Scientia Iranica, Vol. 15, No. 2, pp 218–222

SCIENTIA
IRANICA

Research Note

On Domination and its Forcing in Mycielski's Graphs

D.A. Mojdeh* and N. Jafari Rad[1]

In this paper, for a given graph, G, some domination parameters and the forcing domination number of the graph, $M(G)$, obtained from G arising in Mycielski's construction, are studied.

INTRODUCTION

A vertex in a graph, G, dominates itself and its neighbors. A set of vertices, S, in a graph, G, is a dominating set, if each vertex of G is dominated by some vertex of S. The minimum cardinality of a dominating set of G is the domination number, $\gamma(G)$, of G and the maximum cardinality of a minimal dominating set of G is the upper domination number, $\Gamma(G)$. A dominating set that is independent is called an independent dominating set of G. The independent domination number, $i(G)$, of G is the minimum cardinality of an independent dominating set of G. A dominating set that is connected is called a connected dominating set of G. The connected domination number, $\gamma_c(G)$, of G is the minimum cardinality of a connected dominating set of G. A dominating set, S, is called a total dominating set, if each vertex of G is dominated by some vertices of S. The total domination number, $\gamma_t(G)$, of G is the minimum cardinality of a total dominating set of G. A dominating set, S, of G is called a strong dominating set, if each vertex, x, of $V(G) \setminus S$ is dominated by some vertices, y, of S, with $\deg(y) \geq \deg(x)$. The strong domination number, $\gamma_s(G)$, of G is the minimum cardinality of a strong dominating set of G, [2-6]. A $\gamma(G)$-set is referred to as a dominating set for G of size $\gamma(G)$, a $i(G)$-set to an independent dominating set for G of size $i(G)$, a $\gamma_t(G)$-set to a total dominating set for G of size $\gamma_t(G)$ and a $\gamma_c(G)$-set to a connected dominating set for G of size $\gamma_c(G)$.

A subset, F, of a minimum dominating set, S, is a forcing subset for S, if S is the unique minimum dominating set containing F. The forcing domination number, $f(S,\gamma)$, of S is the minimum cardinality among the forcing subsets of S and the forcing domination number, $f(G,\gamma)$, of G is the minimum forcing domination number to be found among the minimum dominating sets of G [1].

The open neighborhood of a vertex, v, in a graph, G, denoted by $N_G(v)$, is the set of all vertices of G, which are adjacent to v. Also, $N_G[v] = N_G(v) \cup \{v\}$ is called the closed neighborhood of v in the graph, G.

In this paper, by G, one means a connected graph. From a graph, G, by Mycielski's construction, one can get a graph, $M(G)$, with $V(M(G)) = V \cup U \cup \{w\}$, where:

$$V = V(G) = \{v_1, \cdots, v_n\}, \quad U = \{u_1, \cdots, u_n\},$$

and:

$$E(M(G)) = E(G) \cup \{u_i v : v \in N_G(v_i) \cup \{w\}, \quad i = 1, \cdots, n\}.$$

For each $0 \leq i \leq n$, v_i and u_i are called the corresponding vertices of $M(G)$ and denote $C(v_i) = u_i$, $C(u_i) = v_i$. Moreover, for subsets $A \subseteq U$, $B \subseteq V$, one denotes:

$$C(A) = \{C(u_i) : u_i \in A\},$$

$$C(B) = \{C(v_i) : v_i \in B\}.$$

Also, $x \leftrightarrow y$ is denoted, when $\{x, y\}$ is an edge.

The following is made use of.

Theorem A [3]

For any graph, G, $\gamma(M(G)) = 1 + \gamma(G)$, $\gamma_t(M(G)) = 1 + \gamma_t(G)$.

*. *Corresponding Author, Department of Mathematics, University of Mazandaran, P.O. Box 47416-1467, Babolsar, I.R. Iran.*

1. *Department of Mathematics, University of Mazandaran, P.O. Box 47416-1467, Babolsar, I.R. Iran.*

Some domination parameters are studied with respect to $M(G)$ and some properties of $\gamma(M(G))$-sets. Then, the forcing domination number of $M(G)$ is studied, with respect to some given properties of $\gamma(G)$-sets.

