Scientia Iranica, Vol. 15, No. 3, pp 389–397

SCIENTIA
IRANICA

A Methodology for Optimizing Statistical Multi-Response Problems Using Fuzzy Goal Programming

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This paper presents a method for optimizing statistical multi-response problems. The method is based on fuzzy goal programming and it enjoys a strong mathematical foundation. In this method, the decision maker's comments are considered objectively. The LINGO programming environment is used to test the developed method. The method performance is evaluated by comparing the results with those of other existing methods.

INTRODUCTION

In multi objective decision making environments, a problem of interest is to select a set of input conditions (or independent variables), which results in a product with a desirable set of outputs (or response variables). In fact, this problem is about simultaneous optimization of the response variables, $Y_1, \cdots, Y_m$, each of which depends upon a set of independent variables, $X_1, \cdots, X_n$. Here, it is desirable to select the levels of the independent variables such that all the response variables are optimized [1].

For example, as mentioned in [1], in quality control environments, the goal may be to find the levels of the input variables (quality characteristics) of the process, so that the quality of the product has the desired characteristics. Also, in Response Surface Methodology (RSM), the levels of the input variables are adjusted until the set of outputs are optimized.

Since goods have more than one qualitative attribute, the simultaneous improvement of these qualitative attributes is very important. A common problem in the simultaneous optimization of multi-response problems is that optimizing one attribute affects the other qualitative attributes. In other words, a set of conditions which is optimized for one attribute is not necessarily optimized for other attributes. Therefore, designing a method, which can offer an acceptable product considering different aspects is important [2]. Multi-response optimization in the framework of RSM is an attempt to reach this end.

Classic methods, such as bounded objectives or a combination of bounded objectives and lexicography, are used in multi-response optimization. Myers and Carter used the method of bounded objectives for the first time and they proposed optimizing the main solution [3]. Biles extended this method for more than two solutions [4]. Myers, Khari and Vining extended these results by combining Myers and Carter's method with Taguchi's method. They used deviation and response effects as two independent solutions in their optimization [5].

The loss function method was used by Tang and Lo [6], Pignatello [7], Winston [8], Artiles-Leon and Ross [9,10], Kapur and Cho [11] and Robert and Richard [12]. The basis of this method is Taguchi's loss function, in which a second order loss function, encompassing qualitative attributes and, in some cases, qualitative attributes variance, is offered. Taguchi's loss function is a second order function of the deviation of the desirable qualitative attribute from the target value, as shown by Relation 1:

$$\text{Loss}(Y(X)) = k(Y(X) - T). \tag{1}$$

In Relation 1, k is the loss coefficient factor and T is the desired value of the qualitative attribute. Extending Taguchi's loss function, Artiles-Leon offered the loss function shown by Equation 2:

$$L(Y, X, T) = 4\sum_{i=1}^{k}\left[\frac{Y_i(X) - T_i}{\text{USL}_i - \text{LSL}_i}\right]^2. \tag{2}$$

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Derringer and Suich [13] introduced a useful class of desirability functions. There are two types of transformation from Y_i to $d_i(Y_i)$, namely, one-sided and two-sided. The one-sided transformation is applied when Y_i is to be maximized and the two-sided transformation is used when Y_i is to be assigned a target value.

In a two sided transformation, assume l_i and u_i to be the lower and upper limits of the response, Y_i, respectively. Also, assume that t_i be the target value of the response, Y_i, respectively, such that $l_i < t_i < u_i$. The desirability function is defined by the following equation:

$$d_i(Y_i) = \begin{cases} 0, & Y_i < l_i, \\ \left(\frac{Y_i - l_i}{t_i - l_i}\right)^s, & l_i \leq Y_i \leq t_i, \\ \left(\frac{Y_i - u_i}{t_i - u_i}\right)^t, & t_i \leq Y_i \leq u_i, \\ 0, & Y_i > u_i, \end{cases} \tag{3}$$

