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Detection of a Band-Limited Signal Using an Orthonormal, Fully-Decimated Filter-Bank

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In this paper, two methods are proposed for the detection of a band-limited signal in unknown variance white Gaussian noise. The complex amplitude and the frequency of the signal and the noise variance are assumed as unknown parameters. Using wavelet concepts, an orthonormal, fully-decimated filter-bank is employed to decompose the signal into its subband components. It is shown that, in this process, the noise is also decomposed into orthonormal zero-mean components. In the output, if a band-limited target signal is present, the respective single subband component (or two components in marginal cases) containing the target signal presents a non-zero mean. The presence of a non-zero mean component(s) in this canonical form is tested using a well-known Generalized Likelihood Ratio (GLR) solution (F-test), which is based on the ratio between the output power of one (or two) subband(s) and the average output power of the other subbands (estimating the noise variance). Comparing to a threshold, a Constant False Alarm Rate (CFAR) detector is constructed. Since the target signal's central frequency is unknown, the proper subband(s) is selected as the one (or two) maximizing the F-test statistic and a GLR test, namely a Wavelet Detector (WD), is obtained. It turns out that the performance of WD depends on the frequency of the signal. For instance, a lowpass signal is detected better than a bandpass signal by this detector. To overcome this problem, the frequency band, where the signal may exist, is estimated, and the signal is down-converted such that the detection is always accomplished at the lowest subband in the new detector, a Modified WD (MWD). The performance of the proposed methods is evaluated in solving two well-known problems, compared with the existing DFT detector. A sinusoid with unknown amplitude, phase and frequency is detected by these detectors as an approximately band-limited signal. The proposed detectors are also applicable for the detection of a signal composed of a white component and an approximately band-limited component. A sinusoid, with unknown phase and frequency and Rayleigh-distributed amplitude, is also detected as such a signal.

INTRODUCTION

In this paper, detectors are developed for the detection of a band-limited signal in unknown variance white Gaussian noise. Band-limited signal detection has several applications in system-condition diagnosis, such as: Industrial process monitoring, astrophysics, underwater surveillance and acoustic signal processing. The detection of a narrow-band signal in noise is studied in literature (see e.g., [1,2] and the references therein).

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A band-limited signal detector can also be employed for the detection of some approximately band-limited signals. For instance, the detection of a sinusoid signal with unknown amplitude, phase and frequency, which is approximately a band-limited signal, can be performed using a band-limited signal detector. Detection of such a signal in unknown Gaussian noise has various applications in different areas, e.g., in coherent radar detection with unknown Doppler shift. It is a challenging problem and different approaches with different assumptions about the signal and noise parameters are proposed to solve the problem. An Average Likelihood Ratio (ALR) detector and a GLR detector are proposed in [3], assuming the noise variance as a known parameter, however, closed form solutions are not derived. It is shown that the Uniformly Most Powerful (UMP) and Uniformly Most Powerful Invariant (UMPI) tests do not exist for this problem (see e.g., [4,5] and the references therein). However, assuming that the Signal-to-Noise Ratio (SNR) is given, a UMPI test is derived in [5] that also provides an upper performance bound for any invariant test in unknown SNR, such as the GLR test. Simulations illustrate that this GLR test is very close to UMPI. The Discrete Fourier Transform (DFT) detector is also proposed for the detection of a sinusoid signal with unknown parameters and, generally, for a periodic signal, e.g., [6,7]. Using the DFT matrix to transform the problem to the canonical form, a GLR detector is proposed, based on the F-test in [8], named the Modified DFT (MDFT).

A band-limited signal detector can also be used for the detection of some signals with special formats. It seems that, for some signals, an implementable classic detector, such as ALR or GLR, cannot be derived, due to the signal complicated parameter space. If these signals can be represented as the composition of a white part and an approximately band-limited component, they can be detected by a band-limited signal detector. For instance, it will be shown that the authors' method can be applied for the coherent detection of a rapid fluctuating radar signal. The detection of a sinusoid signal with the Rayleigh-distributed variable amplitude (e.g., rapid fluctuating signal) is outlined in some references (see e.g., [9,10] and the references therein). However, due to the complicated detector structure of this problem, it is assumed that, in [9,10], the noise variance and Doppler frequency are known as parameters.

