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A Meshless Boundary Element Method Formulation for Transient Heat Conduction Problems with Heat Sources

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In the boundary element formulation of heat conduction, the heat source effect imposes an additional domain integral term on the system of integral equations. With this term, an important advantage of the boundary element method as a boundary-only formulation will be lost. This paper presents an accurate method for the evaluation of heat source domain integrals, with no need of domain discretization. Transformation of the domain integral into the corresponding boundary integral is carried out using Green's theorem. Both time-dependent and time-independent fundamental solutions are considered. The methodology can be implemented in general and for similar situations. Numerical examples will be presented to demonstrate the accuracy and efficiency of the presented method.

INTRODUCTION

The Boundary Element Method (BEM) is an efficient and accurate numerical technique for heat conduction analysis. The BEM formulation of heat conduction includes several boundary and domain integrals. One domain integral is associated with time rate formulation in the form of initial conditions, and the other one includes the contribution due to heat sources. The direct numerical evaluation of domain integrals necessitates the discretization of the domain into internal elements. Although domain discretization does not introduce additional unknowns to the final equations, an important advantage of the boundary element method as a boundary only formulation will disappear. Hence, a method is required to transform the domain integral to the boundary. For the transformation of such integrals in BEM, several schemes have been proposed.

For a steady state analysis of the heat conduction problem, the Dual Reciprocity Method (DRM) and the Multiple Reciprocity Method (MRM) were presented by Nardini & Brebbia [1] and Nowak & Brebbia [2,3], respectively. In addition, Tang et al. [4] used the Fourier series to represent particular solutions in order to transform integrals to the boundary.

For the transient analysis of heat conduction problems, two classes of formulation have been considered. In the first class [5-9], a time-dependent fundamental solution is used, and in the second class (including DRM [10] and MRM [11], particular integrals [12] and the radial integration method [13]), a time-independent fundamental solution is used. In DRM [10], the domain terms are approximated by a series of localized particular solutions. This approximation permits the domain integral to be converted into boundary integrals by a simple integration by parts. In MRM [11], high order fundamental solutions are used. Although DRM and MRM are computationally cost effective for transient analysis, they do not include the advantages of the time dependent fundamental solution. In particular, when time steps are selected to be small, DRM and MRM show unstable behavior. Yang et. al [12] employed the particular integrals method for the transformation domain integrals of a heat conduction problem to boundary integrals. Gao [13] used the radial integration method (RIM) for the transformation of domain integrals into boundary integrals. In RIM, the integrand of the domain integral is expressed in a radial form. The domain integral is, then converted to a one-dimensional integral and a boundary integral.

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In other fields of analysis, the transformation of the domain integrals has played an efficient role. In elasticity, researchers have introduced several schemes for transforming the domain integrals. Among these schemes is the Galerkin tensor method, which was introduced by Cruse [14] and which was applied to a limited range of body force problems. Danson [15] has also presented a unified treatment for the transformation of body force domain integrals into surface integrals, using the concept of the Galerkin vector. Rizzo and Shippy [16] and Karami and Kuhn [17] presented an efficient method based on replacing body force and temperature changes by a scalar potential function. Dual reciprocity and multiple reciprocity techniques were presented by Nardini and Brebbia [18] and Neves and Brebbia [19]. A particular integral concept in boundary element modeling, in conjunction with employing a global shape function for the temperature distribution, was presented by Henry & Banerjee [20]. Other solutions include the exact transformation of anisotropic domain integrals (see for example [21]).

In this paper, by using Green's lemma, the domain integrals associated with non-uniform heat sources are transformed to the boundary. The method to be presented here is general enough to be used by employing both either time-dependent or time-independent fundamental solutions. The method is also successful in solving problems with multiple connected regions and can be extended to other two dimensional problems with arbitrary domain loading or non-uniform body forces.

It is clear that the method to be presented here is more complex, from a mathematical point of view, than conventional BEM. However, because of the meshless nature of the present approach, the method will be much simpler from a user's point of view.

THE BOUNDARY INTEGRAL EQUATION OF TRANSIENT HEAT CONDUCTION

The governing equation of transient heat conduction with constant material parameters is expressed as follows:

$$\nabla^2 T(X,t) + \frac{1}{k} g(X,t) = \frac{1}{\alpha}\frac{\partial}{\partial t} T(X,t), \tag{1}$$

where $T(X,t)$ is the temperature at point X at time t. The thermal diffusivity, α, is equal to $\frac{k}{\rho c}$, in terms of conductivity, k, density, ρ and specific heat, c, respectively. $g(X,t)$ is the heat source function and is considered to be a function of position and time. The boundary integral form of Equation 1, for a problem with domain Ω and boundary Γ, can be expressed as follows [5]:

$$\begin{aligned} C(P)T(P,t_f) &+ \alpha \int_\Gamma \int_{t_0}^{t_f} T(Q,\tau) q^*(P,t_f;Q,\tau) d\tau d\Gamma \\ &= \alpha \int_\Gamma \int_{t_0}^{t_f} T(Q,\tau) T^*(P,t_f;Q,\tau) d\tau d\Gamma \\ &+ \frac{\alpha}{k} \int_\Omega \int_{t_0}^{t_f} g(Q,\tau) T^*(P,t_f;Q,\tau) d\tau d\Omega \\ &+ \int_\Omega T(Q,t_0) T^*(P,t_f;Q,t_0) d\Omega, \end{aligned} \tag{2}$$

where P and Q are, respectively, source and field points within the domain or on the boundary and $C(P)$ is a coefficient, related to the local geometry at point P. The initial and final time are represented by t_0 and t_f, respectively. T^* is the time dependent fundamental solution for the diffusion equation, which has the following form [22]:

$$T^*(P,t;Q,\tau) = \frac{1}{4\pi\alpha(t-\tau)} \exp\left[\frac{-r^2}{4\alpha(t-\tau)}\right] H(t-\tau),$$

where H is the Heaviside function. In the above equation, r is the Euclidian distance between P and Q. The normal derivative of the fundamental solution is denoted by q^*, where:

$$\begin{aligned} q^*(P,t;Q,\tau) &= \frac{\partial}{\partial n} T^*(P,t;Q,\tau) \\ &= \frac{-r}{8\pi\alpha^2(t-\tau)^2} \exp\left[\frac{-r^2}{4\alpha(t-\tau)}\right] H(t-\tau)\frac{\partial r}{\partial n}. \end{aligned}$$