In Equation 3, the exponents, s and t, determine how strictly the target value is desired and the user must specify their values [1]. Derringer and Suich offered Relation 4 for computing the decision function:

$$D(Y) = [d_1(Y_1).d_2(Y_2) \cdots\cdots\cdots d_k(Y_k)]^{1/k}. \tag{4}$$

Zimmermann offered a method for solving multi-objective linear programming problems using fuzzy logic [14]. Later, Cheng and his colleagues extended the Zimmermann method for optimizing statistical multi-response problems [15]. Keeny and Raiffa presented a method for multiple objective problems using preferences and value tradeoffs [16].

Noorossana and his colleagues offered a method for extracting and using the decision function in optimizing multi-response problems [17] and Pasandideh and Niaki [1] modeled a multi-response statistical optimization problem through the desirability function approach, where they applied four GA methods to solve this model by simulation. They also studied the performance of each method through different simulation replications and statistically compared them via a performance measure.

The solutions of most of the above existing methods take a long time to generate. This weakness is due to the rapid increase in solution time, as the number of qualitative attributes and objectives increase. For this reason, designing optimization algorithms for statistical multi-response problems that do not have this disadvantage is of special importance.

In this paper, a methodology for optimizing statistical multi-response problems, using fuzzy goal programming, is presented. Through some examples, the desirable execution time of the developed algorithm will be shown when the number of factors and/or objectives increase. The paper is organized as follows: First, definitions are presented. Next, the model and the proposed methodology are explained and numerical examples and comparison with other existing methods will, subsequently, be shown. Finally, the conclusion and the nomenclature are given.

DEFINITION 1

The mathematical model of the multiobjective problem is defined as follows:

$$\max Z_j = Y_j(X_1, X_2, \cdots, X_n), \qquad j = 1, 2, \cdots, m$$

$$-1 \leq X_j \leq 1, \qquad j = 1, 2, \cdots, n.$$

DEFINITION 2

If, in a multi objective problem with m objectives, each objective function is solved independently, then, one has m independent optimal solutions. By replacing each optimal solution in the other objective functions, a lower and upper limit for each objective function will be gained.

For the ith objective function, the following problem is solved separately ($i = 1, 2, \cdots, m$). The results are shown in Table 1, in which Z_{ij} is the value of the jth objective function, in terms of optimal variables of the ith objective function problem; X_{ij} is the optimal value of variable X_j in the ith objective function.

DEFINITION 3

For any objective function, there is a fuzzy membership function (shown in Figure 1) as follows:

$$\mu(Z_j) = \begin{cases} 0 & Z_j < U_j - \Delta_j = L_j \\ \frac{Z_j - (U_j - \Delta_j)}{\Delta_j} & U_j - \Delta_j \leq Z_j \leq U_j \\ 1 & Z_j \geq U_j \end{cases} \tag{5}$$

Table 1. The range of objective functions.

	Z_1	Z_2	$\cdots$	Z_m	X_1	X_2	$\cdots$	X_n
$\max(Z_1)$	$Z_{11} = Z_1^*$	Z_{12}	$\cdots$	Z_{1m}	X_{11}	X_{12}	$\cdots$	X_{1n}
$\max(Z_2)$	$Z_{21} = Z_1$	$Z_{22} = Z_2^*$	$\cdots$	Z_{2m}	X_{21}	X_{22}	$\cdots$	X_{2n}
$\vdots$	$\vdots$				$\vdots$			
$\max(Z_m)$	$Z_{m1} = Z_1$	Z_{m2}	$\cdots$	$Z_{mm} = Z_m^*$	X_{m1}	X_{m2}	$\cdots$	X_{mn}

