Scientia Iranica, Vol. 15, No. 2, pp 175–181

SCIENTIA
IRANICA

Analytical Mode Distributions and Resonant Frequencies in Ladder Networks Used in Power Combining Oscillator Arrays

M.H. Akbarpour[1], A. Banai[1] and F. Farzaneh*

An analysis of resonant modes in ladder networks is performed using difference equations. Linear and loop configurations are studied and analytical relations are derived for the mode distributions and resonant frequencies of these networks, having arbitrary elements. Furthermore, it is shown that a) Modes with an exponential distribution along network nodes may exist in some cases and b) The mode distributions in loop networks are independent of network elements. Finally, a simple criterion is obtained for a loop structure to control the spacing between resonant frequencies.

INTRODUCTION

LADDER networks are used in strongly coupled oscillator networks for power combining [1-5]. In these arrays, oscillators, having nonlinear elements as their active parts and passive elements as their resonator and load, are coupled using coupling networks. The coupling of several oscillators in this manner results in the multimode oscillation of these arrays.

An analysis of these modes and their stability has been performed, using averaging and specifically averaged potential [1,6,7], which is a perturbational method. In this method, mode frequencies and distribution are determined by the analysis of a linear lossless network, obtained by removing active and resistive (lossy) elements of the network, and the stability of modes is studied using an averaged potential function. A mode analysis of these networks has been performed in previous works for some special cases. For example in [1], the networks having transmission lines as their coupling networks are studied and, in [7], two-dimensional networks, having lumped inductors as coupling elements, are analyzed.

In this paper, a method for determining the resonant frequencies and mode distribution of ladder networks is proposed, using difference equations. Mode distributions obtained by this method can be used for stability analysis in oscillator networks, obtained by adding nonlinear and other lossy elements to the circuit, using the averaged potential method. Using difference equations, analytical relations can be derived for resonant frequencies and for mode distributions in networks having arbitrary resonators and coupling networks.

Linear and loop configurations will be studied and it will be shown that resonant modes with an exponential mode distribution can exist in networks having a linear configuration, which were previously often neglected.

In [8] these distributions are studied in forced ladder networks, but in self oscillating networks, these modes are neglected. Furthermore, it will be shown that, in networks having a loop configuration, mode distributions are independent of resonator type and coupling elements. Using this property in these networks, it is easy to obtain simple criteria for controlling the separation between mode frequencies for cases in which single mode oscillation is desired. Also, having analytical results makes it easier to analyze the stability of modes in oscillator networks having this configuration.

NETWORKS HAVING LINEAR CONFIGURATION

A ladder network is shown in Figure 1. Suppose that this network is composed of lossless reactive elements. This network has n resonators with node voltages, $v_1, v_2, \cdots, v_n$. These resonators are elements having input admittance y_r and coupling networks, which are

1. *Department of Electrical Engineering, Sharif University of Technology, Tehran, I.R. Iran.*
*. *Corresponding Author, Department of Electrical Engineering, Sharif University of Technology, Tehran, I.R. Iran.*

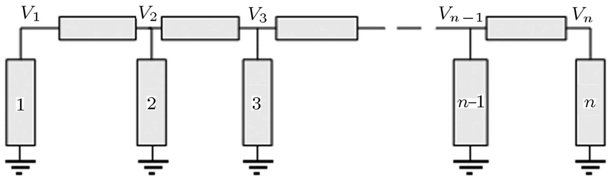

Figure 1. Ladder network with linear configuration.

reciprocal and symmetrical two-port networks with an admittance matrix as follows:

$$Y = \begin{bmatrix} y_s & y_m \\ y_m & y_s \end{bmatrix}. \tag{1}$$

Throughout the analysis, it is supposed that all resonators are the same and that coupling networks are identical. Note that it is not necessary to know y_r, y_s and y_m, analytically. These parameters may be obtained from simulation or measurement data. Supposing $y_m \neq 0$, one can write KCL for network nodes as follows:

$$V_{p+1} + K_2 V_p + V_{p-1} = 0, \qquad p=2,3,\cdots,n-1, \tag{2a}$$

$$K_1 V_1 + V_2 = 0, \tag{2b}$$

$$K_1 V_n + V_{n-1} = 0, \tag{2c}$$

in which:

$$K_l = \frac{y_r + l y_s}{y_m}. \tag{3}$$

Equation 2a is a second order difference equation and is valid for inner network nodes. Equations 2b and 2c are valid for boundary nodes, which can be used as a boundary condition for the difference equation system.