SOME DOMINATION PARAMETERS, WITH RESPECT TO $M(G)$

In this section, $i(M(G))$, $\gamma_s(M(G))$, $\gamma_c(M(G))$, $\Gamma(M(G))$ and $\beta_0(M(G))$ are studied. It is well known that, for any graph, G, $\gamma(G) \leq i(G)$. Also, for $K_{m,n}$, with $\min\{m,n\} > 1$, this inequality is strict. In the following, the relation between the independent domination number of $M(G)$ and the independent domination number of G is obtained.

Theorem 1

For any graph, G, $i(M(G)) = 1 + i(G)$.

Proof

For any $i(G)-$set D, $D \cup \{w\}$ is an independent dominating set of $M(G)$, hence, $i(M(G)) \leq 1 + i(G)$.

If $|V(G)| \leq 2$, the equality, $i(M(G)) = 1 + i(G)$, is obvious. So, suppose that $|V(G)| > 2$. Assume that $i(M(G)) \leq i(G)$ and S is a $i(M(G))$-set of $M(G)$. Clearly, $w \notin S$, so, $S \cap U \neq \emptyset$. It is easily seen that $S \cap V \neq \emptyset$ and, also, for each $v_t \in S \cap V$, one has $u_t \in S \cap U$. If, for each $u_{k'} \in S \cap U$, one has $v_{k'} \in S$, then, $S \cap V$ is an independent dominating set of G, which is a contradiction. So, suppose that there is some vertex, $u_k \in S \cap U$, such that $v_k \notin S$; let then, $A = \{u_i \in S \cap U : v_i \notin S\}$, $A' = C(A)$ and $B = S \cap V$. Let u_{x_1} be a vertex of A, which has maximum neighbors in A', then, $D_1 = (A' \setminus \{v_{x_1}\}) \cup B$ is a dominating set of G. If D_1 is not independent, then, choose $u_{x_2} \in A \setminus \{u_{x_1}\}$, with maximum neighbors in $A' \setminus \{v_{x_1}\}$ and let $D_2 = (A' \setminus \{v_{x_1}, v_{x_2}\}) \cup B$. By continuing this method, there is an integer, m, such that D_m is an independent dominating set of G with size less than $i(G)$, which is a contradiction. Hence, $i(M(G)) \geq 1 + i(G)$, which implies the equality.■

Similarly, there is the following result, for which the proof is omitted.

Theorem 2

$$\gamma_s(M(G)) = 1 + \gamma_s(G).$$

Now, the connected dominating sets can be studied. Clearly, $\gamma_c(K_n) = 1$ and no two vertices of $M(K_n)$ can be a connected dominating set. Also, by considering $\{w, u_1, v_2\}$, one can verify that:

$$\gamma_c(M(K_n)) = 3 = \gamma_c(K_n) + 2 \quad \text{for} \quad n \geq 2.$$

Also, it is easily seen that no m vertex of $M(P_8)$, with $m \leq 4$, can form a connected dominating set and by $\{w, u_2, v_3, u_7, v_6\}$, one obtains:

$$\gamma_c(M(P_8)) = 5 = \gamma_c(P_8) - 1.$$

But, for $\gamma_c(G) \geq 3$, let S be a minimum connected dominating set for G and $\{v_x, v_y, v_z\} \subseteq S$, with $v_x \leftrightarrow v_y, v_y \leftrightarrow v_z$. Then, $(S \setminus \{v_y\}) \cup \{u_y, w\}$ is a connected dominating set for $M(G)$. So, one has the following bound, which is a strict of equality for many graphs, for example, $P_n, C_n, n \geq 7$.

Proposition 1

If $\gamma_c(G) \geq 3$, then, $\gamma_c(M(G)) \leq 1 + \gamma_c(G)$.

It is clear that U is a minimal dominating set of $M(G)$, so $\Gamma(M(G)) \geq |V(G)|$. Also, for many graphs, such as P_4, the equality, $\Gamma(M(G)) = |V(G)|$, holds and for many graphs, such as the following example, $\Gamma(M(G)) > |V(G)|$.

Consider the graph, $K_{1,n}$ for $n \geq 2$. Let x be the vertex with $\deg(x) = n$ and connect x to any vertex of the graph, $K_m, m \geq 4$, to obtain a graph, G^*. Then, by considering the vertices of $K_{1,n} \setminus \{x\}$, together with $C(K_{1,n} \setminus \{x\})$ and also $C(K_m)$, it is concluded that $\Gamma(M(G^*)) > |V(G^*)|$.

