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Research Note

A Variational Iteration Method for Solving Systems of Partial Differential Equations and for Numerical Simulation of the Reaction-Diffusion Brusselator Model

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In this work, systems of linear and nonlinear partial differential equations and the reaction-diffusion Brusselator model are handled by applying the variational iteration method. The Variational iteration method has the advantage of being more concise for analytical and numerical purposes. The results reveal that the method is very effective and convenient.

INTRODUCTION

Systems of partial differential equations have attracted much attention in studying evolution equations describing wave propagation [1-3], in investigating shallow water waves [1-3] and in examining the chemical reaction-diffusion model of a Brusselator [4-6]. The general idea of these systems is of wide applicability. In [1-3], the characteristics method and the Riemann invariants method are used to handle systems of partial differential equations. These methods contain a large size of computation, especially when the system involves several partial differential equations. In [5], a method, based on a combination of the waveform relaxation method and multigrid, was implemented to solve nonlinear systems. The Brusselator model is solved with the periodic multigrid waveform relaxation method [5], using five multigrid levels. Parallel processors were used to carry out the large size of calculations. Recently, some new methods, differing from the above methods for nonlinear equations, have attracted broad attention, for example; the variational iteration method [7-14], the homotopy perturbation method [15-18] and the F-expansion method [19-22]. This paper applies the variational iteration method to the discussed problem.

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VARIATIONAL ITERATION METHOD

To illustrate the basic concepts of the variational iteration method, the following system of differential equations is considered:

$$L_1 u(x,t) + N_1(u(x,t), v(x,t)) = f(x,t),$$

$$L_2 v(x,t) + N_2(u(x,t), v(x,t)) = g(x,t), \qquad (1)$$

where L_1 and L_2 are linear differential operators, with respect to time; N_1 and N_2 are nonlinear operators and $f(x,t)$ and $g(x,t)$ are given functions. According to the variational iteration method, a correction functionals can be constructed as follows [7-10,23,24]:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1(\tau)\{L_1 u_n(x,\tau) + N_1(\widetilde{u}_n(x,\tau), \widetilde{v}_n(x,\tau)) - f(x,\tau)\}d\tau, \qquad (2)$$

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2(\tau)\{L_2 u_n(x,\tau) + N_2(\widetilde{u}_n(x,\tau), \widetilde{v}_n(x,\tau)) - g(x,\tau)\}d\tau, \qquad (3)$$

where λ_1 and λ_2 are general Lagrange multipliers, which can be identified, optimally, via a variational theory [25-28]. The second term on the right-hand side in Equations 2 and 3 is called the correction and the subscript, n, denotes the nth order approximation.

Under suitable restricted variational assumptions (i.e. $\widetilde{u}_n$ and $\widetilde{v}_n$ are considered as a restricted variation), one can assume that the above correction functionals are stationary (i.e. $\delta u_{n+1} = 0$ and $\delta v_{n+1} = 0$), then, the Lagrange multipliers can be identified.

APPLICATION

First, one starts with systems of linear partial differential equations.

Linear Partial Differential Systems

Example 1

First, the linear system:

$$u_t + v_x = 0, \qquad v_t + u_x = 0, \tag{4}$$

is considered with the initial data:

$$u(x,0) = e^x, \qquad v(x,0) = e^{-x}. \tag{5}$$

To apply the variational iteration method, the following correction functionals are constructed:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1(\tau)\{(u_n)_\tau + (\widetilde{v}_n)_x\}d\tau, \tag{6}$$

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2(\tau)\{(v_n)_\tau + (\widetilde{u}_n)_x\}d\tau, \tag{7}$$

where λ_1 and λ_2 are Lagrange multipliers and $\widetilde{u}_n$ and $\widetilde{v}_n$ denote the restricted variations (i.e. $\delta\widetilde{u}_n = \delta\widetilde{v}_n = 0$). Making the above correction functionals stationary to find the optimal value of λ_1 and λ_2;