In the band-limited signal detector in [2], Wang et al. considered the filtered white noise as the narrow-band signal in known variance additive white Gaussian environmental noise. They used wavelet systems to estimate some of the unknown parameters and derived a GLR test for the problem. In contrast, in this paper, no assumption is made of the band-limited signal format and the noise variance and the detector is derived for a general band-limited signal with unknown parameters. The proposed method, at first, converts the band-limited signal detection problem to the canonical form. A filter-bank is used, consisting of filters with special orthogonal properties, so that the output noise remains independent and identically distributed (i.i.d.). It is shown that a filter-bank, based on an orthonormal wavelet system, satisfies these conditions. The detector is used for the detection of a sinusoid with unknown deterministic parameters as an approximately band-limited signal. An efficient detector is also proposed for a sinusoid with unknown phase and frequency and Rayleigh-distributed variable amplitude, as a signal with approximately band-limited non-white component.

The remainder of this paper is organized as follows. First, the formulation of the detection problem is presented. Then, the general linear hypothesis is reviewed; the DFT and MDFT detector are briefly explained; the bases of the detector, i.e., the Wavelet Detector (WD) are outlined and the Modified Wavelet Detector (MWD) is introduced, considerably improves the performance. After that, two well-known examples are presented that can be solved by the authors' methods and the performance results of the authors' algorithms are compared with the existing algorithm (MDFT) for these problems. Finally, the conclusions are summarized.

DETECTION PROBLEM STATEMENT

The detection of a band-limited signal with unknown parameters in complex white Gaussian noise is considered, where the noise has an unknown variance, σ^2. The hypothesis testing problem is:

$$\begin{cases} \mathcal{H}_0 : \mathbf{r} = \mathbf{n} \\ \mathcal{H}_1 : \mathbf{r} = \mathbf{s} + \mathbf{n} \end{cases}, \tag{1}$$

where $\mathbf{r} = [r[0], \cdots, r[N-1]]^T$, $\mathbf{s} = [s[0], \cdots, s[N-1]]^T$ and $\mathbf{n} = [n[0], \cdots, n[N-1]]^T$ are the vectors of the received signal samples, the band-limited signal samples and identically distributed complex white Gaussian noise samples, respectively. The frequency band, where the band-limited signal may exist, is assumed to be unknown. The noise samples are assumed to be zero-mean with the unknown common variance, σ^2; $\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \sigma^2 I)$, where $\mathbf{0}$ is a zero-vector of length N.

PROPOSED DETECTORS

General Linear Hypothesis

The general linear hypothesis uses the principle of invariance in hypothesis testing [11,12]. One form of this problem concerns independent real random variables, $\mathbf{u} = [u[1], \cdots, u[n]]^T$, having normal distributions with the mean, $E[u[i]] = \mu_i$, $i = 1, \cdots, n$ and common variance σ^2. If $\mu_{k+1} = \cdots = \mu_n = 0$, $k < n$, this problem is called the canonical form of the general linear hypothesis and it can be shown that the rejection region for the F-test, which is UMPI (and also GLR [13,14]), in testing $\mathcal{H}_0 : (\mu_1, \cdots, \mu_r) = \mathbf{0}$ versus $\mathcal{H}_1 : (\mu_1, \cdots, \mu_r) \neq \mathbf{0}$ for $r < k$ is, as follows (see e.g., [11,12,15]):

$$\mathcal{F} = \frac{\frac{1}{r}\sum_{i=1}^{r} u^2[i]}{\frac{1}{n-k}\sum_{i=k+1}^{n} u^2[i]} > \eta, \tag{2}$$

where η is chosen, such that the probability of a false alarm requirement is satisfied and $\mathbf{0}$ is a zero-vector of length r. For instance, for $r=1$ in testing $\mathcal{H}_0 : \mu_1 = 0$ against $\mathcal{H}_1 : \mu_1 \neq 0$, where the observation is of the mean $[\mu_1, 0, \cdots, 0]$ and the unknown variance, $\sigma^2 I$, the decision criteria for the test will be $\frac{u^2[1]}{\sum_{i=2}^{N} u^2[i]}$. Note that, if $E[u[i]] = \mu_i \neq 0, \forall i$, Equation 2 cannot be applied directly. However, if the problem can be converted to the canonical form; e.g., by using a transformation, then, the test in Equation 2 is applicable.