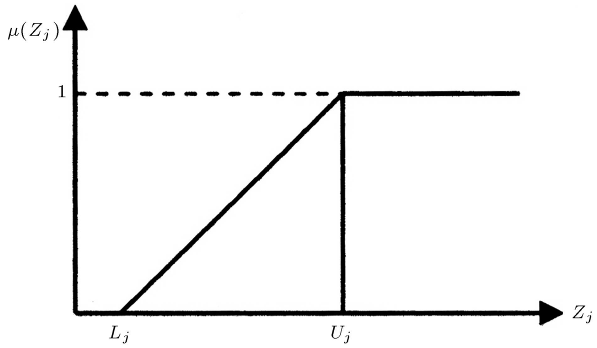

Figure 1. Membership function of Z_j function.

where:

$$U_j = Z_j^*, \qquad L_j = \min_i(Z_{ij}), \qquad \Delta_j = U_j - L_j.$$

THE MODEL AND THE PROPOSED METHODOLOGY

In this paper, the following assumptions are made:

1. All the factors that make up the input of the problem, are the independent variables $X_1, X_2, \cdots, X_n$;
2. The lower and upper bounds of the independent variables are -1 and 1, where X_j is a coded variable, such that $-1 \leq X_j \leq 1$;
3. The output of the problem is the response variables denoted by $Y_1, Y_2, \cdots, Y_k$;
4. For every objective function Z_j, one has:

   $$L_j \leq Z_j \leq U_j.$$
5. The one-sided or two-sided transformation for each response depends on the nature of the objective of the problem.

The mathematical model of the problem becomes:

$$\max : Z_j = Y_j(X_1,\ X_2,\ \cdots,\ X_n),$$

$$j = 1, 2, \cdots, m,$$

s.t:

$$-1 \leq X_j \leq 1, \qquad j = 1, 2, \cdots, n.$$

Theorem 1

Consider the following response variable problems:

$$\max : Z_j = Y_j(X_1,\ X_2,\ \cdots,\ X_n),$$

$$j = 1, 2, \cdots, m,$$

s.t:

$$-1 \leq X_j \leq 1, \qquad j = 1, 2, \cdots, n.$$

Assuming an identical importance for the objectives and an identical access level to the optimal point of each objective, the decision maker's desirable solution is found by solving the following mathematical programming model:

$$\max : \alpha,$$

s.t:

$$\frac{Z_j}{\Delta_j} + n_j - p_j = \frac{U_j}{\Delta_j}, \qquad j = 1, 2, \cdots, m,$$

$$\alpha + n_j \leq 1, \qquad j = 1, 2, \cdots, m,$$

$$-1 \leq X_j \leq 1, \qquad j = 1, 2, \cdots, n,$$

$$\alpha \in \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$n_j \geq 0, \qquad p_j \geq 0, \qquad j = 1, 2, \cdots, m, \tag{6}$$

where:

- p_j: the function positive deviation, Z_j/Δ_j,
- n_j: the function negative deviation, Z_j/Δ_j,
- α: access level to the optimum of any objective function.

Proof

In Figure 1 there is:

$$\mu(Z_j) = \begin{cases} 0 & Z_j < U_j - \Delta_j = L_j \\ \frac{Z_j - (U_j - \Delta_j)}{\Delta_j} & U_j - \Delta_j \leq Z_j \leq U_j \\ 1 & U_j \leq Z_j \end{cases},$$

$$\alpha = \min_j \mu(Z_j) \Rightarrow \max : \alpha,$$

(a)

$$\alpha \leq \frac{Z_j}{\Delta_j} - \frac{U_j}{\Delta_j} + 1, \qquad \frac{U_j}{\Delta_j} - 1 \leq \frac{Z_j}{\Delta_j} \leq \frac{U_j}{\Delta_j},$$

for some j (n_j: negative deviation),

(b)

$$\frac{U_j}{\Delta_j} \leq \frac{Z_j}{\Delta_j} \leq \frac{U_j}{\Delta_j} + 1, \qquad \text{for some } j.$$