If G has a maximum minimal independent dominating set, $D = \{v_{d_1}, \cdots, v_{d_t}\}$, of size $\Gamma(G) = t$, then $D \cup \{u_{d_1}, \cdots, u_{d_t}\}$ is a minimal dominating set of $M(G)$ and, so $\Gamma(M(G)) \geq 2\Gamma(G)$. So, if G has a maximum minimal independent dominating set, then $\Gamma(M(G)) \geq \max\{2\Gamma(G), |V(G)|\}$.

The above bound can be strict. For example, see the above graph, G^*.

Similarly, one has $\beta_0(M(G)) \geq \max\{|V(G)|, 2\beta_0(G)\}$, whose bound can be strict.

SOME PROPERTIES OF $\gamma(M(G))$-SETS

In this section, more conclusions of $\gamma(M(G))$-sets and the relationship between them and $\gamma(G)$-sets are studied. It is seen that, for many graphs, such as K_n, $K_{m,n}$, $K_{n_1,\cdots,n_m}$, P_n, C_n, $K_2 \times P_n (n \geq 5)$, $P_3 \times P_n (n \geq 5)$, $P_4 \times P_{3n+1}$ and $P_5 \times P_{2n+1}$, every $\gamma(G)$-set is either independent or has just two adjacent vertices.

Proposition 2

If $|V(G)| \neq 2$ and $\gamma(G) = 1$, then the $\gamma(M(G))$-sets are precisely $\{w, v_k\}$ and $\{v_k, u_k\}$, where $\{v_k\}$ is a minimum dominating set of G.

Proof

For each $\gamma(G)$-set $\{v_i\}$ of G, it is clear that both $\{w, v_i\}$ and $\{v_i, u_i\}$ are $\gamma(M(G))$-sets. Now, let S be a $\gamma(M(G))$-set. The following cases exist as follows:

1. If $w \in S$, then, $S = \{w, u_k\}$ for some k and, clearly, the vertex, v_k, is not dominated by S, so that $S = \{w, v_{k'}\}$ for some integer k', where $\{v_{k'}\}$ is a $\gamma(G)$-set;
2. If $w \notin S$, let $u_j \in S \cap U$ for some j. When $|V(G)| = 1$, clearly $v_j \in S$.

Suppose that $|V(G)| \geq 3$ and $v_j \notin S$, then, $N(v_j) \cap S \neq \emptyset$. If $v_t \in N(v_j) \cap S$ for some t, then, u_t is not dominated by S and, if $u_i \in N(v_j) \cap S$ for some i, then, $U \setminus S$ is not dominated by S. Hence, $v_j \in S$.■

Note that, when $G \cong K_2$ and $V(G) = \{v_1, v_2\}$, then, the 2-sets are $\{w, v_1\}$, $\{v_1, u_1\}$ and $\{u_1, u_2\}$.

Proposition 3

If $\gamma(G) \geq 2$ and every $\gamma(G)$-set is independent, then, every $\gamma(M(G))$-set is also independent and contains w.

Proof

It may be assumed that $w \notin S$. Let S be a $\gamma(M(G))$-set, then, $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$. Let $u_k \in S$ for some k, then the following cases exist:

1. If $v_k \in S$ and $t \neq k$ exists, such that $u_t \in S$, then, $C(S \setminus \{u_k, u_t\}) \cap U)$, together with $V \cap S$, form a dominating set of G, a contradiction;
2. If $v_k \in S$ and, for each $t \neq k$, $u_t \notin S$, then, $v_s \in V \cap S$ for some $s \neq k$, but $u_s \notin S$, so $N(v_s) \cap V \neq \emptyset$. Now $C(S \setminus \{u_k\})$ together with $S \cap V$ form a $\gamma(G)$-set with two adjacent vertices, which is a contradiction;
3. If $v_k \notin S$ and $u_l \in N(v_k) \cap S$ exists for some l, then, one considers $v_{l'} \in S \cap V$ for some l'. If $u_{l'} \in S$, then, $C(S \cap U) \setminus \{u_k, u_{l'}\})$, together with $S \cap V$, form a dominating set of G with a size less than $\gamma(G)$, which is a contradiction. If $u_{l'} \notin S$, then, $C(S \cap U) \setminus \{u_k\})$, together with $S \cap V$, form a dominating set of G with two adjacent vertices, which is a contradiction;
4. If $v_k \notin S$ and $v_t \in S \cap N(v_k)$ exists for some t, then, $N[u_t] \cap S \neq \emptyset$. Now, by considering $C((S \cap V) \setminus \{u_k\})$ or $C((S \cap V) \setminus \{u_k, u_t\})$, one gets a contradiction (see above).■

