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Research Note

Lyapunov-Krasovskii Approach for Delay-Dependent Stability Analysis of Nonlinear Time-Delay Systems

A.R. Vali[1] and S.K.Y. Nikravesh*

In this paper, a new approach for constructing quadratic Lyapunov-Krasovskii functionals for a class of nonlinear time-delay systems is developed. The functionals are then used to derive delay-dependent stability conditions. These conditions are sufficient and local. Numerical example shows that the results obtained using the proposed method are less conservative than those obtained by the existing methods.

INTRODUCTION

It is well known that most physical systems contain inherent time-delay phenomena. Hence, the stability problem for time-delay systems has received considerable attention and has been one of the most interesting topics in control systems. The topic is of theoretical interest, as well as providing a powerful tool for practical system analysis and design. Delay phenomena are encountered in such diverse areas as mechanics, physics, biology, medicine, economics and engineering systems. Moreover, time delay is frequently a source of instability and oscillation in many practical systems. Consequently, the stability analysis of time-delay systems is worth considering from both practical and theoretical points of view [1-4].

Stability criteria for time-delay systems can be classified into two categories, depending on whether the stability criterion contains the delay argument as a parameter or not, namely, delay-independent and delay-dependent stability conditions. Generally speaking, the latter is less conservative, but the former is also important when the effect of time-delay is small. Hence, there have been numerous interesting developments in the search for stability criteria of time-delay systems, but most of them have been restricted to searching for delay-independent and delay-dependent criteria for the stability of linear systems with discrete delays [5-10].

It is well known that there are several challenges in extension of Lyapunov-Krasovskii functionals for the stability analysis of nonlinear system with time-delay. Further complication arises when delay-dependent stability is considered. There are a few works that have investigated delay-independent stability and delay-dependent stability, however, with more conservative criteria [11-14].

In this paper, a new algorithm in generating Lyapunov-Krasovskii functionals is developed that enables one to study delay-dependent stability for classes of nonlinear time-delay systems [15,16]. In [16], a class of nonlinear systems of the form $\dot{x}(t) = bf(x(t)) + cf(x(t-h))$ was considered. As can be seen, the state function, f, for both current and delayed states are the same and separated. The purpose of this paper is to propose delay-dependent stability criteria for a more general form of nonlinear time-delay systems in a scalar case (Equation 4). The system considered here contains different and separated functions for delay-independent and delay-dependent terms.

The structure of the paper is, as follows: First, definitions and the problem statement are given. Then, the new algorithm for generating Lyapunov-Krasovskii functionals, in order to investigate delay-dependent stability, is derived. After that, a numerical example is included, in order to demonstrate the capability of the proposed method. Finally, the conclusions and some propositions are presented.

1. *Department of Electrical Engineering, Amirkabir University of Technology, Tehran, I.R. Iran.*
*. *Corresponding Author, Department of Electrical Engineering, Amirkabir University of Technology, Tehran, I.R. Iran.*

SYSTEM DESCRIPTION

Generally, time-delay systems are represented, as follows [2]:

$$\begin{cases} \dot{x}(t) = f(t, x_t(\theta)), & t \geq 0 \\ x_t(\theta) = x(t+\theta), & h \leq \theta \leq 0 \end{cases}, \tag{1}$$

where $x \in \Re^n$ is the state vector of the system, $h \in \Re^+$ is the delay, f is a function of class C, $x(t) = \varphi(t)$ is the initial condition of the system for the time interval $h \leq \theta \leq 0$ and x_t is of class C for $t \geq 0$. Having an appropriate Lyapunov functional $V : \Re_+ \times C \to \Re$, the sufficient condition for stability of Equation 1 is, as follows:

$$\text{i) } u(|\varphi(0)|) \leq V(t, \varphi), \tag{2}$$

$$\text{ii) } \dot{V}(t, \varphi) \leq 0, \tag{3}$$

where $|\varphi(0)| = \max|\varphi(\tau)|\ \ _{h \leq \tau \leq 0}$ and $u : \Re^+ \to \Re^+$ is a function, for which $u(0) = 0$ and $u(s) > 0$ $(s \in \Re^+)$. In order to establish the stability condition, the selected Lyapunov functional should meet Conditions 2 and 3 for $t \geq 0$.