For the constraints of section (a), suppose that:

$$\frac{Z_j}{\Delta_j} = \frac{U_j}{\Delta_j} - n_j.$$

Now, one has:

$$\alpha + n_j \leq 1, \qquad \frac{Z_j}{\Delta_j} + n_j = \frac{U_j}{\Delta_j}. \tag{7}$$

For the constraints of section (b), suppose that:

$$\frac{Z_j}{\Delta_j} = p_j + \frac{U_j}{\Delta_j}, \qquad (p_j: \text{ positive deviation}).$$

Now, one has:

$$\frac{Z_j}{\Delta_j} - p_j = \frac{U_j}{\Delta_j}. \tag{8}$$

Now, Constraints 8 and 9 are combined and, finally:

$$\max : \alpha$$

$$\frac{Z_j}{\Delta_j} + n_j - p_j = \frac{U_j}{\Delta_j}, \qquad j = 1, 2, \cdots, m,$$

$$\alpha + n_j \le 1, \qquad j = 1, 2, \cdots, m,$$

$$-1 \le X_j \le 1, \qquad j = 1, 2, \cdots, n,$$

$$\alpha \in [0 \quad 1], \; n_j \ge 0, \;\; p_j \ge 0, \;\; j = 1, 2, \cdots, m.$$

Corollary 1

If the access level to the optimum state of each objective is different and the objectives are not identically important from the viewpoint of the decision maker, then, the desirable solution of the decision maker is obtained by solving the following mathematical model:

$$\max : \sum_{j=1}^{m} W_j \alpha_j,$$

$$\frac{Z_j}{\Delta_j} + n_j - p_j = \frac{U_j}{\Delta_j},$$

$$\alpha_j + n_j \le 1,$$

$$-1 \le X_j \le 1,$$

$$\alpha_j \in [0 \quad 1],$$

$$n_j \ge 0, \qquad p_j \ge 0. \tag{9}$$

Proof

This is the result of Theorem 1, with the following assumptions (shown in Figure 2):

$$\alpha_j \le \mu(Z_j).$$

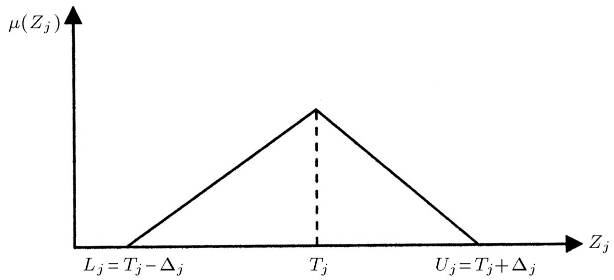

Figure 2. Membership function of Z_j objective function.

Corollary 2

If the response variables have the best nominal value (two-sided transformation), the objectives are not identically important from the view point of the decision maker and the access level to the optimum status of each objective is different, then, the decision maker's desirable solution is obtained by solving the following mathematical model:

$$\max : \sum_{j=1}^{m} W_j \Delta_j,$$

$$\frac{Z_j}{\Delta_j} + n_j - p_j = \frac{T_j}{\Delta_j},$$

$$\alpha_j + n_j + p_j \le 1,$$

$$-1 \le X_j \le 1,$$

$$\alpha_j \in [0 \quad 1], \qquad p_j \ge 0, \qquad n_j \ge 0, \tag{10}$$

where T_j is nominal value of objective function Z_j.

Proof

This is the result of Theorem 1 with the following assumptions:

$$\mu(Z_j) = \begin{cases} 0 & Z_j < T_j - \Delta_j \\ \frac{Z_j - (T_j - \Delta_j)}{\Delta_j} & T_j - \Delta_j \le Z_j < T_j \\ \frac{T_j + \Delta_j - Z_j}{\Delta_j} & T \le Z_j < T_j + \Delta_j \\ 0 & Z_j \ge T_j + \Delta_j \end{cases}$$