DFT Detector

The DFT detector is used for the detection of a sinusoid signal (see e.g., [6-8]) and, potentially, can be used for the detection of a band-limited signal. Since DFT decomposes a signal to its frequency components, employing this transformation for a band-limited signal results in zero and non-zero components, where the zero-components reflect the signal-free frequency bands of the band-limited signal. Therefore, DFT converts the problem of band-limited signal detection to the canonical form, while the transformed noise is still white [8]. For resulting such a detection problem in canonical form, one can use the F-test and propose the following GLR detector:

$$\max_{1 \leq k \leq N} \frac{\left| \sum_{n=1}^{N} r[n] e^{-j\frac{2\pi}{N}kn} \right|^2}{\sum_{n=1}^{N} |r[n]|^2} = \max_{1 \leq k \leq N} \frac{|R[k]|^2}{\sum_{n=1}^{N} |r[n]|^2},$$

where $R[k], k = 0, \cdots, N$ is the DFT of $r[k], k = 1, \cdots, N$. Apparently, some performance loss would happen at marginal frequencies that are multipliers of $\frac{\pi}{N}$, where N is the DFT order. To overcome this problem, a method in [8] is proposed, termed a Modified DFT (MDFT) detector,

$$\max_{1 \leq k \leq N} \frac{|R[k]|^2 + |R[k+1]|^2}{\sum_{n=1}^{N} |r[n]|^2}.$$

It must be noted that both DFT and MDFT detectors are CFAR [8].

Wavelet Detector

It is mentioned in [8] that DFT acts by applying the signal to a filter-bank that converts the problem to the canonical form, while it remains noise in white form. In this section, a fully decimated filter-bank, based on an orthogonal wavelet system, is used for signal decomposition. It will be seen that this transformation performs a better filtering on the received signal and provides more independent data for noise estimation compared to DFT, while the transformed noise is still white.

Referring back to detection Problem 1, $r[n]$ consists of two terms, $s[n]$ and $n[n]$, under hypothesis $\mathcal{H}_1$. The output of the wavelet system to each input term, signal and noise is studied. Suppose that the noise term, $n[n]$, is the input to the filter-bank in Figure 1a and $w_{i,q}[n]$ is the output noise of the ith branch at the qth stage of processing ($i = 0, \cdots, 2^q - 1$). It will be shown, in Appendix A that $w_{i,q}[n], 0 \leq n \leq N - 1, 0 \leq i \leq 2^q - 1$ are independent Gaussian R.V.s of variance σ^2. It can be seen that all the orthogonal properties of the wavelet filters are used to prove that the output noise is i.i.d.

The fully-decimated filter-bank system in Figure 1a is equivalent to the structure in Figure 1b [16] and the frequency response of the resulting filter-bank is schemed in Figure 1c. The noise and the signal components are demonstrated in Figures 1a and 1b, respectively, to introduce notations. If a band-limited signal, with the bandwidth B, is applied to the system of Figure 1a and $B \leq \frac{\pi}{2^q}$ ($\frac{\pi}{2^q}$ is the bandwidth of each subband filter), then, the output of the associated subband, i.e., the output corresponding to the qth branch of the filter-bank in Figure 1, contains the signal and has non-zero mean, while other bands are signal-free noise samples. The equivalent problem on the system output is:

$$\begin{cases} \mathcal{H}_0 : \mathbf{z} = \mathbf{w} \\ \mathcal{H}_1 : \mathbf{z} = \mathbf{y} + \mathbf{w} \end{cases}, \tag{3}$$

where $\mathbf{z} = [z_{0,q}[0], \cdots, z_{0,q}[D_q - 1], \cdots, z_{2^q-1,q}[0], \cdots, z_{2^q-1,q}[D_q - 1]]^T$ is the output of the wavelet structure in Figure 1a, $\mathbf{y} = [y_{0,q}[0], \cdots, y_{0,q}[D_q - 1], \cdots, y_{2^q-1,q}[0], \cdots, y_{2^q-1,q}[D_q - 1]]^T$ is the signal term and $\mathbf{w} = [w_{0,q}[0], \cdots, w_{0,q}[D_q - 1], \cdots, w_{2^q-1,q}[0], \cdots, w_{2^q-1,q}[D_q - 1]]^T$ is the output noise. Note that $z_{i,q}[n]$ is the output of the ith branch at the qth stage of processing ($i = 0, \cdots, 2^q - 1$), $y_{i,q}[n] = x_{i,q}[2^q n]$, $x_{i,q}[n] = s[n] * h'_i[n]$, $*$ is the convolution and $h'_i[n], i = 0, \cdots, 2^q - 1$ are the filters in Figure 1b. D_q is the output length of each branch at the qth stage and, assuming that the wavelet filters, $h_0[n]$ and $h_1[n]$, are of length L and $n[n]$ and $s[n]$ are of length N, one has (see Appendix B):

$$D_q = \left\lceil \frac{N + L_q^{\text{total}} - 1}{2^q} \right\rceil,$$

$$L_q^{\text{total}} = (2^q - 1)(L - 1) + 1, \tag{4}$$

where L_q^{total} is the length of $h'_i[n]$ in Figure 1b and $\lceil x \rceil$ is the integer part of x. It will be shown, in Appendix A, that the components of $\mathbf{w}$ are independent Gaussian R.V.s with common variance σ^2. If a target signal