In this paper, a more general nonlinear time-delay system than the one given in [16] is considered:

$$\dot{x}(t) = bf(x(t)) + cg(x(t \quad h)), \qquad t \geq 0, \tag{4}$$

where $x \in \Re$ is the state variable, $h \in \Re^+$ is delay and b and $c \in \Re$ are the real parameters of the system. $x = 0$ is the equilibrium point of System 4, f and $g \in C$ are continuous functions and the relationships:

$$0 < \frac{f(y)}{y} \leq m_1, \qquad 0 < \frac{g(y)}{y} \leq m_2,$$

hold for $\forall y \in \Re \quad \{0\}$ and $m_i \in \Re^+$, $i = 1, 2$ are the assumptions made for the system (partial Lipschitz condition).

In the next section, first, a new algorithm for constructing a proper Lyapunov-Krasovskii functional is introduced and, then, the delay-dependent stability of System 4 is analyzed.

PROBLEM FORMULATION

For System 4, the proposed Lyapunov functional, $V : C \to \Re$ is assumed to be, as follows:

$$V(x_t) = V_1(x_t) + V_2(x_t),$$

where:

$$V_1 = \frac{1}{2}\left[x(t) + \int_{t \quad h}^{t} r(t, u) g(x(u)) du\right]^2,$$

where $r(t, u)$ is an unknown time varying function, which will be determined later. If this $r(t, u)$ is positive, then, V_1 will also be positive definite and meet Condition 2. On the other hand, if V_1 can be a candidate for the Lyapunov functional (i.e., Conditions 2 and 3 are met), one can just derive the delay-independent stability condition. In other words, if no limit on $r(t, u)$ is set, then V_1 may not satisfy Condition 2. In order to solve this problem, one may set some assumption on $r(t, u)$, so that, V_1 become positive definite. Also, one can select V_2 in such a way that the Lyapunov functional, V, satisfy Condition 2. Now, one should first calculate the derivative of V_1 before reaching the final conclusion. Therefore, one will have:

$$\dot{V}_1 = \left[x(t) + \int_{t \quad h}^{t} r(t, u) g(x(u)) du\right] \left[\int_{t \quad h}^{t} \frac{\partial r(t, u)}{\partial t} g(x(u)) du + (b + r(t, t)) f(x(t)) + (c \quad r(t, t \quad h)) g(x(t \quad h))\right].$$

For simplicity, it is assumed that:

$$r(t, t \quad h) = c. \tag{5}$$

In 3D space, $r(t, u)$ is a curve with independent variables t and u. For each $t \in \Re^+$, this curve would be of distance c from the $t \quad u$ plane at points $(t, u) = (t, t \quad h)$.

In order to represent $\dot{V}_1$ in second order form, the first step is to state $\frac{\partial r}{\partial t}$ in terms of r. Therefore, $\frac{\partial r}{\partial t}$ is considered, as follows:

$$\frac{\partial r(t, u)}{\partial t} = \Psi(t) r(t, u),$$

where $\Psi(t)$ is an unknown matrix and, for simplicity, it is assumed that $\Psi(t)$ is a constant matrix:

$$\frac{\partial r(t, u)}{\partial t} = q r(t, u).$$

Applying Condition 5, the function r would be considered as:

$$r(t, u) = c e^{q(u \quad t + h)},$$

where q is determined, based on the condition obtained for the stability of System 4. For the sake of simplicity, the following two assumptions are made:

$$g_1(x(t)) = \int_{t-h}^{t} r(t,u)g(x(u))du$$

$$= c\int_{t-h}^{t} e^{q(u-t+h)}g(x(u))du,$$

$$c_1 = ce^{qh}.$$

Then, $\dot{V}_1$ would be, as follows:

$$\dot{V}_1 = c_1x(t)g(x(t)) + bx(t)f(x(t)) - qx(t)g_1(x(t)) + c_1g(x(t))g_1(x(t)) + bf(x(t))g_1(x(t)) - qg_1^2(x(t)).$$

Remark 1

q is the only arbitrary parameter in V_1 and $\dot{V}_1$ which can be adjusted. Since the sign of $\dot{V}_1$ is not of any importance, the structure of q is determined in such a way that V_1 meet Condition 2. Therefore, if Condition 2 is satisfied by V, it must now be satisfied by V_2 as well. Hence, the optimum choice for V_2 can be obtained, based on the following lemma.