$$\alpha_j \le \mu(Z_j). \tag{11}$$

Corollary 3

If there are k response variables on "the more – the better" (one-sided transformation), the $m - k$ response variables are on the better nominal value (two-sided transformation) and the goals are not identically important from the viewpoint of the decision maker, then, the desired solution, from the viewpoint of the decision maker, is obtained by the following mathematical model:

$$\max: \sum_{j=1}^{m} W_j \alpha_j$$

$$\frac{Z_j}{\Delta_j} + n_j - P_j = \frac{U_j}{\Delta_j}, \qquad j = 1, 2, \cdots, k,$$

$$\frac{Z_j}{\Delta_j} + n_j - P_j = \frac{T_j}{\Delta_j}, \qquad j = k+1,\ k+2, \cdots, m,$$

$$\alpha_j + n_j \leq 1, \qquad j = 1, 2, \cdots, k,$$

$$\alpha_j + n_j + p_j \leq 1, \qquad j = k+1,\ k+2, \cdots, m,$$

$$-1 \leq X_j \leq 1, \qquad j = 1, 2, \cdots, n,$$

$$\alpha_j \in \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$n_j \geq 0, \qquad p_j \geq 0. \tag{12}$$

The proof is obtained from Corollaries 1 and 2.

Based on the proof of this theorem, an algorithmic procedure for calculating the response surface is now proposed.

Algorithm

Step 1: Define the following response variable problems:

$$\max: Z_j = Y_j(X_1,\ X_2,\ \cdots,\ X_n),$$

$$j = 1, 2, \cdots, k$$

$$\min: r_j = R_j(X_1,\ X_2,\ \cdots,\ X_n),$$

$$j = k+1,\ k+2, \cdots, m,$$

$$\text{s.t}: -1 \leq X_j \leq 1, \qquad j = 1, 2, \cdots, n.$$

Step 2: Change the response variable problems as follows:

$$\max: Z_j = Y_j(X_1,\ X_2,\ \cdots .\ X_n),$$

$$j = 1, 2, \cdots, m,$$

$$\text{s.t}: -1 \leq X_j \leq 1, \qquad j = 1, 2, \cdots, n.$$

Step 3: Solve the problem for each objective function separately and obtain the solutions. Put the solutions in the objective functions. For each objective function obtain two lower limit (L_j) and upper limit (U_j) as the best and the worst case and, then, obtain $\Delta_j = U_j - L_j$.

Step 4: Obtain the objective weights, W_j, from the decision maker and then solve the following mathematical programming model:

$$\max: \sum_{j=1}^{m} W_j \alpha_j,$$

$$\frac{Z_j}{\Delta_j} + n_j - P_j = \frac{U_j}{\Delta_j}, \qquad j = 1, 2, \cdots, m,$$

$$\alpha_j + n_j \leq 1, \qquad j = 1, 2, \cdots, m,$$

$$-1 \leq X_j \leq 1, \qquad j = 1, 2, \cdots, n,$$

$$\alpha_j \in \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad j = 1, 2, \cdots, m,$$

$$p_j \geq 0, \qquad n_j \geq 0, \qquad j = 1, 2, \cdots, m.$$

NUMERICAL EXAMPLES AND COMPARISON

Example 1

This example concerns research that has been done by Noorossana [17] and has four design variables: The ammoniac (X_1), the thickness of lead and expulsion alloys on the radiator pipe X_2), the temperature (X_3) and the percentage of expulsion in the alloy of the radiator pipe (X_4).

The responses are corrosion (Y_1) and adhesiveness (Y_2). The design is a central composite design. The data are given in Table 2.

