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Research Note

A New Technique for Approximate Solutions of the Nonlinear Volterra Integral Equations of the Second Kind

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In this paper, a different approach for finding an approximate solution of the Nonlinear Volterra Integral Equations (NVIE) of the Second Kind is presented. In this approach, the nonlinearity of the kernel has no serious effect on the convergence of the solution. The author's approach is simple and direct for solving the (NVIE). The solution of the original problem is obtained, by converting the problem into an optimal moment problem. The moment problem is modified into one consisting of the minimization of a positive linear functional over a set of Radon measures. Then, an optimal measure is obtained, which is approximated by a finite combination of atomic measures and, by using atomic measures, this one is changed into a semi-infinite dimensional nonlinear programming problem. The latter is approximated by a finite dimensional linear programming problem. Finally, the approximated solution for some examples is found.

INTRODUCTION

The problem of finding numerical solutions for integral equations is an important case and many computational methods have been proposed in this area (see [1-4]). It seems that the idea of solving NVIEs by reducing them to an optimization problem gives rise to a fascinating result.

Consider the following nonlinear Volterra integral equation:

$$u(x) = f(x) + \int_a^x k(t, x, u(t))dt, \tag{1}$$

where $x \in [a, b]$, a.e. The necessary and sufficient conditions of existence and uniqueness of the solution for the above problem can be found in [5]. Here, it is assumed that the problem, which is being considered, has a unique solution, but it is not always the case (see [3,6]). In the next section, it is shown that one can convert the NVIE to an optimal moment problem.

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OBTAINING THE MOMENT PROBLEM

Let the interval $[a, b]$ be divided to M subintervals, $J_i = [x_{i\ \ 1}, x_i]$, $i = 1, \cdots, M$, where $x_0 = a$ and $x_M = b$ and $[a, b] = \bigcup_{i=1}^{M} [x_{i\ \ 1}, x_i]$.

Now, Equation 1 on $[a, b]$ is converted into the following system:

$$\begin{cases} \int_{x_0}^{x_1} k(t, x_1, u(t))dt \quad u(x_1) = \quad f(x_1), \\ \int_{x_0}^{x_2} k(t, x_2, u(t))dt \quad u(x_2) = \quad f(x_2), \\ \vdots \\ \int_{x_0}^{x_M} k(t, x_M, u(t))dt \quad u(x_M) = \quad f(x_M), \end{cases} . \tag{2}$$

System 2 can be written, equivalently, in the following form:

$$\begin{cases} \int_{x_0}^{x_1} k(t, x_1, u(t))dt \quad u(x_1) = \quad f(x_1), \\ \int_{x_0}^{x_1} k(t, x_2, u(t))dt + \int_{x_1}^{x_2} k(t, x_2, u(t))dt \quad u(x_2) = \quad f(x_2), \\ \vdots \\ \sum_{i=1}^{M} \int_{x_{i\ \ 1}}^{x_1} k(t, x_M, u(t))dt \quad u(x_M) = \quad f(x_M), \end{cases} \tag{3}$$

Now, the following minimization problem is defined:

$$\text{minimize} \sum_{i=1}^{M} \int_{x_{i-1}}^{x_i} g_i(t,u(t))dt, \tag{4}$$

subject to Problem 3, where the auxiliary functions $g_i(t,u(t))$ are arbitrary known functions. In fact, one only needs to find the feasible solution of system in Problem 3. Thus, finding a solution for System 2 is equivalent to finding a solution of the optimization Problems 3 and 4.

Definition 1

The trajectory function, $u(.) : [a,b] \to U \subset R$, is admissible if it is absolutely continuous and the constraints of Problem 3 are satisfied.

It is assumed that the set of all values, $u(.)$, denoted by U, is non-empty. Now, let the following be defined:

$$a_i = u(x_i) - f(x_i), \qquad (i = 1,2,\cdots,M). \tag{5}$$

Hence, the optimization Problems 3 and 4 are reduced to the following problem:

$$\text{minimize} \sum_{i=1}^{M} \int_{x_{i-1}}^{x_i} g_i(t,u(x))dt, \tag{6}$$

subject to:

$$\sum_{j=1}^{i} \int_{x_{j-1}}^{x_j} k(t,x_i,u(t))dt = a_i, \qquad (i = 1,2,\cdots,M). \tag{7}$$

This problem is known as an optimal moment problem and it is considered in the next section.