So, under the hypothesis, Proposition 3, the $\gamma(M(G))$-sets have the following forms:

$$D \cup \{w\},$$

where D is a $\gamma(G)$-set.

Proposition 4

I) If $\gamma(G) = 2$ and every $\gamma(G)$-set contains just two adjacent vertices, then, the $\gamma(M(G))$-sets have one of the following forms:

$$\{v_x, v_y, w\}, \{u_x, u_y, w\}, \{v_x, u_y, w\}, \{v_x, v_y, u_t\},$$

where $\{v_x, v_y\}$ is a $\gamma(G)$-set and $v_y \leftrightarrow u_t$;

II) If $\gamma(G) = 3$ and every $\gamma(G)$-set contains just two adjacent vertices, then, the $\gamma(M(G))$-sets have one of the following forms:

1) $\{v_x, v_y, v_z, w\}$, $\{v_x, u_y, v_z, w\}$, $\{v_x, u_y, u_z, w\}$, $\{v_x, v_y, v_z, u_x\}$,
2) $\{u_x, u_{x'}, v_y, v_z\}$ with $v_x \leftrightarrow u_{x'}$, when $|(N(v_x) \cap U) \setminus (N(v_y) \cup N(v_z))| \leq 1$.

In both items 1 and 2, $\{v_x, v_y, v_z\}$ is a $\gamma(G)$-set and $v_y \leftrightarrow v_z$.

Proof

I) Clearly, for a $\gamma(G)$-set $\{v_i, v_j\}$, all the sets, $\{v_i, v_j, w\}$, $\{u_i, u_j, w\}$, $\{v_i, u_j, w\}$, $\{v_i, v_j, u_l\}$, are $\gamma(M(G))$-sets with $v_j \leftrightarrow u_l$. Suppose that S is a $\gamma(M(G))$-set. If $w \in S$, then, by replacing the vertices of $U \cap S$ with $C(U \cap S)$, one gets a $\gamma(G)$-set, hence, S is one of the sets, $\{v_x, v_y, w\}$, $\{u_x, u_y, w\}$, $\{v_x, u_y, w\}$, where $\{v_x, v_y\}$ is a $\gamma(G)$-set. If $w \notin S$, then, $S \cap U \neq \emptyset$ and, by Theorems A and 1, $S \cap V \neq \emptyset$. Let $v_k \in S$ for some k. If $u_k \in S$, then, it is easily seen that S has one of the above forms. If $u_k \notin S$, then, $N(v_k) \cap S \cap V \neq \emptyset$ and suppose that $v_{k+1} \in N(v_k) \cap S \cap V$. Also, let $u_{x'}$ be the third vertex of S. If $u_{x'}$ is adjacent neither to v_k nor to v_{k+1}, then, $v_{x'}$ is not dominated by S, which is a contradiction, so that $u_{x'}$ is adjacent to at least one of the vertices, v_k and v_{k+1}.