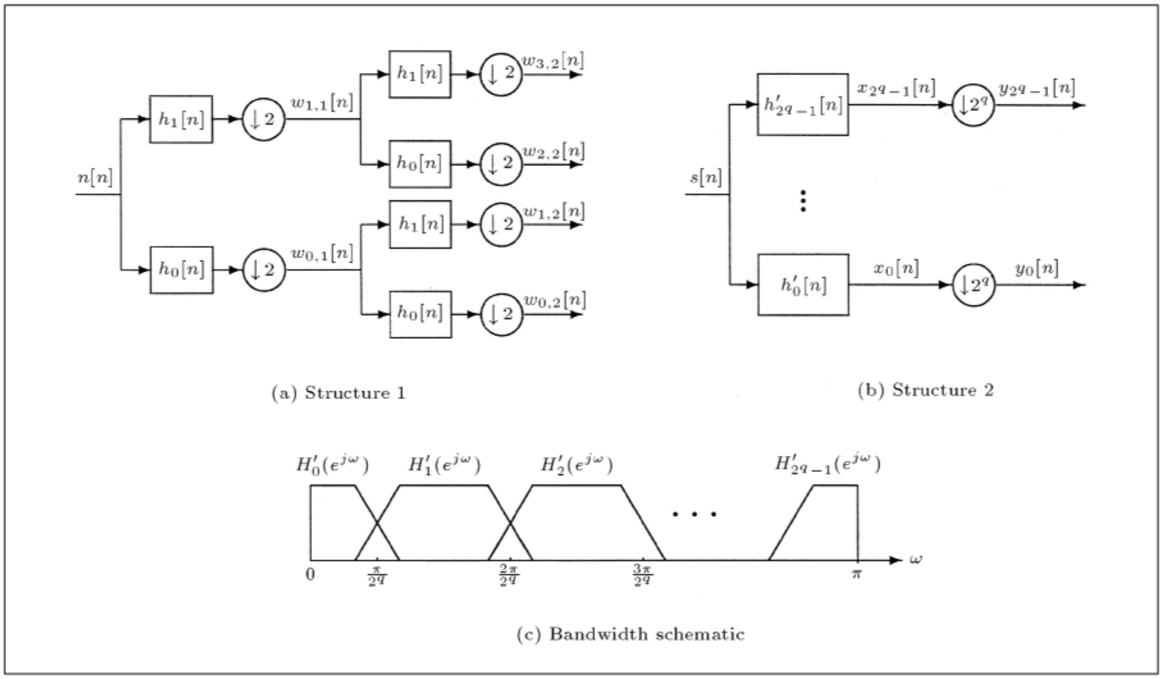


Figure 1. Equivalent wavelet system structures and subband filters in $(0, \pi)$; the diagrams in (a) and (b) are the equivalent wavelet structures. The schematic in (c) shows the bandwidths of the subband filters constructed by the filter-bank at positive frequencies. The schematic is symmetric for $(-\pi, \pi)$.

presents, the respective subband components (kth subband) of the signal term will provide nonzero values:

$$\mathbf{y} = \left[\underbrace{0, \cdots, 0}_{kD_q}, y_{k,q}[0], \cdots, y_{k,q}[D_q - 1], \underbrace{0, \cdots, 0}_{(2^q - k - 1)D_q}\right]^T, \tag{5}$$

where the length of $\mathbf{y}$ is $2^q D_q$. It is interesting that applying the received signal, $\mathbf{r}$, which has non-zero mean under the hypothesis, $\mathcal{H}_1$, to the wavelet system, results in the output that contains some signal-free zero mean components in the detection Problem 3. So, this detection problem is in the canonical form and, according to the previous sections, the following UMPI test statistics can be proposed that are equivalent to GLR in this problem [14], if the central frequency (frequency subband) of the signal is known:

$$F_{k,q}(\mathbf{z}) = \frac{\sum\limits_{l=0}^{D_q-1} |z_{k,q}[l]|^2}{\sum\limits_{\substack{i=0 \\ i \neq k}}^{2^q-1} \sum\limits_{l=0}^{D_q-1} |z_{i,q}[l]|^2}. \tag{6}$$

However, since it is assumed that the signal frequency band, k, is unknown, its ML estimate is used and the GLR test with the following rejection region will result:

$$F_q(\mathbf{z}) = \max_{0 \leq k \leq 2^q - 1} F_{k,q}(\mathbf{z}) > \eta_{\mathrm{GLR}}, \tag{7}$$

where the threshold, η_{GLR}, is selected, such that the probability of a false-alarm (P_{fa}) requirement is satisfied. The threshold may be obtained either analytically, by using the distribution of $F_q(\mathbf{z})$, or, by Monte-Carlo simulation, when no target signal is present.