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t\lambda_1(\tau)\{(u_n)_\tau+(\widetilde{v}_n)_x\}d\tau, \tag{8}$$

$$\delta v_{n+1}(x,t)=\delta v_n(x,t)+\delta\int_0^t\lambda_2(\tau)\{(v_n)_\tau+(\widetilde{u}_n)_x\}d\tau, \tag{9}$$

yields the following stationary conditions:

$$1 + \lambda_1(\tau)\mid_{\tau=t}= 0, \qquad \lambda_1'(\tau) = 0, \tag{10}$$

$$1 + \lambda_2(\tau)\mid_{\tau=t}= 0, \qquad \lambda_2'(\tau) = 0. \tag{11}$$

Therefore, the Lagrange multipliers can be defined as follows:

$$\lambda_1 = -1, \qquad \lambda_2 = -1.$$

And, one gets the following iterations:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \{(u_n)_\tau + (v_n)_x\}d\tau, \tag{12}$$

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \{(v_n)_\tau + (u_n)_x\}d\tau. \tag{13}$$

Starting with the initial approximations $u_0(x,t) = u(x,0)$ and $v_0(x,t) = v(x,0)$, and by Equations 12 and 13, one will have:

$$u_1 = e^x + e^{-x}t,$$

$$v_1 = e^{-x} - e^x t,$$

$$u_2 = e^x\left(1 + \frac{t^2}{2!}\right) + e^{-x}t,$$

$$v_2 = e^{-x}\left(1 + \frac{t^2}{2!}\right) - e^x t,$$

$$u_3 = e^x\left(1 + \frac{t^2}{2!}\right) + e^{-x}\left(t + \frac{t^3}{3!}\right),$$

$$v_3 = e^{-x}\left(1 + \frac{t^2}{2!}\right) - e^x\left(t + \frac{t^3}{3!}\right),$$

$$u_4 = e^x\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!}\right) + e^{-x}\left(t + \frac{t^3}{3!}\right),$$

$$v_4 = e^{-x}\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!}\right) - e^x\left(t + \frac{t^3}{3!}\right), \tag{14}$$

and so on. Using Equations 12 and 13, one obtains:

$$u(x,t) = e^x\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) + e^{-x}\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right),$$

$$v(x,t) = e^{-x}\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) - e^x\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right), \tag{15}$$

which has an exact analytical solution of the form:

$$u(x,t) = e^x\cosh(t) + e^{-x}\sinh(t),$$

$$v(x,t) = e^{-x}\cosh(t) - e^x\sinh(t), \tag{16}$$

or, equivalently,

$$u(x,t) = \cosh(x-t) + \sinh(x-t),$$

$$v(x,t) = \cosh(x-t) - \sinh(x-t). \tag{17}$$

Example 2

Consider the linear system of partial differential equations:

$$u_t + u_x - 2v = 0, \qquad v_t + v_x + 2u = 0, \tag{18}$$

with the initial data:

$$u(x,0) = \sin(x), \qquad v(x,0) = \cos(x). \tag{19}$$

In the same manner, for the system of partial differential equations (Equations 18), one obtains $\lambda_1 = \lambda_2 = -1$. Therefore, the following iterations are obtained:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \{(u_n)_\tau + (u_n)_x - 2v\}d\tau,$$

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \{(v_n)_\tau + (v_n)_x + 2u\}d\tau. \tag{20}$$

By setting $u_0 = u(x,0)$ and $v_0 = v(x,0)$ and by Equations 20, one will have:

$$u_1 = \sin(x) + \cos(x)t,$$

$$v_1 = \cos(x) - \sin(x)t,$$

$$u_2 = \sin(x)\left(1 - \frac{t^2}{2!}\right) + \cos(x)t,$$

$$v_2 = \cos(x)\left(1 - \frac{t^2}{2!}\right) - \sin(x)t,$$

$$u_3 = \sin(x)\left(1 - \frac{t^2}{2!}\right) + \cos(x)\left(t - \frac{t^3}{3!}\right),$$

$$v_3 = \cos(x)\left(1 - \frac{t^2}{2!}\right) - \sin(x)\left(t - \frac{t^3}{3!}\right),$$

$$u_4 = \sin(x)\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!}\right) + \cos(x)\left(t - \frac{t^3}{3!}\right),$$

$$v_4 = \cos(x)\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!}\right) - \sin(x)\left(t - \frac{t^3}{3!}\right). \tag{21}$$