II) Clearly, for a $\gamma(G)$-set $\{v_i, v_j, v_k\}$ with $v_j \leftrightarrow v_k$, all of the above sets are $\gamma(M(G))$-sets. Now, let S be a $\gamma(M(G))$-set. If $w \in S$, then, by replacing the vertices of $S \cap U$ by $C(S \cap U)$, one obtains a $\gamma(G)$-set $D = \{v_x, v_y, v_z\}$ with $v_y \leftrightarrow v_z$. Since the vertex, v_x, is dominated by some vertex in S, hence, $v_x \in S$. If $w \notin S$, then, $S \cap U \neq \emptyset$, so by Theorem A, $S \cap V \neq \emptyset$. Let $u_t \in S \cap U$. By deleting u_t and replacing the other vertices of $S \cap U$ by $C(S \cap U \setminus \{u_t\})$, one gets a $\gamma(G)$-set $D = \{v_x, v_y, v_z\}$ G with $v_y \leftrightarrow v_z$. If $v_x \in S$, then, $u_x = u_t$ and it is easily seen that $\{v_y, v_z\} \subseteq S$. If $v_x \notin S$, then, $u_x \in S$ and $u_t \leftrightarrow v_x$, so that $|(N(v_x) \cap U) \setminus (N(v_y) \cup N(v_z))| \leq 1$. Now, it is easily seen that $\{v_y, v_z\} \subseteq S$.■

Proposition 5

If $\gamma(G) \geq k+2$ for some k and every $\gamma(G)$-set induces a $P_k + (\gamma(G) - k)K_1$, then, the $\gamma(M(G))$-sets have one of the forms $(D \setminus M) \cup C(M) \cup \{w\}$, where D is a $\gamma(G)$-set and $M \subseteq V(P_k)$.

Proof

Let D be a $\gamma(G)$-set and D induces a $P_k+(\gamma(G)-1)K_1$, in which $V(P_k) = \{v_1, v_2, \cdots, v_k\}$. For any subset, $M \subseteq V(P_k)$, $(D \setminus M) \cup C(M) \cup \{w\}$ is a dominating set of $M(G)$, which is minimum by Theorem A. Now, suppose that S is a $M(G)$-set. There are two cases:

1. If $w \in S$, then, $D = C(S \cap U) \cup (S \cap V)$ is a dominating set for G, which is minimum by Theorem A. So, D induces a $P_k + (\gamma(G) - 1)K_1$ and one may let $V(P_k) = \{v_{i1}, v_{i2}, \cdots, v_{ik}\}$ and $D\backslash V(P_k) = \{v_{j1}, v_{j2}, \cdots, v_{j(\gamma(G)-1)}\}$. If there is an integer, t, such that $v_{jt} \in D\backslash S$, then, $u_{jt} \in S$. But, then, v_{jt} is not dominated by S and this contradicts the fact that S is a minimum dominating set of $M(G)$. Hence, $\{v_{j1}, v_{j2}, \cdots, v_{j(\gamma(G)-1)}\} \subseteq S$. Now, since there is no integer, t', such that $\{u_{t'}, v_{t'}\} \subseteq S$, there is a subset, $M' \subseteq \{v_{i1}, v_{i2}, \cdots, v_{ik}\}$, such that:
$$(\{v_{i1}, v_{i2}, \cdots, v_{ik}\}\backslash M') \cup C(M') = S\backslash\{v_{j1}, v_{j2}, \cdots, v_{j(\gamma(G)-1)}\}.$$
2. If $w \notin S$, then, $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$. Moreover, $|S \cap U| \geq 2$ and there is no integer l, such that $\{u_l, v_l\} \subseteq S$. Let $w_i \in S \cap U$, then, $D = C((S\cap U)\backslash\{w_i\}) \cup (S\cap V)$ is a minimum dominating set for G, which induces a $P_k + (\gamma(G) - k)K_1$. Let $\{v_{t1}, v_{t2}\} \subseteq D\backslash V(P_k)$, then, $\{v_{t1}, v_{t2}, u_{t1}, u_{t2}\}$ is not dominated by S. This is a contradiction.■

Corollary 1

If $\gamma(G) \geq 4$ and $\gamma(G)$-set has just two adjacent vertices, then, the $\gamma(M(G))$-sets have one of the following forms:
$$D \cup \{w\}, (D \setminus \{v_k\}) \cup \{u_k, w\},$$
and:
$$(D \setminus \{v_k, v_l\}) \cup \{w, u_k, u_l\},$$
where D is a $\gamma(G)$-set and v_k and v_l are the two adjacent vertices of D.