There are two domain integrals in Equation 2, which include the contribution, due to the heat source and initial conditions. If one assumes that the domain integral associated with the initial condition vanishes (i.e., the initial temperature field is constant or stationary [7]), by assuming $t_0 = 0$ and $t_f = t_{n+1}$ (t_n: current time, t_{n+1}: new time), one may obtain the following:

$$\begin{aligned} C(P)T(P,t_{n+1}) &+ \alpha \int_\Gamma \int_{t_n}^{t_{n+1}} T(Q,\tau) q^*(P,t_{n+1};Q,\tau) d\tau d\Gamma \\ &= \alpha \int_\Gamma \int_{t_n}^{t_{n+1}} q(Q,\tau) T^*(P,t_{n+1};Q,\tau) d\tau d\Gamma \\ &+ \frac{\alpha}{k} \int_\Omega \int_0^{t_{n+1}} g(Q,\tau) T^*(P,t_{n+1};Q,\tau) d\tau d\Omega \\ &+ \alpha \int_\Gamma \int_0^{t_n} [q(Q,\tau) T^*(P,t_{n+1};Q,\tau) \\ &- T(Q,\tau) q^*(P,t_{n+1};Q,\tau)] d\tau d\Gamma. \end{aligned} \tag{3}$$

In the above equation, the only domain integral is the third integral that is as follows:

$$I_{g_{n+1}}=\frac{\alpha}{k}\int_{\Omega}\int_{0}^{t_{n+1}}g[Q(x_1,x_2),\tau]$$
$$T^*[P,t_{n+1};Q(x_1,x_2),\tau]d\tau d\Omega, \tag{4}$$

where (x_1,x_2) indicate the coordinates of the field point. For a direct numerical evaluation of the domain integral (Equation 4), the domain under consideration should be discretized. In such a case, the important advantage of the boundary element method as a boundary solution technique would be considerably lost. In this paper, an attempt is made to transfer the domain integral, $I_{g_{n+1}}$, to the boundary, in order to derive a boundary-only formulation.

EVALUATION OF THE HEAT SOURCE INTEGRAL

In this section, an accurate and relatively efficient method to evaluate the heat source domain integral is presented. To do so, Green's theorem is used, which relates the domain integral of a two-dimensional region to the boundary. Green's theorem is expressed as follows [23]:

$$\int_{\Omega}\frac{\partial}{\partial x_1}R(x_1,x_2)d\Omega=\int_{\Gamma}R(x_1,x_2)dx_2.$$

This relation can be applied to simply or multiply connected regions, provided that the function, S, is continuous at all points on Ω and on the boundary, Γ. By setting $R(x_1,x_2)$ as follows:

$$R(x_1,x_2)=\frac{\alpha}{k}\int_{a}^{x_1}\int_{0}^{t_{n+1}}g[Q(x',x_2),\tau]$$
$$T^*[P,t_{n+1};Q(x',x_2),\tau]d\tau dx,' \tag{5}$$

the following relation is obtained:

$$I_{g_{n+1}}=\frac{\alpha}{k}\int_{\Gamma}\int_{a}^{x_1}\int_{0}^{t_{n+1}}g[Q(x',x_2),\tau]$$
$$T^*[P,t_{n+1};Q(x',x_2),\tau]d\tau dx'dx_2. \tag{6}$$

The constant a in Equations 5 and 6 is an arbitrary constant. As a suitable value for a, one can set the following:

$$a=\frac{x_{1\,\min}+x_{1\,\max}}{2},$$

where $x_{1\,\min}$ and $x_{1\,\max}$ are, respectively, the minimum and maximum values of the first coordinate of the boundary points.

As can be seen, the time integral in Equation 6 must be calculated from zero to t_{n+1}, the fact of which causes too much computation, which, subsequently, becomes ineffective and expensive. To overcome this difficulty, one can use the fact that the fundamental solution decays with time and, therefore, one can eliminate several initial time steps. The same procedure can also be used to evaluate the last integral in Equation 3 [24]. In the present study, for better accuracy, all time steps are considered for evaluation of the time integral. For cases where the heat generation function is time independent, this difficulty can be effectively eliminated. In this respect, one can write as follows:

$$I_{g_{n+1}}-I_{g_n}=\frac{\alpha}{k}\int_{\Gamma}\int_{a}^{x_1}\int_{0}^{t_{n+1}}g[Q(x',x_2)]$$
$$\frac{\exp\left[\frac{-r^2(x',x_2)}{4\alpha(t_{n+1}-\tau)}\right]}{4\pi\alpha(t_{n+1}-\tau)}d\tau dx'dx_2$$
$$-\frac{\alpha}{k}\int_{\Gamma}\int_{a}^{x_1}\int_{0}^{t_n}g[Q(x',x_2)]$$
$$\frac{\exp\left[\frac{-r^2(x',x_2)}{4\alpha(t_n-\tau)}\right]}{4\pi\alpha(t_n-\tau)}d\tau dx'dx_2. \tag{7}$$

By introducing a new variable, $w=t_{n+1}-\tau$, for the first integral and $w=t_n-\tau$ for the second integral in Equation 7, one can write the following:

$$I_{g_{n+1}}-I_{g_n}=\frac{-1}{4\pi k}\int_{\Gamma}\int_{a}^{x_1}\int_{t_{n+1}}^{0}g[Q(x',x_2)]$$
$$\frac{\exp\left[\frac{-r^2(x',x_2)}{4\alpha w}\right]}{w}dwdx'dx_2$$
$$+\frac{1}{4\pi k}\int_{\Gamma}\int_{a}^{x_1}\int_{t_n}^{0}g[Q(x',x_2)]$$
$$\frac{\exp\left[\frac{-r^2(x',x_2)}{4\alpha w}\right]}{w}dwdx'dx_2,$$

or:

$$I_{g_{n+1}}=I_{g_n}+\frac{1}{4\pi k}\int_{\Gamma}\int_{a}^{x_1}\int_{t_n}^{t_{n+1}}g[Q(x',x_2)]$$
$$\frac{\exp\left[\frac{-r^2(x',x_2)}{4\alpha w}\right]}{w}dwdx'dx_2.$$

As seen, the contribution due to the heat generator can be computed by integration at only one time interval.