METAMORPHOSIS

Let $\Omega = \bigcup_{i=1}^{M} \Omega_i$, where $\Omega_i = J_i \times U$, $J = \bigcup_{i=1}^{M} J_i$, where J_i is defined as before and the set U is compact. Suppose $C(\Omega_i)$ to be the space of all continuous functions on Ω_i. Now, the moment minimization Problems 6 and 7 are replaced to another one, as follows:

1. The mapping $\Lambda_i : h_i \to \int_{x_{i-1}}^{x_i} h_i(t,u(t))dt$, $\forall h_i \in C(\Omega_i)$, defines a positive bounded linear functional on $C(\Omega_i)$, $(i = 1,2,\cdots,M)$;
2. By the Riesz representation theorem (see [7]), there exists a unique positive Radon measure, μ_i on Ω_i, such that:

$$\Lambda_i(h_i) = \int_{\Omega_i} h_i d\mu \equiv \mu_i(h_i), \qquad \forall h_i \in C(\Omega_i). \tag{8}$$

These measures, μ_i, are required to have certain properties. First, by Equation 8,

$$|\mu_i(h_i)| \le T_i \sup |h_i(t,u(t))|,$$

where $T_i = x_i - x_{i-1}$. Hence:

$$\mu_i(1) \le T_i.$$

Also, by Equations 7 and 8, one has:

$$\sum_{j=1}^{i} \mu_j(k(t,x_i,u(t))) = a_i, \qquad (i = 1,2,\cdots,M).$$

Finally, functions $\theta_i \in C(\Omega_i)$ are considered, which do not depend on u, that is, $\theta_i(t,u_1) = \theta_i(t,u_2)$, for all $t \in [x_{i-1},x_i]$, $u_1,u_2 \in U$, where $u_1(.) \ne u_2(.)$. Then,

$$\int_{\Omega_i} \theta_i d\mu = \int_{x_{i-1}}^{x_i} \theta_i(t,u(t))dt = \alpha_{\theta_i},$$

where u is an arbitrary element of U. Furthermore, α_{θ_i} is the integral of $\theta_i(.,u)$ over $[x_{i-1},x_i]$. Let $M^+(\Omega_i)$ be the set of all positive Radon measures on Ω_i. The set, Q, is defined as a subset of $M^+(\Omega)$, such that:

$$Q = S_1 \cap S_2 \cap S_3,$$

where:

$$S_1 = \left\{ (\mu_1,\mu_2,\cdots,\mu_M) \in \prod_{i=1}^{M} M^+(\Omega_i) : \mu_i(1) \le T_i, (i = 1,2,\cdots,M) \right\},$$

$$S_2 = \left\{ (\mu_1,\mu_2,\cdots,\mu_M) \in \prod_{i=1}^{M} M^+(\Omega_i) : \sum_{j=1}^{i} \mu_j(k(t,x_i,u(t))) = a_i, (i = 1,2,\cdots,M) \right\},$$

$$S_3 = \left\{ (\mu_1,\mu_2,\cdots,\mu_M) \in \prod_{i=1}^{M} M^+(\Omega_i) : \mu_i(\theta_i) = \alpha_{\theta_i}, \theta_i \in C(\Omega_i) \text{ and independent of } u \right\}.$$

By using the Alaoglu Theorem (see [8]) and the Tychonoff Theorem (see [9]), the set, S_1, is compact. The set, S_2, can be written, as follows:

$$S_2 = \bigcap_{i=1}^{M} \left\{ (\mu_1,\mu_2,\cdots,\mu_M) \in \prod_{i=1}^{M} M^+(\Omega_i) : \sum_{j=1}^{i} \mu_j(k(t,x_i,u(t))) = a_i \right\} = \bigcap_{i=1}^{M} M_i,$$

where:

$$M_i = \left\{ (\mu_1, \mu_2, \cdots, \mu_M) \in \prod_{i=1}^{M} M^+(\Omega_i) \right.$$

