125-137 (2003).
9. Carpinteri, A., Chiaia, B. and Cornetti, P. "On the mechanics of quasi-brittle materials with a fractal microstructure", *Engineering Fracture Mechanics*, **70**, pp 2321-2349 (2003).
10. Barenblatt, G.I. "The formation of equilibrium cracks during brittle fracture: General ideas and hypotheses", *J. Appl. Math. Mech.*, **23**, pp 622-36 (1959).
11. Barenblatt, G.I. "The mathematical theory of equilibrium of cracks in brittle fracture", *Advanced Applied Mechanics*, **7**, pp 55-129 (1962).
12. Dugdale, D.S. "Yielding of steel sheets containing slits", *Journal of Mechanics and Physics of Solids*, **8**, pp 100-108 (1960).
13. Hillerborg, A., Modéer, M. and Petersson, P.E. "Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements", *Cement and Concrete Research*, **6**, pp 773-782 (1976).
14. Petersson, P.-E. "Crack growth and development of fracture zone in plane concrete and similar materials", Report No. TVBM-1006, Division of Building Materials, Lund Institute of Technology, Lund, Sweden (1981).
15. Cornelissen, H.A., Hordijk, D.A. and Reinhardt, H.W. "Experimental determination of crack softening characteristics of normal and lightweight concrete", *Heron*, **31**(2), pp 45-46 (1986).
16. Cornelissen, H.A., Hordijk, D.A. and Reinhardt, H.W., *Experiments and Theory for the Application of Fracture Mechanics to Normal and Lightweight Concrete*, F.H. Wittman, Ed. Elsevier, Amsterdam, pp 565-575 (1986).
17. Gopalaratnam, V.S. and Shah, S.P. "Softening response of plain concrete in direct tension", *ACI J.*, **82**(3), pp 310-323 (1985).
18. Planas, J. and Elices, M., *Towards a Measure of G_F: An Analysis of Experimental Results*, F.H. Wittman, Ed., Elsevier, Amsterdam, pp 381-390 (1986).
19. Planas, J. and Elices, M. "Fracture criteria for concrete: Mathematical approximations and experimental validation", *Engineering Fracture Mechanics*, **35**, pp 87-94 (1990).
20. Slowik, V., Villmann, V., Bretschneider, N. and Villmann, T. "Computational aspects of inverse analyses for determining softening curves of concrete", *Comput. Methods. Appl. Mech. Engrg. Article*, **195**(52), pp 7223-7236 (2005).
21. Reinhardt, H.W. "Fracture mechanics of fictitious crack propagation in concrete", *Heron*, **29**(2), pp 3-42 (1984).
22. Reinhardt, H.W. "Fracture mechanics of an elastic softening material-like concrete", *Heron*, **29**(2), pp 3-42 (1984).
23. Hordjik, D.A. "Local approach to fatigue of concrete", Doctoral Thesis, Delft University of Technology, Delft, The Netherlands (1991).
24. Planas, J. and Elices, M. "Shrinkage Eigenstresses and structural size effect", In *Fracture Mechanics of Concrete Structures*, Z.P. Baûant, Ed., Elsevier Applied Science, London, pp 939-950 (1992).