NUMERICAL IMPLEMENTATION

As T and q vary considerably more slowly than T^* and q^*, one can assume that they are constant over a small interval of time, ($\Delta t = t_{n+1} - t_n$). Hence, q and T may be taken out of the time integration. By this formulation, only the geometrical boundary must be discretized. If boundary Γ is discretized into N linear boundary elements, the integral Equation 3, may be written as:

$$C(P_i)T(P_i,t_{n+1}) + \alpha\sum_{j=1}^{N}\int_{\Gamma_j} T(Q_j,t_{n+1})\int_{t_n}^{t_{n+1}} q^*(P_i,t_{n+1};Q_j,\tau)d\tau d\Gamma_j = \alpha\sum_{j=1}^{N}\int_{\Gamma_j} q(Q_j,t_{n+1})\int_{t_n}^{t_{n+1}} T^*(P_i,t_{n+1};Q_j,\tau)d\tau d\Gamma + \alpha\sum_{m=1}^{n}\sum_{j=1}^{N}\int_{\Gamma_j}\left[q(Q_j,t_m)\int_{t_{m-1}}^{t_m} T^*(P_i,t_{n+1};Q_j,\tau)d\tau - T(Q_j,t_m)\int_{t_{m-1}}^{t_m} q^*(P_i,t_{n+1};Q_j,\tau)d\tau\right]d\Gamma_j + \frac{\alpha}{k}\sum_{m=1}^{n+1}\sum_{j=1}^{N}\int_{\Gamma_j}\int_a^{x_1}\int_{t_{m-1}}^{t_m} g[Q(x',x_2),\tau]T^*(P_i,t_{n+1};Q_j,\tau)d\tau dx' dx_2. \quad (8)$$

Subscript i is an index to represent the node number on the boundary.

The time integrals containing T^* and q^* can be evaluated, analytically as follows:

$$g_{ij} = \alpha\int_{t_n}^{t_{n+1}} T^*(P_i,t_{n+1};Q_j,\tau)d\tau = \frac{1}{4\pi}Ei\left(\frac{r^2}{4\alpha\Delta t}\right),$$

$$h_{ij} = \alpha\int_{t_n}^{t_{n+1}} q^*(P_i,t_{n+1};Q_j,\tau)d\tau = \frac{-1}{2\pi r}\exp\left(\frac{-r^2}{4\alpha\Delta t}\right)\frac{\partial r}{\partial n},$$

$$g_{ij}^m = \alpha\int_{t_{m-1}}^{t_m} T^*(P_i,t_{n+1};Q_j,\tau)d\tau = \frac{1}{4\pi}\left[Ei\left(\frac{r^2}{4\alpha(t_{n+1}-t_{m-1})}\right) - Ei\left(\frac{r^2}{4\alpha(t_{n+1}-t_m)}\right)\right],$$

$$h_{ij}^m = \alpha\int_{t_{m-1}}^{t_m} q^*(P_i,t_{n+1};Q_j,\tau)d\tau = \frac{1}{2\pi r}\left[\exp\left(\frac{-r^2}{4\alpha(t_{n+1}-t_{m-1})}\right) - \exp\left(\frac{-r^2}{4\alpha(t_{n+1}-t_m)}\right)\right]\frac{\partial r}{\partial n},$$

where $Ei()$ is the exponential integral function, i.e. as follows:

$$Ei(a) = \int_a^{\infty}\frac{\exp(-x)}{x}dx.$$

The last time integral in Equation 8 cannot be evaluated by analytical means. To evaluate this integral, numerically, constant elements over time are used, i.e. g and T^* are set to be constant over $t_{m-1} < \tau < t_m$ and are evaluated at midpoint ($\tau = t_m - 0.5\Delta t$). Consequently, Equation 6 takes the following form:

$$C(P_i)T(P_i,t_{n+1}) + \sum_{j=1}^{N}\int_{\Gamma_j} h_{ij}T(Q_j,t_{n+1})d\Gamma_j = \sum_{j=1}^{N}\int_{\Gamma_j} g_{ij}q(Q_j,t_{n+1})d\Gamma_j + \sum_{m=1}^{n}\sum_{j=1}^{N}\int_{\Gamma_j}\left[g_{ij}^m q(Q_j,t_m) - h_{ij}^m T(Q_j,t_m)\right]d\Gamma_j + \frac{\alpha\Delta t}{k}\sum_{m=1}^{n+1}\sum_{j=1}^{N}\int_{\Gamma_j}\int_a^{x_1} g[Q(x',x_2),t_m - 0.5\Delta t]T^*(P_i,t_{n+1};Q_j,t_m - 0.5\Delta t)dx'dx_2. \quad (9)$$

Regular boundary integrals in Equation 9 are evaluated by a standard 6-point Gauss quadrature method. Since the integrand of the inner integral in Equation 9 may have a severe variation in the integral interval, ($[a \quad x_1]$), the integral must be evaluated with special attention. Here, in this research, this integral is evaluated using an adaptive version of the Simpson method. The adaptive Simpson integration method is described in the next section. By adaptive integration methods, one can find a complicated integral with a desired accuracy.

The evaluation of singular integrals appearing in the formulation is not a difficult task. For this matter, the treatment given in [25] is employed.

Equation 9 can be written in matrix form as follows:

$$\mathbf{H}\mathbf{T}_{n+1} = \mathbf{G}\mathbf{q}_{n+1} + \mathbf{F}_{n+1} + \mathbf{L}_{n+1},$$

where $\mathbf{H}$ and $\mathbf{G}$ are coefficient matrices, $\mathbf{T}_{n+1}$ and $\mathbf{q}_{n+1}$ are temperature and heat flux vectors at time n+1, respectively, $\mathbf{F}_{n+1}$ is a vector associated with a history of the previous steps and $\mathbf{L}_{n+1}$ represents a vector associated with heat sources.

For cases where the heat source function is not a function of time, one can write:

$$L_{i_{n+1}} = L_{i_n} + \frac{\Delta t}{2\pi k} \sum_{j=1}^{N} \int_{\Gamma_j} \int_a^{x_1} g[Q(x', x_2)] \frac{\exp\left[\frac{-r^2(x', x_2)}{2\alpha(t_n + t_{n+1})}\right]}{(t_n + t_{n+1})} dx' dx_2.$$

In writing the above equation, the integral over $w = t_n$ to $w = t_{n+1}$ is approximated by a constant element, i.e. the integrand is set to be constant over the time element and is evaluated at midpoint ($w = \frac{t_n + t_{n+1}}{2}$).