$$\left. : \sum_{j=1}^{i} \mu_j(k(t, x_i, u(t))) = a_i \right\},$$

$$i = 1, 2, \cdots, M,$$

is closed, because it is the inverse image of a closed singleton set on the real line, the set $\{a_i\}$, under a continuous map. By a similar argument, it can be shown that S_3 is closed. If one topologizes the space, $M^+(\Omega)$ by the weak *-topology, since Q is a closed subset of the compact set, S_1, therefore, Q is compact. By definition of a convex set, it is concluded that the sets, S_1, S_2 and S_3, are convex, thus, Q is a compact convex set (see [10]). By the Krein-Milman theorem (see [11]), Q has extreme points.

Now, the original minimization problem is replaced by one in which the minimum of:

$$I\left(\sum_{i=1}^{M} \mu_i\right) = \int_{\Omega_i} \sum_{i=1}^{M} g_i d\mu \equiv \sum_{i=1}^{M} \mu_i(g_i), \tag{9}$$

is sought over the compact, convex set, Q.

Theorem 1

The measure-theoretic problem, which consists of finding the minimum of the functional in Equation 9 over the set, Q, of $M^+(\Omega)$, attains its minimum, $\mu^* = \sum_{i=1}^{M} \mu_i^*$, in the set, Q.

Proof

The proof is clear, since $\mu^* = \sum_{i=1}^{M} \mu_i^*$ is continuous and Q is a compact, convex set.

AN APPROXIMATION OF OPTIMAL TRAJECTORY BY AN OPTIMAL MEASURE

In this section, an approximation to the optimal measure, $\mu^* \in M^+(\Omega)$, is obtained that is defined in Theorem 1. For this purpose, the minimization of the functional in Equation 9 over the set, Q, is considered, as follows:

$$\text{minimize } I(\sum_{i=1}^{M} \mu_i) = \mu_i(\sum_{i=1}^{M} g_i), \tag{10}$$

subject to:

$$\begin{cases} \mu_i(1) \leq T_i, & (i = 1, 2, \cdots, M), \\ \sum_{j=1}^{i} \mu_j(k(t, x_i, u(t))) = a_i, & (i = 1, 2, \cdots, M), \\ \mu_i(\theta_i) = \alpha_{\theta_i}, \theta_i \in C(\Omega_i), & (i = 1, 2, \cdots, M) \\ & \text{and independent of } u. \end{cases} \tag{11}$$

It is known that this problem is one of linear programming; all the functions in Problems 10 and 11 are linear in the variable, μ_i, and, furthermore, the measure, μ_i, is required to be positive.

The linear programming described above is infinite-dimensional and an approximate solution to this problem can be obtained by solving a finite dimensional linear programming associated with Problems 10 and 11. In the above linear programming, only a finite number, G_i, of functions θ_i $(i = 1, 2, \cdots, M)$ are chosen, as follows:

$$\begin{cases} \theta_{1,1} & \theta_{1,2}, \ , \cdots, \ \theta_{1,G_1} \\ \theta_{2,1} & \theta_{2,2}, \ , \cdots, \ \theta_{2,G_2} \\ \vdots \\ \theta_{M,1} & \theta_{M,2}, \ , \cdots, \ \theta_{M,G_M} \end{cases} \tag{12}$$

By this selection, the linear programming (Problems 10 and 11) has $2M + \sum_{i=1}^{M} G_i$ constrained $(i = 1, 2, \cdots, M)$ and is called a semi-infinite dimensional linear programming.

Now, J_i is divided into m_i and U into p subintervals. Thus, a grid is obtained for $\Omega_i = [t_{i-1}, t_i] \times U$ and every subrectangle obtained as Ω_{ik} is renamed. A member from each subrectangle of Ω_{ik} is taken and it is denoted by $z_{ik} = (t_{ik}, u_k)$.