Lemma 1

For the continuous function $k(t)$, one can have [17]:

$$\frac{d}{dt}\left[\int_{-h}^{0}\int_{t+s}^{t} k(u)duds\right] = hk(t) - \int_{-h}^{0} k(t+s)ds$$

$$= hk(t) - \int_{t-h}^{t} k(\tau)d\tau,$$

in which, h is some positive constant.

The following relationship for V_2 could be obtained, according to the above lemma:

$$V_2 = \int_{-h}^{0}\int_{t+s}^{t} pe^{qs}g^2(x(u))duds,$$

where p is a constant unknown parameter and positive. Thus, the derivative of V_2 becomes:

$$\dot{V}_2 = pg^2(x(t))\int_{t-h}^{t} e^{qs}ds - pe^{-qt}\int_{t-h}^{t} e^{qu}g^2(x(u))du.$$

Now, the last term of the above relationship should be converted to a second-order form of g_1. Using Schwarz integral inequality [18], one can get:

$$\left[\int_{t-h}^{t} e^{qu}g(x(u))du\right]^2 \leq \int_{t-h}^{t} e^{qu}du \times \int_{t-h}^{t} e^{qu}g^2(x(u))du,$$

thus:

$$\int_{t-h}^{t} e^{qu}g^2(x(u))du \geq \frac{\left[\int_{t-h}^{t} e^{qu}g(x(u))du\right]^2}{\int_{t-h}^{t} e^{qu}du}$$

$$= \frac{e^{qt-2qh}}{c^2\int_{-h}^{0} e^{qs}ds}g_1^2(x(t)).$$

Therefore, the derivative of V_2 can be obtained as:

$$\dot{V}_2 \leq \frac{p}{q}(1-e^{-qh})g^2(x(t)) - \frac{pqe^{-2qh}}{c^2(1-e^{-qh})}g_1^2(x(t)).$$

Adding $\dot{V}_1$ to $\dot{V}_2$ yields:

$$\dot{V} \leq c_1x(t)g(x(t)) + bx(t)f(x(t)) - qx(t)g_1(x(t)) + c_1g(x(t))g_1(x(t)) + bf(x(t))g_1(x(t))\frac{p}{q}(1-e^{-qh})g^2(x(t)) - q\left(\frac{pe^{-2qh}}{c^2(1-e^{-qh})}+1\right)g_1^2(x(t)).$$

Since $c_1 < 0$, one chooses $q < 0$, then, applying assumptions $0 \leq \frac{f(y)}{y} \leq m$ and $0 \leq \frac{g(y)}{y} \leq m$ yields:

$$\dot{V} \leq \frac{c_1}{m_2}g^2(x(t)) + \frac{b}{m_1}f^2(x(t)) - \frac{q}{m_2}g(x(t))g_1(x(t)) + c_1g(x(t))g_1(x(t)) + bf(x(t))g_1(x(t))\frac{p}{q}(1-e^{-qh})g^2(x(t)) - q(\frac{pe^{-2qh}}{c^2(1-e^{-qh})}+1)g_1^2(x(t)),$$

or:

$$\dot{V} \leq |b|\left[\frac{1}{m_1}f^2(x(t)) - f(x(t))g_1(x(t))\right] + \left(c_1 - \frac{q}{m_2}\right)g(x(t))g_1(x(t)) + \left(\frac{c_1}{m_2} + \frac{p}{q}(1-e^{-qh})\right)g^2(x(t)) - q(\frac{pe^{-2qh}}{c^2(1-e^{-qh})}+1)g_1^2(x(t)).$$

The above relationships can be modified, as follows:

$$\dot{V} \leq (c_1 \quad \frac{q}{m_2})g(x(t))g_1(x(t))$$
$$+ \left(\frac{c_1}{m_2} + \frac{p}{q}(1 \quad e^{\quad qh})\right) g^2(x(t))$$
$$\left(q\left(\frac{pe^{\quad 2qh}}{c^2(1 \quad e^{\quad qh})} + 1\right) + \frac{bm_1}{4}\right) g_1^2(x(t)).$$

In order to convert the above inequality into a quadratic form, in terms of g or g_1, one should have the following inequalities [19]:

i) $q\left[\frac{pe^{\quad 2qh}}{c^2(1 \quad e^{\quad qh})} + 1\right] + \frac{bm_1}{4} \geq 0$

$$\Rightarrow p \geq c^2(e^{qh} \quad e^{2qh})\left[1 + \frac{bm_1}{4q}\right],$$

ii) $\frac{c_1}{m_2} + \frac{p}{q}(1 \quad e^{\quad qh}) \leq 0 \Rightarrow p \leq \frac{qc_1}{m_2(e^{\quad qh} \quad 1)}$.