ADAPTIVE SIMPSON METHOD

Assume that it is desired to compute $I = \int_a^b f(x) dx$, with tolerance ε. This integral can be expressed by Simpson's rule with an interval size, $h = (b-a)/2$ (two intervals, $n = 2$) as follows:

$$I = S(a, b) - \frac{h^5}{90} f^{(4)}(\omega), \qquad \omega \in [a, b], \tag{10}$$

where:

$$S(a, b) = \frac{h}{3}[f(a) + 4f(a+h) + f(b)]. \tag{11}$$

Integral I with four intervals ($n = 4$) can be expressed by the following relation:

$$I = S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right) - \frac{1}{16}\left(\frac{h^5}{90}\right) f^{(4)}(\overline{\omega}), \quad \overline{\omega} \in [a, b]. \tag{12}$$

As an approximation, one assumes $f^{(4)}(\omega) \approx f^{(4)}(\overline{\omega})$ [26]. By considering this assumption, one can show the following:

$$\frac{h^5}{90} f^{(4)}(\omega) \approx \frac{16}{15}\left[S(a, b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right]. \tag{13}$$

Now, the error of Simpson's rule, with 4 intervals, can be estimated as follows:

$$\text{Error} = \left| I - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right) \right| \approx \frac{1}{15}\left| S(a, b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right) \right|. \tag{14}$$

If the error computed by Equation 14 is greater than ε, the procedure is applied separately to intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$, with a tolerance of $\frac{\varepsilon}{2}$ for each subinterval. This halving procedure is continued until each portion is within the required tolerance.

EVALUATION OF THE HEAT SOURCE INTEGRAL IN OTHER FORMULATIONS WITH A TIME INDEPENDENT FUNDAMENTAL SOLUTION

As mentioned previously in the introduction, in some boundary element formulations of heat conduction analysis, a time-independent fundamental solution is employed [10-13]. In such cases, the heat source integral may be expressed as follows:

$$I_g = \frac{1}{k} \int_\Omega g(x_1, x_2) T^*[P; Q(x_1, x_2)] d\Omega, \tag{15}$$

where P and Q are, respectively, source and field points within the domain or on the boundary and T^* is the time independent fundamental solution, which has the following form:

$$T^*(P; Q) = \frac{1}{2\pi} \ln\left[\frac{1}{r(P; Q)}\right],$$

in which r is the Euclidean distance between P and Q.

By applying Green's lemma to Integral 15, the associated boundary integral is obtained as follows:

$$I_g = \frac{1}{k} \int_\Gamma \int_a^{x_1} g(x', x_2) T^*[P; Q(x', x_2)] dx' dx_2.$$

For special cases, where heat generation function, g, is to be considered uniformly distributed ($g = g_0$) over the domain, the integral takes the following form:

$$I_g = \frac{g_0}{2\pi k} \int_\Gamma \int_a^{x_1} \ln\left(\frac{1}{\sqrt{(x' - x_{s1})^2 + (x_2 - x_{s2})^2}}\right) dx' dx_2,$$

where (x_{s1}, x_{s2}) are coordinates of the source point. In this case, the inner integral can be evaluated, analytically. For $a = 0$, it can be expressed as follows:

$$I_g = \frac{-g_0}{2k\pi} \int_\Gamma [(x_1 - x_{s1}) \ln r + (x_2 - x_{s2}) \tan^{-1}\left(\frac{x_1 - x_{s1}}{x_2 - x_{s2}}\right) - (x_1 - x_{s1})] dx_2. \tag{16}$$

When the boundary, Γ, is discretized by linear elements, the boundary integral (Equation 16) can also be computed analytically. This would enhance the accuracy of the solutions, especially when the body is thin or the geometric boundary of the domain is complex.

If one assumes I_{gj} to be the heat source integral over a linear boundary element, Γ_j, with starting and ending points, $P_1(x_{11}, x_{12})$ and $P_2(x_{21}, x_{22})$, then I_{gj} can be written as follows:

$$I_{gj} = \frac{-g_0(x_{22}-x_{12})}{2k\pi}\int_0^1 \Big[(x_1-x_{s1})\ln r + (x_2-x_{s2})\tan^{-1}\left(\frac{x_1-x_{s1}}{x_2-x_{s2}}\right) - (x_1-x_{s1})\Big]d\eta,$$

where η is a local coordinate associated with the element and:

$$x_1 = x_{11}(1-\eta) + x_{21}\eta, \qquad x_2 = x_{12}(1-\eta) + x_{22}\eta,$$

$$r = \sqrt{[x_{11}(1-\eta)+x_{21}\eta-x_{s1}]^2+[x_{12}(1-\eta)+x_{22}\eta-x_{s2}]^2}. \tag{17}$$

Now, one can write as follows:

$$I_{gj} = \frac{-g_0(x_{22}-x_{12})}{2k\pi}(I_{gj_1}+I_{gj_2}+I_{gj_3}).$$

The integrals, I_{gj_1}, I_{gj_2} and I_{gj_3} are evaluated as follows:

$$I_{gj_1} = \int_0^1 (x_1-x_{s1})\ln r d\eta = \int_0^1 (a_1+b_1\eta)\ln\sqrt{c_1\eta^2+d_1\eta+e_1}d\eta, \tag{18}$$

where:

$$a_1 = x_{11}-x_{s1}, \qquad b_1 = x_{21}-x_{11}, \qquad c_1 = r^2_{P_1P_2},$$

$$d_1 = r^2_{sP_2} - r^2_{P_1P_2} - r^2_{sP_1}, \qquad e_1 = r^2_{sP_1}.$$

r_{sP_1} and r_{sP_2} are Euclidian distances from source point to P_1 and P_2, respectively, and $r_{P_1P_2}$ is the length of the element. The analytical evaluation of Integral 18 is described in the Appendix.

The second integral, I_{gj_2}, is expressed as follows:

$$I_{gj_2} = \int_0^1 (x_2-x_{s2})\tan^{-1}\left(\frac{x_1-x_{s1}}{x_2-x_{s2}}\right)d\eta.$$

By substituting x_1 and x_2 from Equations 17 to the above integral, it takes the following form:

$$I_{gj_2} = \int_0^1 (a_2+b_2\eta)\tan^{-1}\left(\frac{c_2\eta+d_2}{e_2\eta+f_2}\right)d\eta, \tag{19}$$

where:

$$a_2 = x_{12}-x_{s2}, \qquad b_2 = x_{22}-x_{12},$$

$$c_2 = x_{21}-x_{11}, \qquad d_2 = x_{11}-x_{s1},$$

$$e_2 = x_{22}-x_{12}, \qquad f_2 = x_{12}-x_{s2}.$$

An analytical evaluation of Integral 19 is given in the Appendix.