According to the result of [12], it is shown that the optimal measure, μ^*, which satisfies the constraints in Problem 11 and minimizes the functional in Problem 10, has the following form:

$$\mu^* = \sum_{i=1}^{M} \sum_{k=1}^{R_i} \beta_{ik}^* \delta(z_{ik}^*), \tag{13}$$

where $R_i = m_i p$ is the number of points in the above partition, $z_{ik}^* \in \Omega_i$, the coefficients, $\beta_{ik}^* \geq 0$, $k = 1, 2, \cdots, R_i$, are unknown and $\delta(z_{ik})$, for each k, denote a unitary atomic measure with the support of the singleton set, $\{z_{ik}^*\}$, such that $\delta(z_{ik}^*)F_i = F_i(z_{ik}^*)$, for all $F_i \in C(\Omega_i)$, $(i = 1, 2, \cdots, M)$.

Let Ω_i be divided into R_i rectangles, Ω_{ik}, $(i = 1, 2, \cdots, M)(k = 1, 2, \cdots, R_i)$. Points $z_{ik} \in \Omega_{ik}$ are chosen and $\sigma_i = \{z_{ik}; k = 1, 2, \cdots, R_i\}$.

Now, let $P(M, \sum_{i=1}^{M} G_i, \varepsilon) \subseteq R^{\left(M + \sum_{i=1}^{M} R_i\right)}$ be the set of all $(\beta_{ik};\ k = 1, 2, \cdots, R_i,\ i = 1, 2, \cdots, M)$ and the variables, $u(x_j)(j=1,2,\cdots,M)$, be defined by:

$$\begin{cases} |\sum_{i=1}^{j}\sum_{k=1}^{R_i}\beta_{ik}k_j(z_{ik}) \quad a_j| \le \varepsilon, & (j=1,2,\cdots,M), \\ |\sum_{k=1}^{R_i}\beta_{ik}\theta_{is}(z_{ik}) \quad \alpha_{\theta_{is}}| \le \varepsilon, & (s=1,2,\cdots,G_i), \\ & (i=1,2,\cdots,M), \\ \sum_{k=1}^{R_i}\beta_{ik} \le T_i, & \\ \beta_{ik}\ge 0, & (i=1,2,\cdots,M), \\ & (k=1,2,\cdots,R_i), \\ u(x_j) \text{ is free,} & (j=1,2,\cdots,M). \end{cases}$$

where k_j is the corresponding kernel, $k(t,x_j,u(t))$ $(j=1,2,\cdots,M)$.

Proposition 1

For every $\varepsilon>0$, the problem of minimizing the function, $\sum_{i=1}^{M}\sum_{k=1}^{R_i}\beta_{ik}g_i(z_{ik})$, $z_{ik}\in\sigma_i$, on the set $P(M,\sum_{i=1}^{M}G_i,\varepsilon)$, has a solution for $R_i=R_i(\varepsilon)(i=1,2,\cdots,M)$ sufficiently large. The solution satisfies:

$$\mu_Q\left(\sum_{i=1}^{M}\sum_{k=1}^{R_i}\beta_{ik}g_i(z_{ik})\right)+\rho(\varepsilon)\le\sum_{i=1}^{M}\sum_{k=1}^{R_i}\beta_{ik}g_i(z_{ik})$$

$$\le\mu_Q\left(\sum_{i=1}^{M}\sum_{k=1}^{R_i}\beta_{ik}g_i(z_{ik})\right)+\varepsilon,$$

where $\rho(\varepsilon)$ tends to zero as ε tends to zero.

Proof

Now, by choosing m_i's $(i=1,2,\cdots,M)$ and p large enough, so that σ_i be approximately dense in Ω_i (see Theorem III in [9]), one can compute $\mu_{Q(M,\sum_{i=1}^{M}G_i)}(g_i)$ (the approximate value of the $\mu^*(g_i)$), by the following linear programming problem:

$$\text{minimize } \sum_{i=1}^{M}\sum_{k=1}^{R_i}\beta_{ik}g(z_{ik}), \tag{14}$$

over the set, $\beta_{ik}\ge 0$, $(i=1,2,\cdots,M)$, $(k=1,2,\cdots,R_i)$ and $u(x_j)$, $(j=1,2,\cdots,M)$, such that:

$$\begin{cases} \sum_{i=1}^{j}\sum_{k=1}^{R_i}\beta_{ik}k_j(z_{ik}) & \\ \quad u(x_j) = \quad f(x_j), & (j=1,2,\cdots,M), \\ \sum_{k=1}^{R_i}\beta_{ik}\theta_{is}(z_{ik})=\alpha_{\theta_{is}}, & (i=1,2,\cdots,M), \\ & (s=1,2,\cdots,G_i), \\ \sum_{k=1}^{R_i+1}\beta_{ik}=T_i, & (i=1,2,\cdots,M), \\ u(x_j) \text{ is free,} & (j=1,2,\cdots,M). \end{cases} \tag{15}$$