FORCING DOMINATION NUMBER

In this section, the forcing domination number of $M(G)$ is studied. It is well known that $f(K_n, \gamma = 1) = 1$ and, for $n \geq 2$, $f(K_{1,n}, \gamma = 1) = 0$. Also, for each $i = 1, \cdots, n$, $\{u_i, v_i\}$ and $\{w, v_i\}$ are minimum dominating sets of $M(K_n)$, so $f(M(K_n)) \geq 1$ for $n > 1$. On the other hand, $\{u_1, v_1\}$ is the only dominating set of $M(K_n)$ containing $F = \{u_1\}$, hence:
$$f(M(K_n), \gamma(M(K_n)) = 1 = f(K_n, \gamma(K_n)).$$
Similarly:
$$f(M(K_{1,n}), \gamma(M(K_{1,n}))) = 1 = 1 + f(K_{1,n}, \gamma(K_{1,n})).$$

Theorem 3

Let $\gamma(G) \geq 2$, D be a $\gamma(G)$-set and F be a minimum forcing set of D with $|F| = f(G, \gamma(G))$. If D is independent, then, $f(M(G), \gamma(M(G))) \leq f(G, \gamma(G))$.

Proof

It is clear that $S = D \cup \{w\}$ is a $\gamma(M(G))$-set. It is shown that this is the unique $\gamma(M(G))$-set containing F. Suppose that S' is another dominating set of $M(G)$ containing F. There are two cases as follows:

1. If $w \in S'$, by replacing the vertices of $S' \cap U$ with $C(S' \cap U)$ one obtains a $\gamma(G)$-set that is equal to D, so, there is a vertex, $v_k \in D$, such that $u_k \in S'$. If $v_k \notin S'$, then, $N(v_k) \cap S' \neq \emptyset$, which is a contradiction. If not, $C((S' \setminus \{w, u_k\}) \cap U)$ form a dominating set of G, which is a contradiction;
2. If $w \notin S'$, then, it is easily seen that $|S' \cap U| \geq 2$. If there exists an integer, j, such that $\{u_j, v_j\} \subseteq S'$, then, u_j and another vertex, $u_{j'}$ of $S' \cap U$ are omitted, so $C((S' \cap U) \setminus \{u_j, u_{j'}\})$, together with $S' \cap V$, form a dominating set of G, which is a contradiction. Otherwise, similarly, contradiction is obtained.■

Corollary 2

If a graph, G, satisfies the conditions of Proposition 3, then:
$$f(M(G), \gamma(M(G))) = f(G, \gamma(G)).$$
Similarly, for any graph:
$$G, f(M(G), i(M(G))) = f(G, i(G)).$$
As an example, for each $m \geq 3$:
$$f(M(K_2), 2) = f(M(M(K_2), 3)) = \cdots = f(M^{m-1}(K_2), m) = 2.$$

It is well known that every pair, a, b, of integers, with b positive and $0 \leq a \leq b$, can be realized as the forcing domination number and domination number, respectively, of some graph [1]. Now, for each pair of integers, a, b, with $0 \leq a \leq b$, if G is a graph satisfying the hypotheses of Proposition 3 and $\gamma(G) = m < b$, $f(G, \gamma(G)) = a$, then, using Mycielski's construction $b - m$ times, one can obtain a graph, G', satisfying the above fact. The following can also be seen:

1. If $|V(G)| \neq 2$ and $\gamma(G) = 1$, then for each integer, m,
$$f(M(G), 2) = \cdots = f(M^{m-1}(G), m) = 1;$$
2. If $\gamma(G) = 2$ and every $\gamma(G)$-set contains just two adjacent vertices, then:
$$f(M(G), \gamma(M(G))) = 2.$$

Proposition 6

Let $\gamma(G) = 3$ and every minimum dominating set of G contains just two adjacent vertices. If G has a minimum dominating set $\{v_x, v_y, v_z\}$, where $v_y \leftrightarrow v_z$ and:

$$|(N(v_x) \cap U) \setminus (N(v_y) \cup N(v_z))| > 1,$$

then, $f(M(G), \gamma(M(G))) = 1$; otherwise, $f(M(G), \gamma(M(G))) = 2$.