The integral, I_{gj_3}, is expressed as follows:

$$I_{gj_3} = -\int_0^1 (x_1-x_{s1})d\eta = x_{s1} - \frac{1}{2}(x_{21}+x_{11}),$$

when the source point coincides with any nodes of the element, I_{gj} can be expressed as follows:

$$I_{gj} = \kappa\frac{g_0(x_{12}-x_{22})}{2k\pi}\Big\{\frac{x_{21}-x_{11}}{8}\left(\ln r_{P_1P_2} - \frac{1}{2}\right) + \frac{1}{2}(x_{22}-x_{12})\tan^{-1}\left(\frac{x_{21}-x_{11}}{x_{22}-x_{12}}\right) - \frac{1}{2}(x_{21}-x_{11})\Big\}.$$

In the above equation, $\kappa = +1$, when the source point coincides with node P_1, and $\kappa = -1$, when the source point coincides with node P_2.

EXAMPLES

Three different examples are presented to show the accuracy of the proposed method. Two examples are concerned with a transient analysis with a time-dependent fundamental solution and the third example is focused on an analysis with a time-independent fundamental solution. The results are compared with those of analytical or Finite Element Method (FEM) solutions. Whenever FEM is employed, the domain of the problem is discretized by a large number of elements (fine mesh) and the Crank-Nicolson scheme [27] is used.

Example 1: Non-Uniform Time-Independent Heat Source

The domain shown in Figure 1 is considered to be subjected to a time-independent heat source, with a distribution of $g(x,y) = 10^6(1+\sin 40x + 10y)$. The initial temperature field is equal to 0.0°C and the boundary condition is of a Robin type, with an ambient temperature of 0.0°C and a heat transfer coefficient of 200 $\frac{\text{W}}{\text{m}^2\,°\text{C}}$. Thermal conductivity and thermal diffusivity are set to be $k = 80.2\ \frac{\text{W}}{\text{m}\,°\text{C}}$ and $\alpha = 2.28\times10^{-5}\ \frac{\text{m}^2}{\text{s}}$, respectively.

Results for the temperature variation, with respect to time, are presented at three different points,

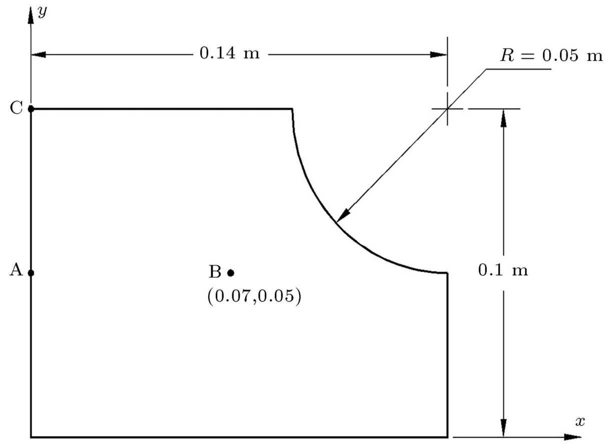

Figure 1. The geometrical domain of the problem.

A, B and C. Points A, B and C are, respectively, a boundary point, an internal point and a corner point. The obtained results are compared with the FEM solution.

For FEM analysis, the domain of the problem is discretized by 512 quadrilateral elements and 559 nodes (Figure 2). For BEM analysis, the boundary of the problem is discretized by 46 linear elements (Figure 3). Figures 4 to 6 compare the FEM solution ($\Delta t = 10$ sec) with results of the present work ($\Delta t = 20$ sec) at the three points, A, B and C. As can be seen, there is a small difference between the solutions of the BEM analysis and FEM. In problems with a strong variation in heat source function, the FEM can produce an acceptable solution, only with a large number of elements, whereas the solutions obtained by BEM will be acceptable, even with a moderate number of boundary elements.

Example 2: Non-Uniform Time-Dependent Heat Source

In this example, the geometry of the problem, the initial conditions and material properties are considered to be identical to the previous example. The only difference is the form of heat generation function, which is assumed to be as follows:

$$g(x, y) = 10^6(1 + \sin 40x + 10y)f(t).$$

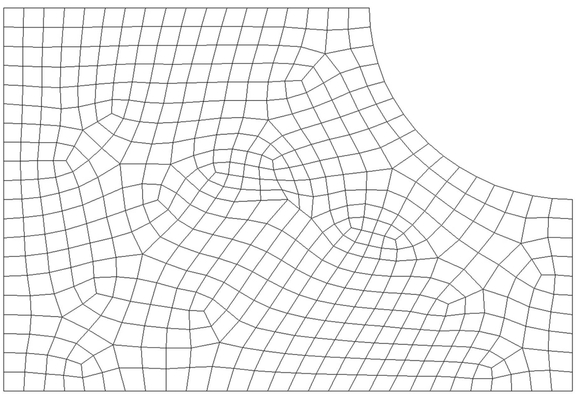

Figure 2. FEM discretization of the domain.

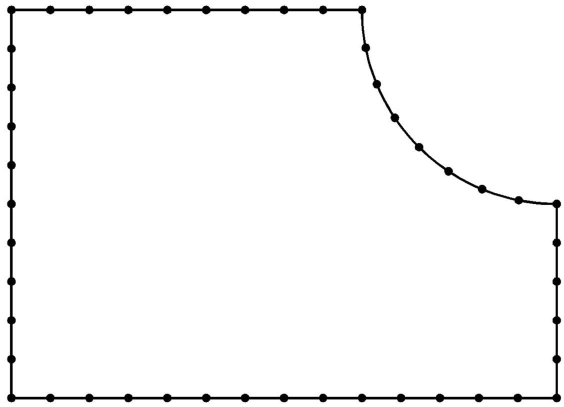

Figure 3. BEM discretization of the domain.

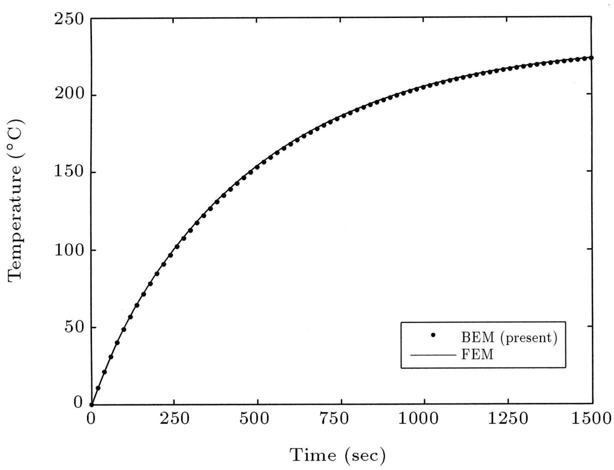

Figure 4. Temperature variation at point A, Example 1.