To solve the difference equation system of Equation 2a, its characteristic equation is written as [9]:

$$s^2 + K_2 s + 1 = 0. \tag{4}$$

This equation is a polynomial equation of second degree with real coefficients, because K_2 is equal to the ratio of two pure imaginary numbers. If one denotes two roots of Equation 4 as s_1 and s_2, one must have $s_1 s_2 = 1$. Using this and the fact that the coefficients of the characteristic equation are real numbers, it is easy to show that the roots of the characteristic equation can uniquely be on the locus shown in Figure 2. This locus contains a real axis and the circle $|s| = 1$.

Depending on the position of the roots, there are five possible types of mode in the circuit:

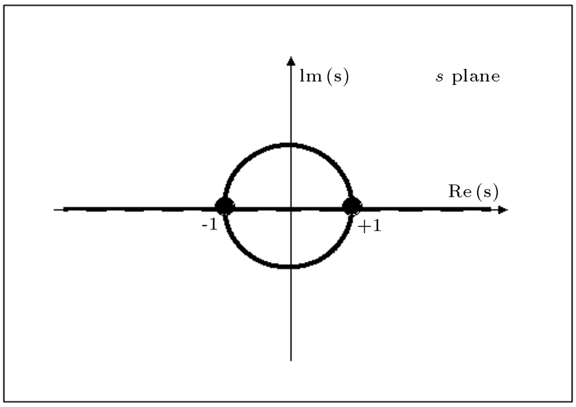

Figure 2. Locus of possible values for s_1 and s_2.

1. $s_1 = s_2 = 1$; in this case, the solution of the difference equation system can be written as a linear function of node indices:

$$v_p = A + Bp. \tag{5}$$

Furthermore, using the sum of the roots, one must have:

$$K_2 = -2, \tag{6}$$

2. $s_1 = s_2 = -1$; the solution of equations in this case takes the form of a linear function of p, but with an alternating sign at network nodes i.e.:

$$v_p = (A + Bp)(-1)^p, \tag{7}$$

with:

$$K_2 = 2, \tag{8}$$

3. $s_1 = e^{\alpha}, s_2 = e^{-\alpha}$ in which α is a real positive number. In this case, the solution of the difference equation can be written as:

$$V_p = A\cosh(\alpha p) + B\sinh(\alpha p), \tag{9}$$

with:

$$K_2 = -2\cosh\alpha, \tag{10}$$

4. $s_1 = -e^{\alpha}, s_2 = -e^{-\alpha}$; again, α is a real positive number in this case. The solution under this condition takes the following form:

$$V_p = (A\cosh(\alpha p) + B\sinh(\alpha p))(-1)^p, \tag{11}$$

with:

$$K_2 = -2\cosh\alpha, \tag{12}$$

5. $s_1 = e^{j\beta}, s_2 = e^{-j\beta}$, in which $0 < \beta < \pi$. In this

case, mode distribution takes the sinusoidal form as:

$$V_p = A\cos(\beta p) + B\sin(\beta p), \tag{13}$$

with:

$$K_2 = -2\cos\beta. \tag{14}$$

In all the above cases, constants A and B are determined from boundary conditions. Using boundary conditions obtained from Equations 2b and 2c and applying the conditions that A and B have nonzero solutions, another equation is obtained between the frequency and mode distribution.

The results are summarized in Table 1. In this table, the parameter, L, is defined as follows:

$$L = \frac{y_s}{y_m}. \tag{15}$$

It can be seen that this parameter depends only on coupling networks.

NETWORKS WITH LOOP CONFIGURATION

The network having a loop configuration is shown in Figure 3. In this network, node equations are the same as equations for the inner nodes in a linear configuration. The only difference between this configuration and the linear configuration is the boundary condition. In a loop network, there are no boundary nodes, but the following condition must be satisfied:

$$V_p = V_{p+n}. \tag{16}$$

Using this condition with the five possible cases mentioned for a linear structure, it can be shown that the modes with exponential distribution cannot exist in these networks.