Combining the results obtained above gives the following:

$$u(x,t) = \sin(x)\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) + \cos(x)\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right),$$

$$v(x,t) = \cos(x)\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) - \sin(x)\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right), \tag{22}$$

so that u and v are known in a closed form by the following:

$$u(x,t) = \sin(x+t), \qquad v(x,t) = \cos(x+t). \tag{23}$$

In the following example, a system of three nonlinear partial differential equations in three unknown functions, $u(x,y,t)$, $v(x,y,t)$ and $w(x,y,t)$, will be investigated.

Nonlinear Partial Differential System

Example 3

Consider the following nonlinear system:

$$u_t + v_x w_y - v_y w_x + u = 0,$$

$$v_t + w_x u_y + w_y u_x - v = 0,$$

$$w_t + u_x v_y + u_y v_x - w = 0, \tag{24}$$

with the initial conditions:

$$u(x,y,0) = e^{x+y},$$

$$v(x,y,0) = e^{x-y},$$

$$w(x,y,0) = e^{y-x}. \tag{25}$$

To apply the variational iteration method, the following correction is constructed:

$$u_{n+1}(x,y,t) = u_n(x,y,t) + \int_0^t \lambda_1(\tau)\{u_{n\tau} + \widetilde{v}_{nx}\widetilde{w}_{ny} - \widetilde{w}_{nx}\widetilde{v}_{ny} + \widetilde{u}_n\}d\tau, \tag{26}$$

$$v_{n+1}(x,y,t) = v_n(x,y,t) + \int_0^t \lambda_2(\tau)\{v_{n\tau} + \widetilde{w}_{nx}\widetilde{u}_{ny} + \widetilde{u}_{nx}\widetilde{w}_{ny} - \widetilde{v}_n\}d\tau, \tag{27}$$

$$w_{n+1}(x,y,t) = w_n(x,y,t) + \int_0^t \lambda_3(\tau)\{w_{n\tau} + \widetilde{u}_{nx}\widetilde{v}_{ny} + \widetilde{u}_{ny}\widetilde{v}_{nx} - \widetilde{w}_n\}d\tau, \tag{28}$$

where λ_1, λ_2 and λ_3 are Lagrange multipliers and $\widetilde{u}_n$, $\widetilde{v}_n$ and $\widetilde{w}_n$ denote the restricted variations (i.e. $\delta\widetilde{u}_n = \delta\widetilde{v}_n = \delta\widetilde{w}_n = 0$). Making the above correction functionals stationary to find the optimal value of λ_1,

λ_2 and λ_3, the followings

$$\delta u_{n+1}(x,y,t) = \delta u_n(x,y,t) + \delta\int_0^t \lambda_1(\tau)\{u_{n\tau} + \widetilde{v}_{nx}\widetilde{w}_{ny} - \widetilde{w}_{nx}\widetilde{v}_{ny} + \widetilde{u}_n\}d\tau, \quad (29)$$

$$\delta v_{n+1}(x,y,t) = \delta v_n(x,y,t) + \delta\int_0^t \lambda_2(\tau)\{v_{n\tau} + \widetilde{w}_{nx}\widetilde{u}_{ny} + \widetilde{u}_{nx}\widetilde{w}_{ny} - \widetilde{v}_n\}d\tau, \quad (30)$$

$$\delta w_{n+1}(x,y,t) = \delta w_n(x,y,t) + \delta\int_0^t \lambda_3(\tau)\{w_{n\tau} + \widetilde{u}_{nx}\widetilde{v}_{ny} + \widetilde{u}_{ny}\widetilde{v}_{nx} - \widetilde{w}_n\}d\tau, \quad (31)$$

yield the following stationary conditions:

$$1 + \lambda_1(\tau)\mid_{\tau=t} = 0, \qquad \lambda_1'(\tau) = 0, \quad (32)$$

$$1 + \lambda_2(\tau)\mid_{\tau=t} = 0, \qquad \lambda_2'(\tau) = 0, \quad (33)$$

$$1 + \lambda_3(\tau)\mid_{\tau=t} = 0, \qquad \lambda_3'(\tau) = 0. \quad (34)$$