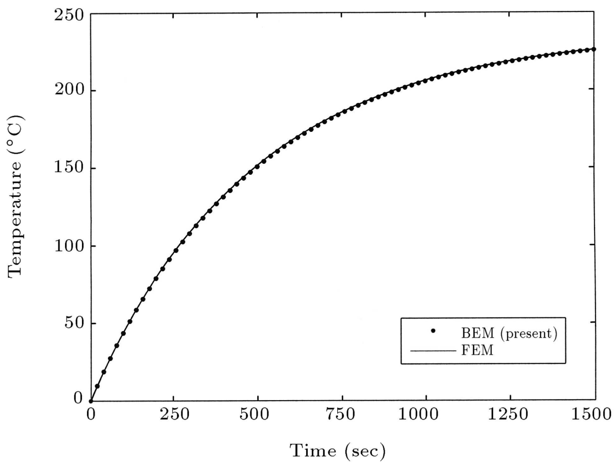

Figure 5. Temperature variation at point B, Example 1.

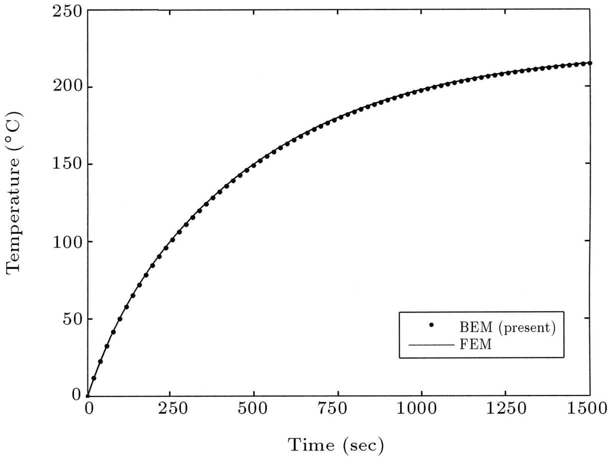

Figure 6. Temperature variation at point C, Example 1.

The function, $f(t)$, is chosen to have a variation over time as shown in Figure 7. The results are presented for the temperature at the three points, A, B and C. Figures 8 to 10 compare the FEM solution (512 quadrilateral elements, $\Delta t = 2.5$ sec) with the results of the present work (46 linear boundary elements, $\Delta t = 5$ sec) at the three points, A, B and C. The temperature variations along the lower edge at two different times ($t = 50$ sec, $t = 100$ sec), are also shown in Figure 11. As seen, the obtained results by the proposed method are in good agreement with the accurate finite element solution.

Example 3: Uniform Heat Source with Time Independent Fundamental Solution

In the two previous examples, the temperature distribution was studied through a simply connected domain, whereas in the example under consideration here, the steady temperature will be studied within a multiply connected domain.

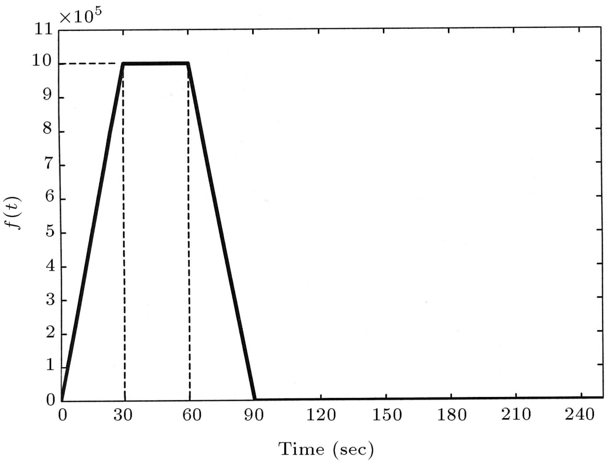

Figure 7. Variation of $f(t)$ with respect to time.

A hollow cylinder, with internal and external radii of r_i and r_o, is subjected to a uniform heat source with intensity g_0. The boundary surfaces at $r = r_i$ and $r = r_o$ are kept at temperatures T_i and T_o, respectively.

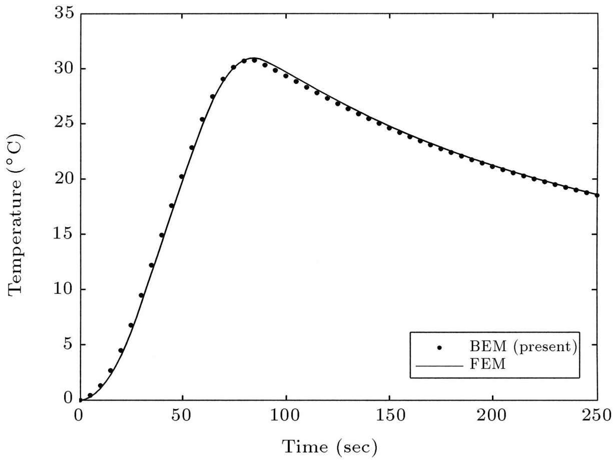

Figure 8. Temperature variation at point A, Example 2.

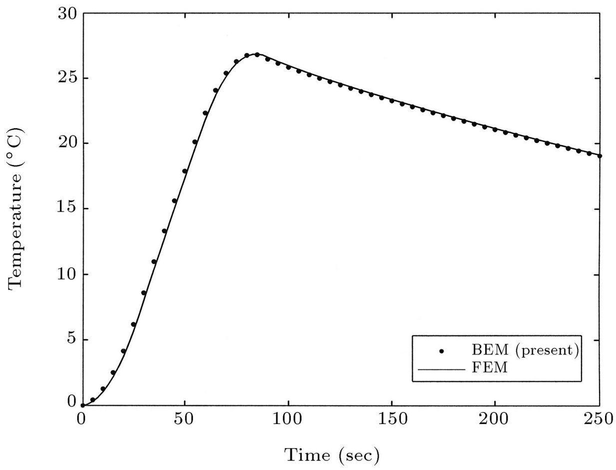

Figure 9. Temperature variation at point B, Example 2.