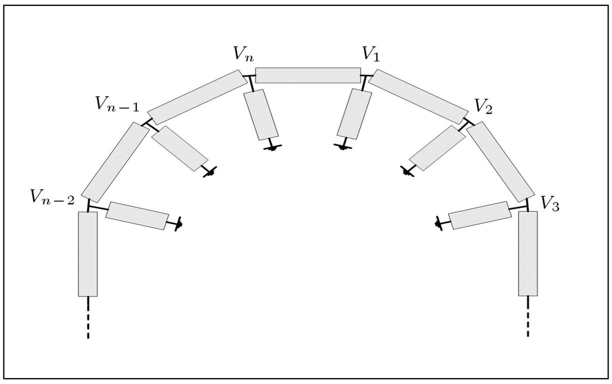

Figure 3. The network with loop configuration.

Another interesting result is that, in the loop configuration, mode distributions can be obtained independent of network elements, because values of β, which satisfy Equation 16, are only dependent on the number of nodes in the circuit. The possible modes and their distributions are summarized in Table 2. In this case, for sinusoidal modes there are two independent constants, i.e. A and B. This is because of the rotational symmetry of the network. This means that the numbering of network nodes can start from anywhere in the loop.

This result can be used in power combining arrays having a loop configuration. A simple criterion is obtained for the separation of mode frequencies from a center frequency. The nearest resonant frequency to the frequency of a power combining mode is obtained by solving the following equation:

$$K_2 = -2\cos(\beta_1) = -2\cos(2\pi/n). \tag{17}$$

Typical variations of K_2 versus frequency are depicted

Table 1. Possible modes in linear ladder network.

Characteristic Equation Roots s_2	Characteristic Equation Roots s_1	Mode Distribution	First Equation	Second Equation
1	1	A (constant)	$K_2 = -2$	$L = -1$
		$A\left(1 - \frac{2}{n+1}p\right)$	$K_2 = -2$	$L = \frac{1+n}{1-n}$
-1	-1	$A(-1)^P$	$K_2 = 2$	$L = 1$
		$A(-1)^p\left(1 - \frac{2}{n+1}p\right)$	$K_2 = 2$	$L = \frac{n+1}{n-1}$
$e^{-\alpha}$	e^{α}	$A\left[\cosh(\alpha p) - \frac{1+L\cosh\alpha}{L\sinh\alpha}\sinh(\alpha p)\right]$	$K_2 = -2\cosh\alpha$	$L^2\sinh(\alpha(n-1)) + 2L\sinh(\alpha n) + \sinh(\alpha(n+1)) = 0$
$-e^{-\alpha}$	$-e^{\alpha}$	$A(-1)^p\left[\cosh(\alpha p) + \frac{1-L\cosh\alpha}{L\sinh\alpha}\sinh(\alpha p)\right]$	$K_2 = 2\cosh\alpha$	$L^2\sinh(\alpha(n-1)) - 2L\sinh(\alpha n) + \sinh(\alpha(n+1)) = 0$
$e^{-j\beta}$	$e^{j\beta}$	$A\left[\cos(\beta p) - \frac{1+L\cos\beta}{L\sin\beta}\sin(\beta p)\right]$	$K_2 = -2\cos\beta$	$L^2\sin(\beta(n-1)) + 2L\sin(\beta n) + \sin(\beta(n+1)) = 0$

Table 2. Possible modes in loop network.

Characteristic Equation Roots		Mode Distribution	Mode Equation
s_2	s_1		
1	1	A (constant)	$K_2 = -2$
-1	-1	$A(-1)^p$ (for even n)	$K_2 = 2$
$e^{-j\beta}$	$e^{j\beta}$	$V_p = A\cos(\beta p) + B\sin(\beta p)$	$K_2 = -2\cos\beta_k$ $\beta_k = \frac{2k\pi}{n}, \begin{cases} k = 1, 2, \cdots, n/2 - 1 \text{ for even } n \\ k = 1, 2, \cdots, (n-1)/2 \text{ for odd } n \end{cases}$