The correlation coefficient and covariance matrix of the responses (Y_1, Y_2) are as follows:

$$R\left(Y_1, Y_2\right) = 0.994,$$

$$S^2\left(Y_1, Y_2\right) = \begin{pmatrix} 1580.79 & 1582.55 \\ 1582.55 & 1599.04 \end{pmatrix}.$$

Therefore, the responses, Y_1 and Y_2, are highly correlated. At first, the above data was coded. Then, a second-order model was fitted for both responses. The response surface curves for Y_1 and Y_2 are as follows:

$$\begin{aligned} Y_1 = {} & 250.34 - 34.26X_1 + 43.91X_2 - 1.55X_3 \\ & + 6.75X_4 - 22.97X_2^2 - 21.84X_3^2 - 23.22X_4^2 \\ & + 4.56X_1X_4 + 16.94X_2X_4 + 25.31X_3X_4, \end{aligned} \tag{13}$$

$$\begin{aligned} Y_2 = {} & 176.75 - 2.01X_1 + 30.6X_2 + 2.59X_3 + 16.87X_4 \\ & - 13.31X_1^2 - 13.19X_2^2 - 12.94X_3^2 - 12.44X_4^2 \\ & + 8.56X_1X_4 - 3.94X_2X_4 - 8.69X_3X_4. \end{aligned} \tag{14}$$

Table 2. Experimental data.

Observation	X_1	X_2	X_3	X_4	Y_1	Y_2
1	2	18	330	25	52	50
2	7	18	360	25	50	45
3	7	23	330	25	120	117
4	2	23	360	25	170	159
4	7	18	330	30	120	110
6	2	18	360	30	94	90
7	2	23	330	30	186	178
8	7	23	360	30	180	176
9	4.5	20.5	345	27.5	166	160
10	4.5	20.5	345	27.5	165	163
11	4.5	20.5	345	27.5	167	165
12	4.5	20.5	345	27.5	161	166
13	4.5	20.5	345	31.04	172	169
14	4.5	20.5	345	23.96	160	157
15	4.5	24.04	345	27.5	173	174
16	4.5	16.96	345	27.5	155	150
17	4.5	20.5	366.2	27.5	171	167
18	4.5	20.5	323.8	27.5	157	159
19	8.035	20.5	345	27.5	169	161
20	.965	20.5	345	27.5	162	159

The studied multi-response problem is as follows:

$$\max: Z_1 = Y_1, \qquad \max: Z_2 = Y_2,$$

s.t:

$$-1 \le X_j \le 1, \qquad j = 1,2,3,4. \tag{15}$$

Proposed Method

First, the two problems are solved independently:

$$\max: Z_1 = Y_1,$$

s.t:

$$-1 \le X_j \le 1, \qquad j = 1,2,3,4, \qquad Z_2 = Y_2,$$

$$\max: Z_2 = Y_2,$$

s.t:

$$-1 \le X_j \le 1, \qquad j = 1,2,3,4, \qquad Z_1 = Y_1. \tag{16}$$

The results are shown in Table 3. Now, one has:

$$\Delta_1 = 311 - 272 = 39, \qquad U_1 = 311,$$

$$\Delta_2 = 198 - 180 = 18, \qquad U_2 = 198.$$

Then, the following problem is solved:

$$\max: W_1\alpha_1 + W_2\alpha_2$$

$$\frac{Z_i}{\Delta_i} + n_i - p_i = \frac{U_i}{\Delta_i}, \qquad i = 1,2,$$

$$n_i + \alpha_i \le 1, \qquad i = 1,2,$$

$$n_i \ge 0, \quad p_i \ge 0, \quad \alpha_i \in \begin{bmatrix}0 & 1\end{bmatrix}, \qquad i = 1,2. \tag{17}$$

The sensitivity analysis on W_1 and W_2 is shown in Table 4.

By increasing W_i, the value of Y_i would be increased. As is clear, the best solution is obtained by considering $W_1 = 0.25$ and $W_2 = 0.75$.

The three foregoing problems were solved using LINGO software. Based on this example, the comparison between the proposed method and existing ones are shown in Table 5.

The above example was presented to evaluate the

Table 3. The range of objective functions.

	Z_1	Z_2	X_1	X_2	X_3	X_4
max(Z_1)	311	180	-1	1	0.297	0.574
max(Z_2)	272	198	0.116	1	-0.099	0.594

Table 4. The range of objective functions for different values of W_1 and W_2.