Proof

By Proposition 4, one has:

$$f(M(G), \gamma(M(G))) \geq 1.$$

Let $\{v_x, v_y, v_z\}$ be a $\gamma(G)$-set with $v_y \leftrightarrow v_z$. If:

$$|(N(v_x) \cap U) \setminus (N(v_y) \cup N(v_z))| > 1,$$

then, $F = \{u_x\}$ is a forcing dominating set for $M(G)$. Otherwise, $F' = \{u_x, v_x\}$ is a forcing dominating set for $M(G)$. Also, if, for any $\gamma(G)-$set $\{v_i, v_j, v_k\}$ with $v_j \leftrightarrow v_k$, $|(N(v_i) \cap U) \setminus (N(v_j) \cup N(v_k))| \leq 1$, then, it is easily seen that no two vertices can uniquely determine a minimum dominating set.■

Theorem 4

If the hypothesis of Corollary 1 holds for G, then;

$$f(G, \gamma(G)) \leq f(M(G), \gamma(M(G))) \leq 2 + f(G, \gamma(G)).$$

Proof

Let F be a minimum forcing dominating set of $M(G)$ and S be the unique minimum dominating set containing it, then, by Corollary 1, S has the form $D \cup \{w\}$, $(D \setminus \{v_k\}) \cup \{u_k, w\}$ and $(D \setminus \{v_k, v_l\}) \cup \{w, u_k, u_l\}$, where D is a $\gamma(G)$-set and $v_k \leftrightarrow v_l$ are the two adjacent vertices of D. Clearly, one of $\{u_k, u_l\}$, $\{v_k, v_l\}$ or $\{v_k, u_l\}$ is contained in F. If $\{u_k, u_l\} \subseteq F$, then, $(F \setminus \{u_k, u_l\}) \cup \{v_k, v_l\}$ is a minimum forcing dominating set of G. If $\{v_k, v_l\} \subseteq F$, then, F is a minimum forcing dominating set of G, and if $\{v_k, u_l\} \subseteq F$, then $(F \setminus \{u_l\}) \cup \{v_l\}$ is a minimum forcing dominating set of G. Hence:

$$f(G, \gamma(G)) \leq f(M(G), \gamma(M(G))).$$

On the other hand, let F' be a minimum forcing dominating set of G and S' be the unique minimum dominating set containing it with two adjacent vertices, v_i, v_j. If $\{v_i, v_j\} \subseteq F'$, then, F' is a minimum forcing dominating set of G. Hence:

$$f(M(G), \gamma(M(G))) \leq f(G, \gamma(G)).$$

If one of the two adjacent vertices, say v_i, belongs to F', then, $F' \cup \{u_j\}$ is a minimum forcing dominating set of $M(G)$. Hence:

$$f(M(G), \gamma(M(G))) \leq 1 + f(G, \gamma(G)).$$

Finally, if none of the above two vertices belong to F', then, $F' \cup \{u_i, u_j\}$ is a minimum forcing dominating set of $M(G)$. Hence:

$$f(M(G), \gamma(M(G))) \leq 2 + f(G, \gamma(G)).$$

■

CONCLUSION

In this paper, the domination number and forcing domination number of $M(G)$ is studied, with respet to some given properties of $\gamma(G)$-sets. However, there are other properties of $\gamma(G)$-sets and $\gamma(M(G))$-sets which can be studied.

ACKNOWLEDGMENT

The authors would like to thank the referee for his (her) useful comments.

REFERENCES

1. Chartrand, G., Gavlas, H., Vandell, R.C. and Harary, F. "The forcing domination number of a graph", *J. Comb. Math. Comb. Comput.*, **25**, pp 161-174 (1997).
2. Cherifi, R., Gravier, S. and Zighem, I. "Bounds on domination number of complete grid graphs", *Ars Combinatoria*, **60**, pp 307-311 (2001).
3. Fisher, D.C., McKenna, P.A. and Boyer, E.D. "Hamiltonicity, diameter, domination, packing and biclique partitions of Myscielski's graphs", *Discrete Appl. Math.*, **84**, pp 93-105 (1998).
4. Gravier, S. and Mollard, M. "On domination numbers of Cartesian product of paths", *Discrete Math.*, **80**, pp 247-250 (1997).
5. Haynes, T.W., Hedetniemi, S.T. and Slater, P.J., Eds., *Domination in Graphs: Advanced Topics*, Marcel Dekker, Inc, New York, NY, USA (1997).
6. Haynes, T.W., Hedetniemi, S.T. and Slater, P.J. Eds. *Fundamental of Domination in Graph: Advanced Topics*, Marcel Dekker, Inc, New York, NY, USA (1998).