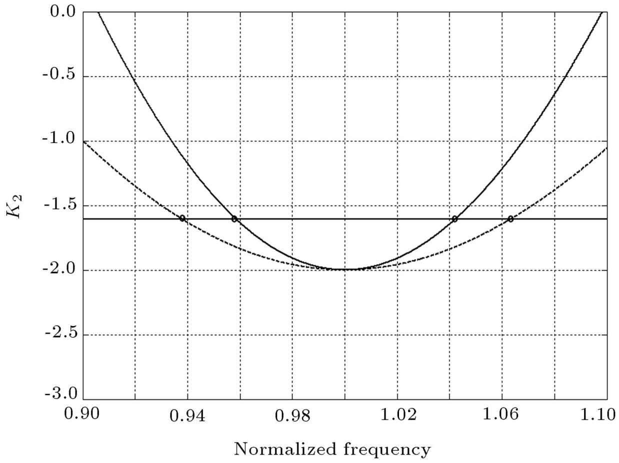

Figure 4. The effect of the changes of K_2 on the separation of undesired mode frequencies from the power combining mode frequency.

in Figure 4. In this figure, the horizontal axis is the axis of frequency normalized to a power combining mode frequency. The frequency of the nearest mode to the power combining mode can be obtained from the points on the K_2 curve, in which Equation 17 is satisfied. These points are the intersection of the K_2 curves with the horizontal line, which has the value of $-2\cos(\beta_1)$ on the vertical axis. The dashed curve changes faster with frequency and, thus, the separation of modes for this curve is less, compared to the solid curve. As can be seen from this figure and by using Equation 17, a simple criterion can be obtained for the separation of undesired mode frequencies from the power combining mode frequency, that is to say, the slower the change of K_2 with frequency, the more the separation between the undesired mode frequencies and the power combining mode frequency.

SIMULATIONS AND COMPARISON

The first network to be studied is shown in Figure 5. This network has parallel LC resonators and transmission lines as its coupling networks. The length of the transmission lines connecting the resonators is $\theta_0 = 2\pi$ at $\omega_0 = 1/\sqrt{L_0C_0}$, in which L_0 and C_0 are the resonant circuit inductance and capacitance, respectively. This network can be used for power combining arrays. First, it is shown that the exponential modes cannot exist in the array having single transmission lines as their coupling networks. For this, consider the second equation for exponential modes. These equations are polynomial equations of a second degree in L. Defining the variable, T, as $L = T - 1$, for the case in which s_1 and s_2 have positive values and rewriting the equation in T, it takes the following form:

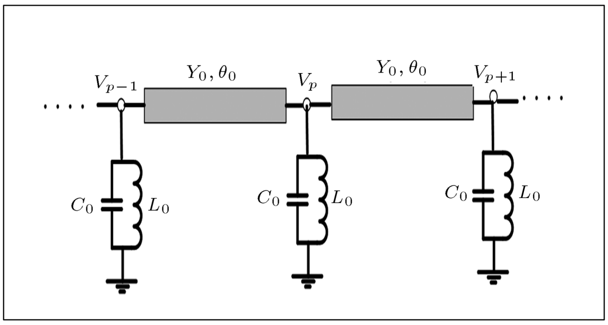

Figure 5. Network with LC resonators and transmission line coupling.

$$T^2[\sinh(\alpha(n-1))] + 4T[\cosh(\alpha(n-1/2))\sinh(\alpha/2)] + 4\sinh(\alpha n)\sinh^2(\alpha/2) = 0. \quad (18)$$

It is easy to show that the solutions of this equation are negative numbers and, thus, solutions for the L must satisfy $L_{1,2} < -1$. In the same way and by defining $L = T + 1$ for the case in which s_1 and s_2 have negative values, it can be shown that, in this case, the solutions for the L must satisfy $L_{1,2} > 1$.

In the network with transmission line coupling, one has:

$$y_s = -jY_0 \cot\left(\frac{\theta_0\omega}{\omega_0}\right), \quad (19)$$

$$y_m = jY_0 \csc\left(\frac{\theta_0\omega}{\omega_0}\right), \quad (20)$$

$$L = \frac{y_s}{y_m} = -\cos(\theta). \tag{21}$$

By the last equation, it is evident that $-1 \leq L \leq 1$ and, hence, the exponential mode cannot exist in this network. The only possible modes are the constant amplitude and sinusoidal modes.