W_1	W_2	Y_1	Y_2	X_1	X_2	X_3	X_4
0.25	0.75	280.4	197.14	-0.10392	1	0.00042	0.4874
0.5	0.5	278.06	197.3	0	1	0.745	0.52
0.75	0.25	278.8	196.67	0	1	0.178	0.565
0.9	0.1	279.3	195.57	0	1	0.276	0.633
0.1	0.9	276.23	197.59	0.0334	1	0	0.53
0.2	0.8	277.33	197.52	0	1	0	0.518
0.8	0.2	278.99	196.4	0	1	0.206	0.583
0.3	0.7	277.45	197.5	0	1	0.111	0.516
0.7	0.3	278.67	196.87	0	1	0.153	0.551
0.6	0.4	278.36	197.14	0	1	0.11	0.531
0.4	0.6	277.76	197.4	0	1	0.42	0.515

Table 5. Optimal results of existing and proposed methods.

	X_1	X_2	X_3	X_4	Y_1	Y_2
Keeney and Raiffa [16]	0	1	0.1764	0.5645	278.8	196.7
Derringer and Suich [13]	0	1	0	0	271.3	194.16
Limited Goals Method	0	1	0.3802	0.7173	279.5	193.8
Noorossana [17]	0	1	0.212	0.58725	279.03	196.35
Pasandideh and Niaki [1]	0.23	0.99	0.0004	0.49	284.6	195.74
Presented Method for $W_1 = 0.25$, $W_2 = 0.75$	-0.10392	1	0.00042	0.4874	280.4013	197.1377

performance of the developed method. As is clear from Table 5, the developed method and the Pasandideh and Niaki method [1] are superior to the existing methods.

Example 2

This example concerns research that has been done by Derringer and Suich [13]. The casting croup is looking for the level of control variables that can minimize the diameter of a hole on the part (Y_1), the size of porosity (Y_2) and different temperatures on the surface of the die (Y_3). The controllable variables are the temperature of the furnace (X_1) and the duration for which the die is being closed (X_2).

After running this experiment, the response level curves are given in the following:

$$Y_1 = 6.79 - 1.67X_1 + 0.5X_2 - 0.167X_1^2,$$

$$Y_2 = 16.89 - 2.67X_1 - 0.5X_2 - 0.33X_1^2 + 1.167X_2^2 + 0.25X_1X_2,$$

$$Y_3 = 94.44 + 10.5X_1 + 3X_2,$$

$$-1 \le X_j \le 1, \qquad j = 1, 2. \tag{18}$$

The studied multi-response problem is as follows:

$$\max: Z_1 = -Y_1, \qquad \max: Z_2 = -Y_2,$$

$$\max: Z_3 = -Y_3,$$

s.t:

$$-1 \le X_j \le 1, \qquad j = 1, 2. \tag{19}$$

Proposed Method

At first, three problems are solved independently:

$$\max: Z_1 = -Y_1,$$

s.t:

$$-1 \le X_j \le 1, \qquad j = 1, 2,$$

$$Z_2 = -Y_2, \qquad Z_3 = -Y_3, \tag{20}$$

$$\begin{aligned}
&\max: Z_2 = Y_2,\\
&\text{s.t:}\\
&-1 \le X_j \le 1, \qquad j = 1, 2,\\
&Z_1 = -Y_1, \qquad Z_3 = -Y_3,
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\max: Z_3 = -Y_3,\\
&\text{s.t:}\\
&-1 \le X_j \le 1, \qquad j = 1, 2,\\
&Z_1 = -Y_1, \qquad Z_2 = -Y_2.
\end{aligned} \tag{22}$$

The solutions are shown in Table 6.