The slack variables, $\beta_{iR_{i+1}}$, are used to put the inequality, $\sum_{k=1}^{R_i}\beta_{ik}\le T_i$, in the form of equality. Note that, $\theta_{is}(t,u)$ is chosen, as follows:

$$\theta_{is}(t,u)=\begin{cases}1 & \text{if } t\in J_{is}\\ 0 & \text{otherwise}\end{cases}$$

where $J_{is}=\left[\frac{(s \quad 1)T_i}{G_i},\frac{sT_i}{G_i}\right]$, $(s=1,2,\cdots,G_i)$, $(i=1,2,\cdots,M)$. So:

$$\alpha_{\theta_{is}}=\int_{J_{is}}\theta_{is}(t,u)dt=\frac{T_i}{G_i},$$

$$(s=1,2,\cdots,G_i),\ (i=1,2,\cdots,M).$$

Thus, $\beta_{iR_{i+1}}=0$, $(i=1,2,\cdots,M)$. Consequently, Problems 14 and 15 change to the following linear programming problem:

$$\text{minimize } \sum_{i=1}^{M}\sum_{k=1}^{R_i}\beta_{ik}g(z_{ik}), \tag{16}$$

subject to:

$$\begin{cases} \sum_{i=1}^{j}\sum_{k=1}^{R_i}\beta_{ik}k_j(z_{ik}) & \\ \quad u(x_j) = \quad f(x_j), & (j=1,2,\cdots,M), \\ \sum_{k=1}^{R_i}\beta_{ik}=T_i, & (i=1,2,\cdots,M), \\ \beta_{ik}\ge 0, & (i=1,2,\cdots,M), \\ & (k=1,2,\cdots,R_i), \\ u(x_j) \text{ is free,} & (j=1,2,\cdots,M). \end{cases} \tag{17}$$

NUMERICAL EXAMPLES

The solution of some nonlinear Volterra integral equations of the second kind has been estimated by using the technique developed here. Before presenting the result, it is necessary to make several comments:

I. The bounded space solution, U, for $u(x)$, is chosen appropriate, such that the approximate solution can be obtained, accurately;

II. The sets of the form $z_{ik}(i=1,2,\cdots,M)$, $(k=1,2,\cdots,R_i)$, were constructed by dividing the appropriate intervals into a number of subintervals, defining, in this way, a grid of points;

III. The solution of Linear Programs 16 and 17, was estimated by means of a home-made revised simplex method (see [13]);

IV. Without loss of generality, in the optimization Problems 3 and 4, the criteria function $g_i(t,u(t))=0$, $(i=1,2,\cdots,M)$ was chosen;

V. By using the slack variables $u(x_i)$, $(i = 1, 2, \cdots, M)$, which have been obtained by solving Linear Programmings 16 and 17, one can get an approximate solution for the original problem (Equation 1), as follows:

$$u(x) = u(x_{i-1}) + \frac{u(x_i) - u(x_{i-1})}{x_i - x_{i-1}}(x - x_{i-1}),$$
$$x \in [x_{i-1}, x_i], \ (i = 1, 2, \cdots, M), \tag{18}$$

where $u(x_0)$ is given;

VI. The computation errors of the exact solution, $u(x)$, and the approximate solution, $u^*(x)$, are compared, as follows:

$$E(u(x), u^*(x)) = \sum_{i=1}^{M} (u(x_i) - u^*(x_i))^2.$$

In the following examples, one assumes that $M = 3$ and $J = [0, 1]$ is divided into $J_1 = [0, 0.3]$, $J_2 = [0.3, 0.7]$ and $J_3 = [0.7, 1]$.