A six element network is analyzed. The values selected for the network elements are $L_0 = 159$ pH, $C_0 = 159$ pF and $Y_0 = 1/70$ S, so, the resonant frequency of the resonators is equal to $f_0 = \omega_0/2\pi = 1$ GHz.

To verify results obtained from the difference equations, two other methods were used. The first method is a time domain simulation. The network is excited using an initial condition to excite all modes. The frequency spectrum of the node voltages has peaks in the position of resonant frequencies.

As the second method, matrix equations obtained from KCL at the nodes were used [1]. This matrix equation takes the following form:

$$\begin{bmatrix} K_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & K_2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & K_2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & K_2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & K_1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_{n-1} \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \tag{22}$$

in which $A_i s$ are the amplitudes of oscillation in nodes for a specific mode. This equation has nonzero solutions only when the determinant of the coefficient matrix is equal to zero. The mode distribution at each resonant frequency can be obtained by the eigenvector associated with the zero eigenvalue of the coefficient matrix. The zeros of the determinant of the coefficient matrix can be found by plotting the determinant of the matrix as a function of the frequency, as depicted in Figure 6.

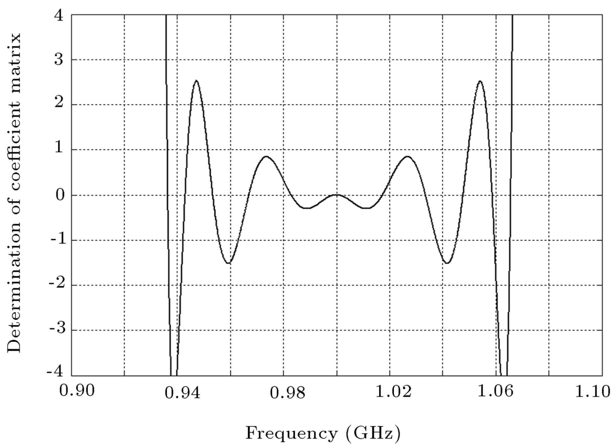

Figure 6. Determinant of coefficient matrix of Equation 22 as a function of frequency.

The solution of the equations in Table 1 can be obtained from the contours of the equation in the $f-\beta$ plane for sinusoidal modes. These contours are plotted in Figure 7. The intersections of two contours give the mode frequencies and the values of β corresponding to each mode. The results for resonant frequencies and the mode distributions of the six element network obtained from the foregoing methods are given in Table 3. As can be seen from Table 3, excellent agreement exists between the results of the resonant frequencies and the mode distribution obtained from these methods.

Another network is analyzed to confirm the existence of exponential modes. In this network, the resonators are the same as in the previous example, but the coupling networks are Π sections, as shown in Figure 8. This Π section consists of a transmission line with a length of $\theta_0 = 2\pi$ at ω_0 and the characteristic admittance, Y_0, and two open ended stubs at two ends having a length of $\theta_{s0} = \pi$ at ω_0 and the characteristic admittance, $Y_{s0} = Y_0$. The quantity, L, for this network is obtained as follows:

$$L = 1 - 2\cos\left(\frac{\theta_0\omega}{\omega_0}\right). \tag{23}$$

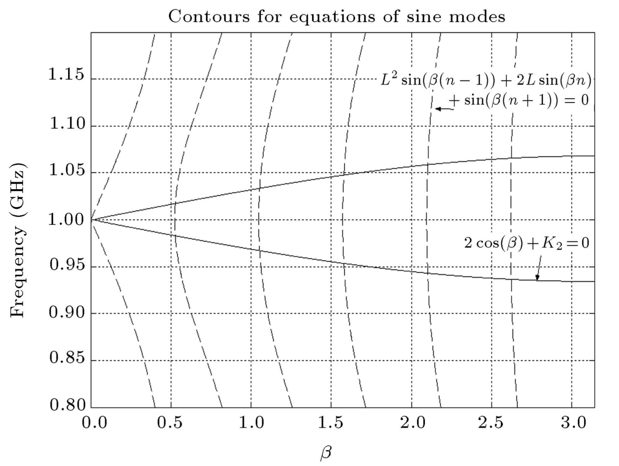

Figure 7. Contours of equations for sinusoidal modes.