Therefore, the Lagrange multipliers can be defined in the following forms:

$$\lambda_1 = -1, \qquad \lambda_2 = -1, \qquad \lambda_3 = -1.$$

And one gets the following iterations:

$$u_{n+1}(x,y,t) = u_n(x,y,t) - \int_0^t \{u_{n\tau} + v_{nx}w_{ny} - w_{nx}v_{ny} + u_n\}d\tau, \quad (35)$$

$$v_{n+1}(x,y,t) = v_n(x,y,t) - \int_0^t \{v_{n\tau} + w_{nx}u_{ny} + u_{nx}w_{ny} - v_n\}d\tau, \quad (36)$$

$$w_{n+1}(x,y,t) = w_n(x,y,t) - \int_0^t \{w_{n\tau} + u_{nx}v_{ny} + u_{ny}v_{nx} - w_n\}d\tau. \quad (37)$$

By setting $u_0 = u(x,y,0)$, $v_0 = v(x,y,0)$ and $w_0 = w(x,y,0)$, one obtains:

$$u_1 = e^{x+y}(1-t),$$
$$v_1 = e^{x-y}(1+t),$$
$$w_1 = e^{y-x}(1+t), \quad (38)$$

$$u_2 = e^{x+y}\left(1 - t + \frac{t^2}{2!}\right),$$
$$v_2 = e^{x-y}\left(1 + t + \frac{t^2}{2!}\right),$$
$$w_2 = e^{y-x}\left(1 + t + \frac{t^2}{2!}\right), \quad (39)$$

$$u_3 = e^{x+y}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!}\right),$$
$$v_3 = e^{x-y}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right),$$
$$w_3 = e^{y-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right). \quad (40)$$

Continuing in this manner, one can find:

$$u_n = e^{x+y}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots + (-1)^n\frac{t^n}{n!}\right),$$
$$v_n = e^{x-y}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots + \frac{t^n}{n!}\right),$$
$$w_n = e^{y-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots + \frac{t^n}{n!}\right). \quad (41)$$

The sequences in Equations 41 converge the following exact solutions:

$$u(x,y,t) = e^{x+y-t},$$
$$v(x,y,t) = e^{x-y+t},$$
$$w(x,y,t) = e^{y-x+t}. \quad (42)$$

REACTION-DIFFUSION BRUSSELATOR MODEL

In this section, the reaction-diffusion Brusselator model is considered [5,6], which models a chemical reaction diffusion process. The two-dimensional Brusselator model [29] is as follows:

$$u_t - u^2v + (A+1)u - \frac{1}{500}(u_{xx} + u_{yy}) - B = 0,$$
$$v_t + u^2v - Au - \frac{1}{500}(u_{xx} + u_{yy}) = 0, \quad (43)$$

with initial data:

$$u(x,y,0)=2+\frac{1}{4}y, \qquad v(x,y,0)=1+\frac{4}{5}x, \tag{44}$$

where $u(x,y,t)$ and $v(x,y,t)$ denote chemical concentrations [5] of intermediate reaction products and A and B are constant concentrations of input reagents where:

$$A=\frac{17}{5}, \qquad B=1. \tag{45}$$

One can construct the following correction functionals:

$$u_{n+1}(x,y,t)=u_n(x,y,t)+\int_0^t \lambda_1(\tau)\{u_{n\tau}-(\widetilde{u}_n)^2\widetilde{v}_n + \left(1+\frac{17}{5}\right)\widetilde{u}_n-\frac{1}{500}(\widetilde{u}_{nxx}+\widetilde{u}_{nyy})-1\}d\tau, \tag{46}$$