Example 1

Consider the following nonlinear Volterra integral equation of the second kind, with the analytical solution, $u(x) = e^x$, on $[0, 1]$:

$$u(x) = e^x - \frac{1}{3}xe^{3x} + \frac{1}{3}x + \int_0^x xu(t)^3 dt, \qquad 0 \le x \le 1.$$

In this example, the appropriate intervals are chosen, as follows:

m_1	m_2	m_3	U	p	R_1	R_2	R_3
10	10	10	[0.06, 46.4]	20	200	200	200

Thus, one has the following linear programming problem:

minimize $0^t\beta$,

subject to:

$$\sum_{k=1}^{R_1} 0.3u_{1k}^3\beta_{1k} + u(x_1) = 1.2039,$$

$$\sum_{k=1}^{R_1} 0.7u_{1k}^3\beta_{1k} - \sum_{k=1}^{R_2} 0.7u_{2k}^3\beta_{2k} + u(x_2) = 0.3416,$$

$$\sum_{k=1}^{R_1} u_{1k}^3\beta_{1k} + \sum_{k=1}^{R_2} u_{2k}^3\beta_{2k} + \sum_{k=1}^{R_3} u_{3k}^3\beta_{3k} - u(x_3) = 3.6436,$$

$$\sum_{k=1}^{R_1} \beta_{1k} = 0.3,$$

$$\sum_{k=1}^{R_2} \beta_{2k} = 0.4,$$

$$\sum_{k=1}^{R_3} \beta_{3k} = 0.3,$$

$$\beta_{ik} \ge 0, \qquad i = 1, 2, 3, \ k = 1, 2, \cdots, R_i,$$

$$u(x_i) \text{ is free}, \qquad i = 1, 2, 3,$$

where $\beta = (\beta_{ik}, \ i = 1, 2, 3, \ k = 1, 2, \cdots, R_i)$. To compare the results, see Table 1.

The error function is $E(u(x), u^*(x)) = 0.0013$. In Figure 1, the suboptimal trajectories of the approximate solution and the exact solution of the above integral equation are compared.

Example 2

Consider the following nonlinear integral equation, with the exact solution, $u(x) = x^2$ on $[0, 1]$:

$$u(x) = x + \frac{1}{2}x(1 - \cosh(x^2)) + \int_0^x xt \sinh(u(t))dt,$$

$$0 \le x \le 1.$$

The summarized results are shown in Table 2. Figure 2 shows the approximate and the exact solutions. The error function is $E(u(x), u^*(x)) = 0.3608 \times 10^{-4}$.

Table 1. The numerical results of Example 1.

x_i	0	0.3	0.7	1
$u^*(x_i)$	1	1.3417	1.9801	2.7262
$u(x_i)$	1	1.3499	2.0138	2.7183

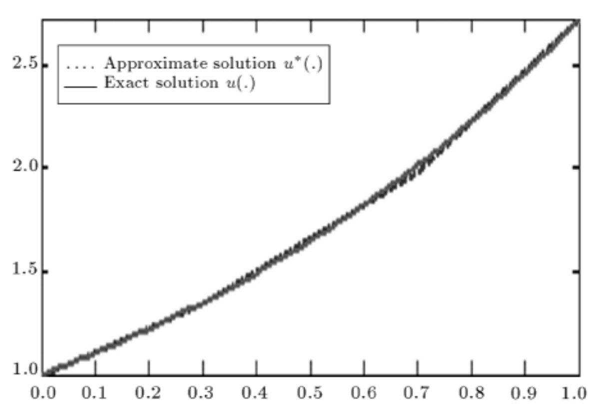

Figure 1. The suboptimal trajectory comparison between the approximate and exact solutions of Example 1.