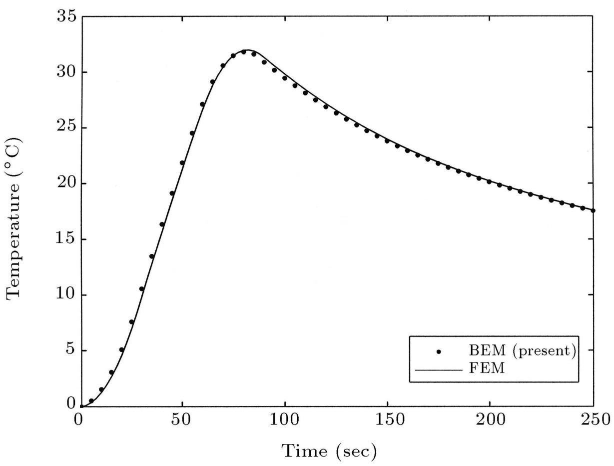

Figure 10. Temperature variation at point C, Example 2.

Table 1. Results of analytical and boundary element solutions.

Distance from Center	Analytical	16 Elements	24 Elements	32 Elements
1.1	1. 644	1.764	1.689	1.661
1.2	2.136	2.239	2.169	2.150
1.3	2.492	2.569	2.509	2.500
1.4	2.727	2.776	2.726	2.727
1.5	2.847	2.876	2.831	2.842
1.6	2.864	2.877	2.833	2.852
1.7	2.782	2.792	2.741	2.765
1.8	2.608	2.631	2.562	2.589
1.9	2.346	2.409	2.309	2.331

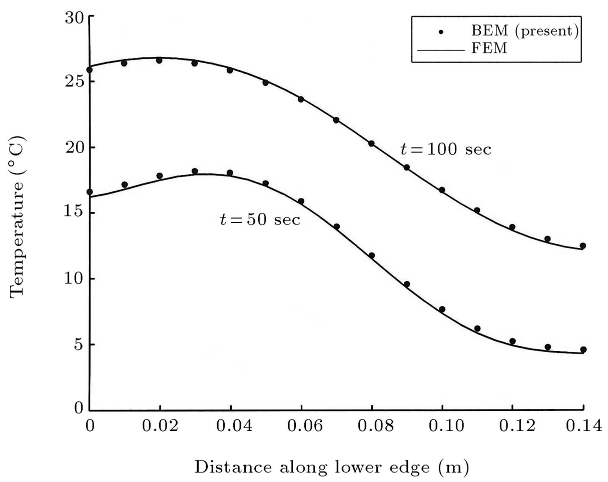

Figure 11. Temperature variation along the lower edge, Example 2.

The governing equation of this problem has the following form:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{g_0}{k}=0.$$

This has an exact solution as follows:

$$T=\frac{-g_0}{4k}r^2+c_1\ln r+c_2,$$

where:

$$c_1=\frac{[T_o-T_i+\frac{g_0}{4k}(r_o^2-r_i^2)]}{\ln\frac{r_o}{r_i}},$$

$$c_2=T_i+\frac{g_0}{4k}r_i^2-c_1\ln r_i.$$

This problem is solved with different numbers of boundary elements with $T_i=1°\text{C}$, $T_o=2°\text{C}$, $r_i=1$ m, $r_o=2$ m, $k=1\ \frac{\text{W}}{\text{m}°\text{C}}$ and $g_0=10\ \frac{\text{W}}{\text{m}^3}$. For boundary element analysis, both outer and inner boundaries must be discretized. Table 1 compares the analytical solution at several points with the numerical results obtained when the boundary of the domain is discretized into 16, 24 and 32 linear boundary elements, respectively. As seen, the obtained results are satisfactory.

CONCLUSION

In this paper, a meshless boundary element method for the analysis of heat conduction problems was presented. The method can be implemented for various kinds of BEM heat conduction formulations, including time-dependent or time-independent fundamental solutions. Although it cannot be used for three-dimensional problems, it can be efficiently employed for the treatment of body-force or domain loading in other related two-dimensional fields of analysis.

For stationary and for transient analysis with time-independent heat sources, the method is completely cost effective, however, it will introduce a computational load for the treatment of time-dependent heat sources. An attractive advantage of the present method is its excellent accuracy, especially for the treatment of domain sources, which have severe variations inside the domain.

In conventional BEM, domain integrals are evaluated by internal discretization and the accuracy of calculated values for domain integrals is directly dependent on the size of internal cells. In the present method, domain integrals are evaluated by a boundary integral and a simple 1-D integral, which is evaluated by an adaptive integration method. By using the adaptive integration method, the integrals can be evaluated with a desired and controllable accuracy.

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APPENDIX

Analytical Evaluation of Integrals I_{gj_1} and I_{gj_2}

I_{gj_1} and I_{gj_2} are expressed in terms of two other integrals, I_1 and I_2. I_1 is introduced as follows:

$$I_1(a,b,c,d,e) = \int_0^1 \frac{a\eta + b}{c\eta^2 + d\eta + e} d\eta,$$

when $d^2 - 4ce = 0$, one can write:

$$I_1(a,b,c,d,e) = \frac{a}{c}\int_0^1 \frac{\eta + \frac{b}{a}}{\left(\eta + \frac{d}{2c}\right)^2} d\eta$$

$$= \frac{a}{c}\left[\int_0^1 \frac{1}{\left(\eta + \frac{d}{2c}\right)} d\eta + \left(\frac{d}{2c} - \frac{b}{a}\right)\int_0^1 \frac{1}{\left(\eta + \frac{d}{2c}\right)} d\eta\right]$$

$$= \frac{a}{c}\left[\ln\left(\eta + \frac{d}{2c}\right) + \left(\frac{d}{2c} - \frac{b}{a}\right)\frac{1}{\left(\eta + \frac{d}{2c}\right)}\right]\Bigg|_{\eta=0}^{\eta=1},$$

and, when $d^2 - 4ce < 0$, one has the following:

$$I_1(a,b,c,d,e) = \frac{a}{2c}\int_0^1 \frac{2\eta + 2\frac{b}{a}}{\eta^2 + \frac{d}{c}\eta + \frac{e}{c}} d\eta$$

$$= \frac{a}{2c}\left[\int_0^1 \frac{2\eta + \frac{d}{c}}{\eta^2 + \frac{d}{c}\eta + \frac{e}{c}} d\eta + \int_0^1 \frac{2\frac{b}{a} - \frac{d}{c}}{\eta^2 + \frac{d}{c}\eta + \frac{e}{c}} d\eta\right]$$

$$= \frac{a}{2c}\left[\ln\left(\eta^2 + \frac{d}{c}\eta + \frac{e}{c}\right)\Bigg|_{\eta=0}^{\eta=1}\right.$$

$$\left. + \left(\frac{2b}{a} - \frac{d}{c}\right)\int_0^1 \frac{1}{\left(\eta + \frac{d}{2c}\right)^2 + \left(\frac{e}{c} - \frac{d^2}{4c^2}\right)} d\eta\right]$$

$$= \frac{a}{2c}\left[\ln\left(\eta^2 + \frac{d}{c}\eta + \frac{e}{c}\right)\right.$$

$$\left. + \left(\frac{2b}{a} - \frac{d}{c}\right)\frac{1}{\sqrt{\frac{e}{c} - \frac{d^2}{4c^2}}}\tan^{-1}\frac{\eta + \frac{d}{2c}}{\sqrt{\frac{e}{c} - \frac{d^2}{4c^2}}}\right]\Bigg|_{\eta=0}^{\eta=1}.$$