$$\Delta_1 = 1.837, \qquad U_1 = -4.953,$$

$$\Delta_2 = 3.014, \qquad U_2 = -13.876,$$

$$\Delta_3 = 10.821, \qquad U_3 = -94.44,$$

Then, the following problem is solved:

$$\begin{aligned}
&\max: W_1\alpha_1 + W_2\alpha_2 + W_3\alpha_3,\\
&\text{s.t:}\\
&\frac{Z_i}{\Delta_i} + n_i - p_i = \frac{U_i}{\Delta_i}, \qquad i = 1, 2, 3,\\
&n_i + \alpha_i \le 1, \qquad i = 1, 2, 3,\\
&Y_i = -Z_i, \qquad i = 1, 2, 3,\\
&n_i \ge 0, \quad p_i \ge 0, \quad \alpha_i \in \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad i = 1, 2, 3.
\end{aligned} \tag{23}$$

The sensitivity analysis on W_1, W_2 and W_3 is shown in Table 7.

By increasing W_i, the value of Y_i would be decreased. LINGO software was used to solve the above four problems. The corresponding comparisons are shown in Table 8. The above results were provided

Table 6. The range of objective functions.

	Z_1	Z_2	Z_3	X_1	X_2
max(Z_1)	-4.953	-13.89	-104.94	1	0
max(Z_2)	-5.0065	-13.876	-105.261	1	0.1071
max(Z_3)	-6.79	-16.89	-94.44	0	0

Table 7. The range of objective functions for different values of W_1, W_2 and W_3.

W_1	W_2	W_3	Y_1	Y_2	Y_3	X_1	X_2
0.336	0.333	0.331	4.953	13.89	104.94	1	0
0.1	0.2	0.7	6.79	16.89	94.44	0.12710^{-6}	0
0.7	0.2	0.1	4.953	13.89	104.94	1	0
0.2	0.2	0.6	6.79	16.89	94.44	0.12410^{-6}	0
0.6	0.2	0.2	4.953	13.89	104.94	1	0
0.3	0.4	0.3	4.953	13.89	104.94	1	0
0.7	0.1	0.2	4.953	13.89	104.94	1	0
0.2	0.7	0.1	4.953	13.89	104.94	1	0
0.1	0.1	0.8	6.79	16.89	94.44	0.12710^{-6}	0

Table 8. Optimal results of the existing methods and the proposed method.

	X_1	X_2	Y_1	Y_2	Y_3
Limited Goals Method	0.19	0.19	6.56	16.33	94.1
Derringer and Suich [13]	0.84	-1	4.77	15.87	100.26
Noorossana [17]	0.815	-1	4.81	15.96	100
Pasandideh and Niaki [1]	0.835	-0.99	4.78	15.86	100.23
Presented Method for $W_1 = 0.7$, $W_2 = 0.2$, $W_3 = 0.1$	1	0	4.953	13.89	104.94
Presented Method for $W_1 = 0.1$, $W_2 = 0.2$, $W_3 = 0.7$	0.12710^{-6}	0	6.79	16.89	94.44

by assuming that $W_1 = 0.7$, $W_2 = 0.2$ and $W_3 = 0.1$, or $W_1 = 0.1$, $W_2 = 0.2$ and $W_3 = 0.7$. The above example was presented to evaluate the performance of the developed method.

CONCLUSION

The proposed method in this paper used fuzzy goal programming to determine the optimal solution of statistical multi-response problems. The proposed method solved $(m+1)$ problems to reach the optimal, m, responses. Two examples were presented to evaluate the performance of the developed method.

NOMENCLATURE

U_j	upper bound of objective function Z_j
L_j	lower bound of objective function Z_j
α_j	access level to the optimum of objective function Z_j
W_j	the importance or weight of objective function Z_j from the viewpoint of the decision maker
$\mu(Z_j)$	the fuzzy membership function of objective function Z_j
Z_j^*	the optimal value of objective function Z_j
X_j^*	the optimal value of variable X_j
Δ_j	the tolerance of objective function Z_j, $\Delta_j = U_j - L_j$

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