Table 2. The numerical results of Example 2.

x_i	0	0.3	0.7	1
$u^*(x_i)$	0	0.0924	0.4928	0.0982
$u(x_i)$	0	0.09	0.49	1

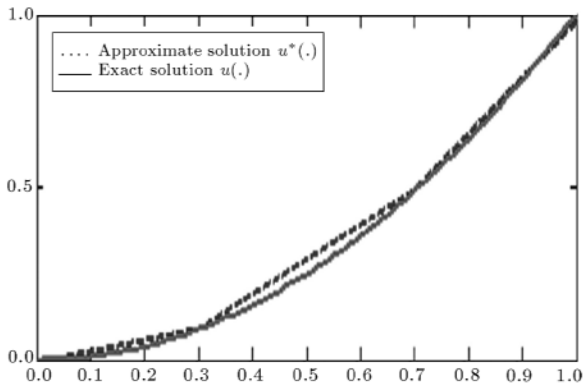

Figure 2. The suboptimal trajectory comparison between the approximate and exact solutions of Example 2.

Example 3

Consider the following integral equation, with the exact solution, $u(x) = e^x$ on $[0, 1]$:

$$u(x) = e^x \quad x\sin(x) + \int_0^x \sin(x)u^3(t)e^{\ 3t}dt,$$

$0 \le x \le 1.$

Table 3 represents the results for this example:

The error function is $E(u(x), u^*(x)) = 0.0031$. The approximate and exact solutions are shown in Figure 3.

Example 4

Consider the following integral equation of the second kind, with the exact solution, $u(x) = x$ on $[0, 1]$:

$$u(x) = \frac{3}{2}x \quad \frac{1}{2}xe^{x^2} + \int_0^x xte^{u^2(t)}dt,$$

$0 \le x \le 1.$

The results for this example are represented in Table 4.

The error function is $E(u(x), u^*(x)) = 0.0011$. Compare the approximate and exact solutions in Figure 4.

Table 3. The numerical results of Example 3.

x_i	0	0.3	0.7	1
$u^*(x_i)$	1	1.3222	1.9956	2.6736
$u(x_i)$	1	1.3499	2.0318	2.7183

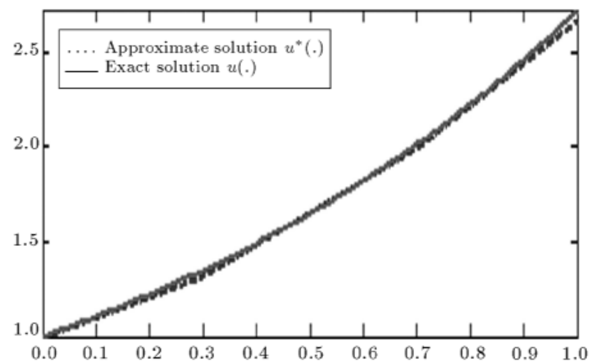

Figure 3. The suboptimal trajectory comparison between the approximate and exact solutions of Example 3.

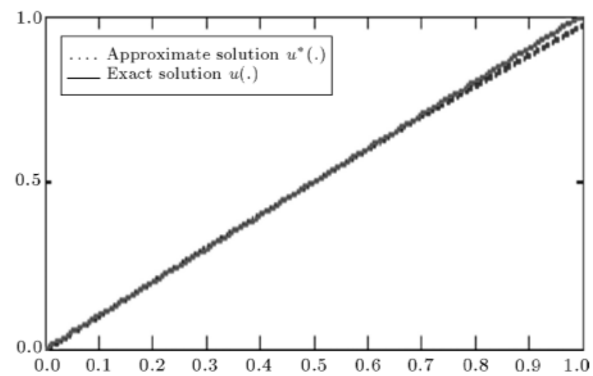

Figure 4. The suboptimal trajectory comparison between the approximate and exact solutions of Example 4.

Table 4. The numerical results of Example 4.

x_i	0	0.3	0.7	1
$u^*(x_i)$	0	0.3075	0.6911	0.9688
$u(x_i)$	0	0.3	0.7	1

CONCLUSIONS

This paper presents a method to find the solution of nonlinear Volterra integral equations by an optimization method that is based on some principles of measure theory, functional analysis and linear programming. In comparison to the other methods, the authors' approach has some advantages. For example, this method is not iterative and it solves the problem directly, without need of any initial guess. It is necessary to mention that the approximate solution for the nonlinear Volterra integral equations, obtained by piecewise lines on subintervals $[x_{i\ 1}, x_i]$, is based on Relation 18.

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