I_2 is introduced as follows:

$$I_2(a,b,c,d,e,f) = \int_0^1 \frac{a\eta^2 + b\eta + c}{d\eta^2 + e\eta + f} d\eta.$$

By integration by part, one obtains the following:

$$I_2(a,b,c,d,e,f) = \frac{a}{d}\eta\Bigg|_{\eta=0}^{\eta=1} + \int_0^1 \frac{\left(b - \frac{ae}{d}\right)\eta + \left(c - \frac{af}{d}\right)}{d\eta^2 + e\eta + f} d\eta$$

$$= \frac{a}{d} + I_1\left(b - \frac{ae}{d}, c - \frac{af}{d}, d, e, f\right).$$

Now, I_{gj_1} can be evaluated as follows:

$$I_{gj_1} = \int_0^1 (a_1 + b_1\eta)\ln\sqrt{c_1\eta^2 + d_1\eta + e_1}\, d\eta.$$

One can write $I_{gj_1} = I'_{gj_1} + I''_{gj_1}$, where:

$$I'_{gj_1} = a_1\int_0^1 \ln\sqrt{c_1\eta^2 + d_1\eta + e_1}\, d\eta$$

$$= a_1\eta\ln\sqrt{c_1\eta^2 + d_1\eta + e_1}\Bigg|_{\eta=0}^{\eta=1}$$

$$- a_1\int_0^1 \frac{\eta(2c_1\eta + d_1)}{2(c_1\eta^2 + d_1\eta + e_1)} d\eta$$

$$= a_1\eta\ln\sqrt{c_1\eta^2 + d_1\eta + e_1}\Bigg|_{\eta=0}^{\eta=1}$$

$$- \frac{a_1}{2} I_2(2c_1, d_1, 0, c_1, d_1, e_1).$$

and:

$$I''_{gj_1} = b_1\int_0^1 \eta\ln\sqrt{c_1\eta^2 + d_1\eta + e_1}\, d\eta$$

$$= \frac{b_1}{2}\eta^2\ln\sqrt{c_1\eta^2 + d_1\eta + e_1}\Bigg|_{\eta=0}^{\eta=1}$$

$$- \frac{b_1}{4}\int_0^1 \frac{\eta^2(2c_1\eta + d_1)}{(c_1\eta^2 + d_1\eta + e_1)} d\eta$$

$$= \frac{b_1}{2}\eta^2\ln\sqrt{c_1\eta^2 + d_1\eta + e_1}\Bigg|_{\eta=0}^{\eta=1}$$

$$- \frac{b_1}{4}\int_0^1 \left[\left(2\eta - \frac{c_1}{d_1}\right) + \frac{\left(\frac{d_1^2}{c_1} - 2e_1\right)\eta + \frac{d_1 e_1}{c_1}}{(c_1\eta^2 + d_1\eta + e_1)}\right] d\eta$$

$$= \left[\frac{b_1}{2}\eta^2\ln\sqrt{c_1\eta^2 + d_1\eta + e_1} - \frac{b_1}{4}\eta^2 + \frac{b_1 d_1}{4c_1}\eta\right.$$

$$\left. - \frac{b_1}{4} I_1\left(\frac{d_1^2}{c_1} - 2e_1, \frac{d_1 e_1}{c_1}, c_1, d_1, e_1\right)\right]\Bigg|_{\eta=0}^{\eta=1}.$$

Another integral, which was introduced previously, is I_{gj_2}, which is expressed as follows:

$$I_{gj_2} = \int_0^1 (a_2\eta + b_2)\tan^{-1}\left(\frac{c_2\eta + d_2}{e_2\eta + f_2}\right) d\eta.$$

By applying integration by part, one obtains the following:

$$I_{gj_2} = \left(\frac{a_2\eta^2}{2} + b_2\eta\right)\tan^{-1}\left(\frac{c_2\eta + d_2}{e_2\eta + f_2}\right)\Bigg|_{\eta=0}^{\eta=1}$$

$$- (c_2 f_2 - e_2 d_2)\int_0^1$$

$$\frac{\left(\frac{a_2}{2}\eta^2 + b_2\eta\right)}{(e_2^2 + c_2^2)\eta^2 + 2(e_2 f_2 + c_2 d_2)\eta + (f_2^2 + d_2^2)} d\eta.$$

When $0 < \frac{-f_2}{e_2} < 1$, one would confront a singularity (denominator in argument of $\tan^{-1}$ becomes zero at $\eta = -\frac{f_2}{e_2}$), which might be easily removed and, thus the solution to I_{gj_2} becomes as follows:

$$I_{gj_2} = \left(\frac{a_2}{2} + b_2\right)\tan^{-1}\left(\frac{c_2 + d_2}{e_2 + f_2}\right) + \theta$$

$$- (c_2 f_2 - e_2 d_2) I_2\Big(\frac{a_2}{2}, b_2, 0, e_2^2 + c_2^2, 2e_2 f_2$$

$$+ 2c_2 d_2, f_2^2 + d_2^2\Big),$$

where:

$$\theta = -\left(\frac{a_2\eta^2}{2} + b_2\eta\right)\tan^{-1}\left(\frac{c_2\eta + d_2}{e_2\eta + f_2}\right)\Bigg|_{\eta=\left(-\frac{f_2}{e_2}\right)^-}^{\eta=\left(-\frac{f_2}{e_2}\right)^+}$$

$$\text{for} \quad 0 < -\frac{f_2}{e_2} < 1 \quad \text{and} \quad 0 \quad \text{for other cases.}$$