$$v_{n+1}(x,y,t)=v_n(x,y,t)+\int_0^t \lambda_1(\tau)\{v_{n\tau}+(\widetilde{u}_n)^2\widetilde{v}_n -\frac{17}{5}\widetilde{u}_n-\frac{1}{500}(\widetilde{u}_{nxx}+\widetilde{u}_{nyy})\}d\tau, \tag{47}$$

where λ_1 and λ_2 are Lagrange multipliers and $\widetilde{u}_n$ and $\widetilde{v}_n$ are restricted variations. The Lagrange multipliers can be identified as $\lambda_1=-1$ and $\lambda_2=-1$ and, therefore, the following iteration formulae can be obtained:

$$u_{n+1}(x,y,t)=u_n(x,y,t)-\int_0^t \{u_{n\tau}-(u_n)^2v_n + \left(1+\frac{17}{5}\right)u_n-\frac{1}{500}(u_{nxx}+u_{nyy})-1\}d\tau, \tag{48}$$

$$v_{n+1}(x,y,t)=v_n(x,y,t)-\int_0^t \{v_{n\tau}+(u_n)^2v_n -\frac{17}{5}u_n-\frac{1}{500}(u_{nxx}+u_{nyy})\}d\tau. \tag{49}$$

Starting with initial approximations, $u_0(x,t)=u(x,0)$ and $v_0(x,t)=v(x,0)$, and by Equations 48 and 49, one will have:

$$u_1=2+\frac{y}{4}-\left(\frac{39}{5}-\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)+\frac{11}{10}y\right)t, \tag{50}$$

$$v_1=1+\frac{4}{5}x-\left(-\frac{34}{5}+\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)-\frac{17}{20}y\right)t, \tag{51}$$

$$\begin{aligned}u_2=&\left\{\frac{1}{4}\left(-\frac{39}{5}+\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)-\frac{11}{10}y\right)^2\right.\\
&\left.\times\left(\frac{34}{5}-\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)+\frac{17}{20}y\right)\right\}t^4\\
&+\left\{\frac{2}{3}\left(2+\frac{y}{4}\right)\left(-\frac{39}{5}+\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)-\frac{11}{10}y\right)\right.\\
&\times\left(\frac{34}{5}-\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)+\frac{17}{20}y\right)+\frac{1}{3}\left(-\frac{39}{5}\right.\\
&\left.\left.+\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)-\frac{11}{10}y\right)^2\left(1+\frac{4}{5}x\right)\right\}t^3\\
&+\left\{\frac{137281}{8000}+\frac{1}{10000}x+\frac{1}{2}\left(2+\frac{y}{4}\right)^2\left(\frac{34}{5}\right.\right.\\
&\left.-\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)+\frac{17}{20}y\right)+\left(2+\frac{y}{4}\right)\left(-\frac{39}{5}\right.\\
&\left.+\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)-\frac{11}{10}y\right)\left(1+\frac{4}{5}x\right)-\frac{11}{5}\\
&\left.\times\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)+\frac{121}{50}y\right\}t^2+\left\{-\frac{39}{5}\right.\\
&\left.+\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)-\frac{11}{10}y\right\}t+2+\frac{y}{4},\end{aligned} \tag{52}$$

$$\begin{aligned}v_2=&\left\{\frac{1}{4}\left(-\frac{39}{5}+\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)-\frac{11}{10}y\right)^2\right.\\
&\left.\times\left(\frac{34}{5}-\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)+\frac{17}{20}y\right)\right\}t^4\\
&+\left\{\frac{2}{3}\left(2+\frac{y}{4}\right)\left(-\frac{39}{5}+\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)-\frac{11}{10}y\right)\right.\\
&\times\left(\frac{34}{5}-\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)+\frac{17}{20}y\right)+\frac{1}{3}\left(-\frac{39}{5}\right.\\
&\left.\left.+\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)-\frac{11}{10}y\right)^2\left(1+\frac{4}{5}x\right)\right\}t^3\\
&\left\{-\frac{106079}{8000}+\frac{1}{10000}x-\frac{1}{2}\left(2+\frac{y}{4}\right)^2\left(\frac{34}{5}\right.\right.\end{aligned}$$