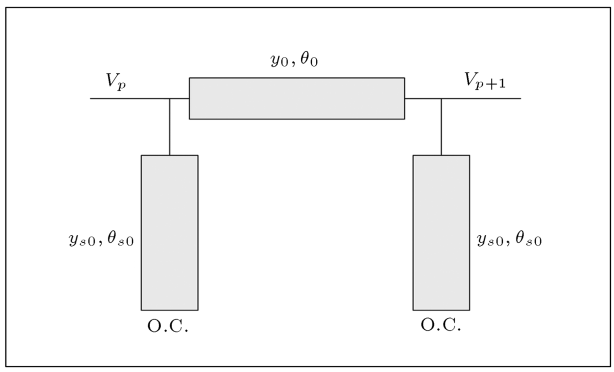

Figure 8. Π section as coupling network.

Table 3. Simulation results for a 6-node linear structure.

Resonant Frequencies (GHz)			Matrix Equation						Difference Equation						
Difference Equation	Matrix Equation	Time Domain Simulation	Mode Amplitude Distributions						Mode Amplitude Distribution						β
0.9364	0.9365	0.9363	0.1539	-0.4090	0.5559	-0.5559	0.4090	-0.1539	0.1538	-0.4086	0.5556	-0.5559	0.4095	-0.1550	2.622
0.943	0.943	0.943	0.2954	-0.5752	0.2861	0.2861	-0.5752	0.2954	0.2952	-0.5751	0.2865	0.2854	-0.5751	0.2963	2.101
0.9534	0.9534	0.9535	0.4140	-0.4024	-0.4082	0.4082	0.4024	-0.4140	0.4140	-0.4024	-0.4082	0.4083	0.4024	-0.4141	1.578
0.967	0.967	0.967	0.5026	0.0054	-0.4973	-0.4973	0.0054	0.5026	0.5026	0.0053	-0.4974	-0.4972	0.0056	0.5027	1.0535
0.9829	0.9829	0.9826	0.5570	0.4090	0.1499	-0.1499	-0.4090	-0.5570	0.5570	0.4090	0.1498	-0.1499	-0.4090	-0.5570	0.5272
1.000	1.000	1.000	0.4082	0.4082	0.4082	0.4082	0.4082	0.4082	0.4082	0.4082	0.4082	0.4082	0.4082	0.4082	——
1.0175	1.0173	1.017	0.5570	0.4090	0.1499	-0.1499	-0.4090	-0.5570	0.5570	0.4090	0.1499	-0.1499	-0.4090	-0.5570	0.5272
1.0335	1.0336	1.0333	0.5027	0.0056	-0.4972	-0.4972	0.0056	0.5027	0.5027	0.0056	-0.4972	-0.4974	0.0053	0.5026	1.0535
1.048	1.0477	1.0473	0.4143	-0.4021	-0.4082	0.4082	0.4021	-0.4143	0.4146	-0.4018	-0.4088	0.4077	0.4029	-0.4135	1.578
1.059	1.0587	1.0583	0.2958	-0.5751	0.2859	0.2859	-0.5751	0.2958	0.2959	-0.5751	0.2858	0.2861	-0.5751	0.2956	2.101
1.0655	1.0657	1.0653	0.1542	-0.4090	0.5558	-0.5558	0.4090	-0.1542	0.1541	-0.4089	0.5557	-0.5558	0.4092	-0.1546	2.622

Table 4. Frequencies and mode distributions of exponential modes for network having Π-section coutpling networks.

f (GHz)	Mode Distribution													
	Matrix Equations							Difference Equations						
0.49601	-0.6685	0.2207	-0.0663	0.0000	0.0663	-0.2207	0.6685	-0.6707	0.2214	-0.0665	0.0002	0.0660	-0.2200	0.6663
0.49602	-0.6648	0.2238	-0.0818	0.0490	-0.0818	0.2238	-0.6648	-0.6690	0.2252	-0.0822	0.0490	-0.0814	0.2224	-0.6607

As can be seen, the exponential modes would exist in this network. An analysis of this network confirms the existence of this mode. The contour for the exponential mode equations in Table 1 is shown in Figure 9 for a 7-element network. This contour shows that these modes exist at two close frequencies. The frequencies and mode distributions obtained from the difference equations are compared with the matrix equation results shown in Table 4. Although this mode is far away from the center frequency in this case, this example shows that the exponential modes are physical modes and can exist in some circuits. The existence of these modes can affect the stability of other modes in some cases.