Finally, $\dot{V}$ is obtained, as follows:

$$\dot{V} \leq \left[\frac{c_1}{m_2} + \frac{p}{q}(1 \quad e^{\quad qh}) + \frac{c^2\left(c_1 \quad \frac{q}{m_2}\right)^2(1 \quad e^{\quad qh})}{4q(pe^{\quad 2qh} + c^2(1 \quad e^{\quad qh})) + bm_1c^2(1 \quad e^{\quad qh})}\right] g^2(x(t)).$$

If $\dot{V}$ is negative definite (i.e., System 4 is Lyapunov stable), then:

$$\left[\frac{c_1}{m_2} + \frac{p}{q}(1 \quad e^{\quad qh}) + \frac{c^2(c_1 \quad \frac{q}{m_2})^2(1 \quad e^{\quad qh})}{4q(pe^{\quad 2qh} + c^2(1 \quad e^{\quad qh})) + bm_1c^2(1 \quad e^{\quad qh})}\right] \leq 0. \tag{6}$$

From (i) and (ii), there exists a p, such that:

$$c^2(e^{qh} \quad e^{2qh})\left[1 + \frac{bm_1}{4q}\right] \leq p \leq \frac{qc_1}{m_2(e^{\quad qh} \quad 1)}, \tag{7}$$

$$c^2(e^{qh} \quad e^{2qh})\left[1 + \frac{bm_1}{4q}\right] \leq \frac{qc_1}{m_2(e^{\quad qh} \quad 1)}. \tag{8}$$

Hence, first q is derived properly from Relation 8 and then p is determined by Relation 7.

NUMERICAL EXAMPLE

As an illustration, consider the scalar equation:

$$\dot{x}(t) = \quad x^3(t) \quad 2x^3(t \quad h)\sin(x(t \quad h)), \qquad t \geq 0, \tag{9}$$

with the initial conditions assumed to be:

$$\varphi(t) = 0.5\sin(t), \qquad h \leq t \leq 0.$$

Here, the object is to investigate the effect of delay (h) on the stability of System 9 and compare the simulation results with the results obtained using the proposed approach. Simulations show that System 9 is stable for h up to $h = 15$. For example, if $h = 2$ is chosen, then one would choose $q = \quad 0.1$ and obtain the range of p from $m_1 = 0.5$ to $m_2 = 0.2$:

$$1.3357 \leq p \leq 3.6979.$$

Choosing $p = 2$, the derivative of the Lyapunov functional would be, as follows:

$$\dot{V} \leq [\quad 2.543]f^2(x(t)).$$

The simulation results show that System 9 is locally stable with the above conditions (Figure 1).

CONCLUSION

This paper presents some explicit expressions for derivation of the Lyapunov-Krasovskii functionals. The functionals are used to analyze delay-dependent stability for a class of nonlinear time-delay systems. The other derivations have been obtained for other nonlinear time-delay systems [15,16]. The simulation results are consistent with those obtained by theory. Less conservatism and derivation of delay-independent stability conditions are the advantages of this procedure.

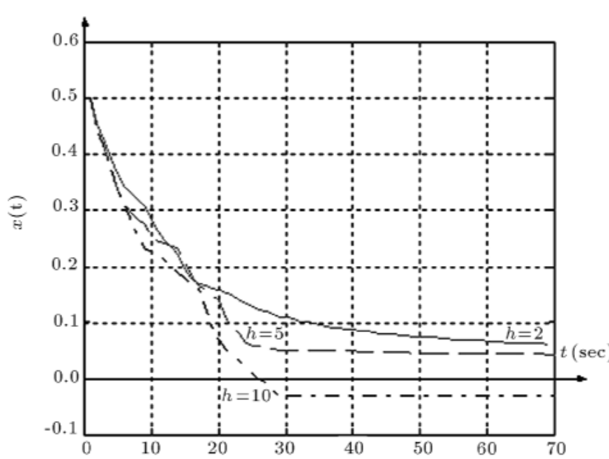

Figure 1. The trajectories of System 9 with $h \leq 1.0$.

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