$$-\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)+\frac{17}{20}y\Bigg)-\left(2+\frac{y}{4}\right)\Bigg(-\frac{39}{5}$$

$$+\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)-\frac{11}{10}y\Bigg)\left(1+\frac{4}{5}x\right)$$

$$+\frac{17}{10}\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)-\frac{187}{100}y\Bigg\}t^2$$

$$+\left\{\frac{34}{5}-\left(2+\frac{y}{4}\right)^2\left(1+\frac{4}{5}x\right)+\frac{17}{20}y\right\}t+1+\frac{4}{5}x. \quad (53)$$

In the same manner, the rest of the components of the iterations (Equations 48 and 49) can be obtained using symbolic packages, such as Maple.

In view of the authors' calculations and setting $x = 0.1$ and $y = 0.1$, one finds:

$$\begin{aligned} u_3 = & -31.71038918t^{13} + 99.44056808t^{12} \\ & -172.6703554t^{11} + 233.5480371t^{10} \\ & -256.9512729t^9 + 231.8344982t^8 \\ & -175.6385484t^7 + 113.4355688t^6 \\ & -63.51254061t^5 + 29.69521318t^4 \\ & -11.7846595t^3 + 5.08162608t^2 \\ & -3.481325t + 2.025, \end{aligned} \quad (54)$$

$$\begin{aligned} v_3 = & 31.71038918t^{13} - 99.44056808t^{12} \\ & +172.6703554t^{11} - 233.5480371t^{10} \\ & +256.9512729t^9 - 231.8344982t^8 \\ & +175.6385484t^7 - 113.4355688t^6 \\ & +62.06380614t^5 - 27.93239187t^4 \\ & +10.10445582t^3 - 3.34069358t^2 \\ & +2.456325t + 1.08. \end{aligned} \quad (55)$$

Figure 1 shows the approximates $u_3(0.1, 0.1, t)$ and $v_3(0.1, 0.1, t)$ of the solutions $u(x, y, t)$ and $v(x, y, t)$ in $x = 0.1$, $y = 0.1$ and $0 \le t \le 1$.

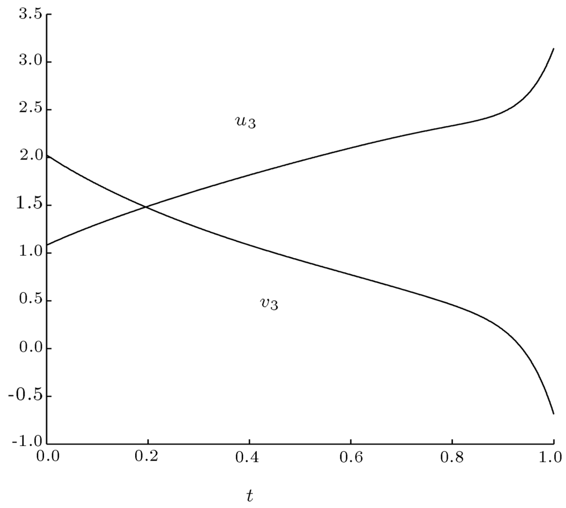

Figure 1. Approximations of $u(0.1, 0.1, t)$ and $v(0.1, 0.1, t)$ by $u_3(0.1, 0.1, t)$ and $v_3(0.1, 0.1, t)$, respectively.

CONCLUSION

Mathematical physics and population growth models, characterized by systems of partial differential equations, such as shallow water waves, the Brusselator model, the Lotka-Volterra model and the Belousov-Zhabotinski reduction model [30], are of wide applicability. The aim of this work has been achieved by formally deriving exact analytical solutions and by obtaining analytical approximations with a high degree of accuracy. In this paper, the variational iteration method has been successfully applied to find the solution of problems (Examples 1 to 3). With attention to the efficiency of the variational iteration method for Example 1 to 3, this method was applied for an analytical approximation solution of the reaction-diffusion Brusselator model.

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