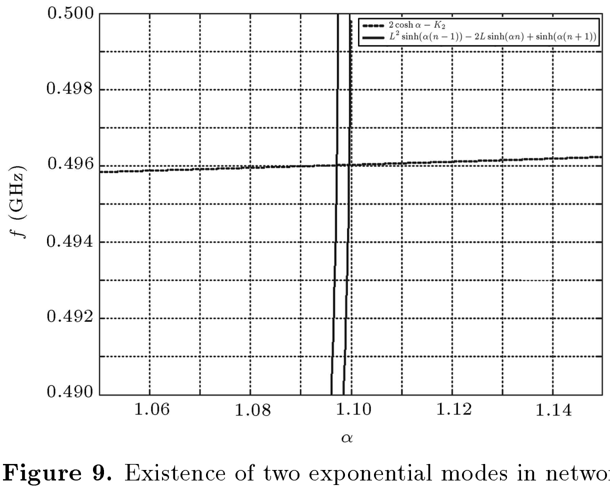

Figure 9. Existence of two exponential modes in network having coupling networks as in Figure 6.

CONCLUSION

A new method was proposed for the analysis of resonant modes in ladder networks intended for linear and loop structure power combiners. It is shown that modes having exponential distributions can exist in linear structures and it is shown that these modes are physical modes and must be considered in the stability analysis of the oscillator networks in certain cases.

In the case of the loop structure, it was shown that the mode distributions are dependent only on the number of oscillator elements. The resonant frequencies are dependent upon the elements themselves. Using this fact, a criterion was obtained to evaluate the separation of undesired mode frequencies from the power combining mode frequency.

Using this method, analytic relations were obtained to evaluate the resonant frequencies and the mode amplitude distributions. These distributions were compared with the numerical results obtained from the matrix equation method.

REFERENCES

1. Nogi, S., Lin, J. and Itoh, T. "Mode analysis and stabilization of a spatial power combining array with strongly coupled oscillators", *IEEE Trans. MTT*, **41**(10), pp 1827-1837 (Oct. 1993).
2. Lin, J. and Itoh, T. "Two-dimensional quasi-optical power-combining arrays using strongly coupled oscillators", *IEEE Trans. MTT*, **42**(4), pp 734-741 (Apr. 1994).
3. Lin, J. and Itoh, T. "A 4 × 4 spatial power-combining array with strongly coupled oscillators in multi-layer structure", *IEEE MTT-S Digest*, pp 607-610 (1993).
4. Mortazawi, A. and De Loach, B.C. "Multiple element oscillators utilizing a new power combining technique", *IEEE Trans. MTT*, **40**(12), pp 2397-2402 (Dec. 1992).
5. Mortazawi, A., Foltz, H.D. and Itoh, T. "A periodic second harmonic spatial power combining oscillator", *IEEE Trans. MTT*, **40**(5), pp 851-856 (May 1992).
6. Kuramitsu, M. and Takase, F. "Analytical method for multimode oscillators using the averaged potential", *Elec. and Comm. In Japan*, **66-a**(4), pp 10-19 (1983).
7. Kuramitsu, M. and Takase, F. "The analysis of the square array of van der Pol oscillators using the averaged potential", *Nonlinear Mechanics*, **20**(56), pp 395-406 (1985).
8. Migulin, V.V., Medvedev, V.I., Mustel, E.R. and Parygin, V.N., *Basic Theory of Oscillations*, Moscow, Mir Publishers (1983).
9. Godunov, S.K., *Difference Schemes: An Introduction to the Underlying Theory*, Elsevier science publications (1987).