According to the discussion in the previous sections, the UMPI test in Equation 6 is also GLR and, so, the test in Equation 7 is GLR. In addition, the GLR test (Equation 7) has the CFAR property. If one divides both numerator and denominator of $F_{k,q}$ in Equation 6 by σ^2, one will see that $F_{k,q}$ and, so, $F_q(\mathbf{z})$ are a function of $\frac{\mathbf{z}_{i,q}}{\sigma}$ and, since $\frac{\mathbf{z}_{i,q}}{\sigma} \sim \mathcal{N}(\mathbf{0}, I)$ under $\mathcal{H}_0$, $F_q(\mathbf{z})$ is independent of the unknown parameters, under the null hypothesis.

In marginal cases, where a signal happens at two consecutive subbands (e.g., its central frequency is on the boundary), clearly, two subband components (e.g., kth and $(k+1)$th subbands) present nonzero means. Therefore, based on the F-test introduced previously, where two components of the mean vector are non-zero ($r = 2$ in the previous sections), the following CFAR detector, termed Wavelet Detector (WD), is proposed that considers both of the signal-candidate subbands:

$$F'_q = \max_{0 \leq k \leq 2^q - 1} \frac{\sum\limits_{l=0}^{D_q-1} |z_{k,q}[l]|^2 + \sum\limits_{l=0}^{D_q-1} |z_{k+1,q}[l]|^2}{\sum\limits_{\substack{i=0 \\ i \neq k,\ k+1}}^{2^q-1} \sum\limits_{l=0}^{D_q-1} |z_{i,q}[l]|^2} > \eta_{\mathrm{WD}}, \tag{8}$$

where η_{WD} is chosen, such that the probability of false alarm is satisfied.

Remark 1

The performance of a WD detector can be improved by increasing the number of processing stages, q. As an intuitive justification, this detector estimates the noise variance in the denominator and the signal energy in the numerator. The number of output samples used for noise estimation grows exponentially with increasing q, whereas the signal candidate sample numbers do not grow so fast (see Equation 4). However, q cannot be increased without limit with monotonic improvement; there exists an optimum value which depends on the signal bandwidth. Since $q \leq \log_2 \frac{\pi}{B}$, where B is the known bandwidth of the signal, the smaller the bandwidth of the signal is, the larger one can choose q, the number of stages.

Remark 2

Note that, for such signals at marginal frequencies, the signal does not completely pass any two consecutive subbands. Therefore, after using WD, there is still some performance loss at those frequencies and the detector does not perform the same for the signals centering at different frequencies. Specially, there is much performance loss at frequencies close to $\frac{\pi}{2}$. As a reason, one can see that $\frac{\pi}{2}$ happens at the boundary of adjacent filters several times during the procedure; e.g., at the first stage in Figure 1a, the frequency band, $(0,\pi)$, is divided by two filters, $h_0[n]$ and $h_1[n]$, and, so, $\frac{\pi}{2}$ is at the boundary. This incidence repeats at each stage. Therefore, at this frequency, one has the most performance loss.

Remark 3

Note that the passband of the lowpass filter is $(-\frac{\pi}{2^q}, \frac{\pi}{2^q})$, whereas other filters have the passband of width $\frac{\pi}{2^q}$. Therefore, for a baseband signal with the bandwidth of B, which is confined to $(-B,B)$, the possible number of processing stages is $q \leq \log_2 \frac{\pi}{B}$, whereas, for its bandpass version, whose bandwidth is $2B$, the limitation on the number of processing stages is $q \leq \log_2 \frac{\pi}{B} - 1$. As mentioned in Remark 1, for a smaller bandwidth signal, one can use larger q. Particularly, the maximum allowable q for a bandpass signal in the wavelet structure (Figure 1a) is one unit less than that of its lowpass equivalent representation. As a result, the detection of a lowpass signal can be done with higher performance, compared to its bandpass equivalent representation. These facts lead one to develop a new detector that is supposed to accomplish the detection at the lowest subband.

Modified Wavelet Detector

To overcome the above mentioned problems in Remarks 2, first, one estimates the frequency sub-band of the band-limited signal using the test statistics in Equation 6, i.e.:

$$\hat{k} = \arg\max_k \sum_{l=0}^{D_q-1} |z_{k,q}[l]|^2 , \tag{9}$$

and, then, down-convert the signal, $r[n]$, in the detection Problem 1 to the first sub-band: $d[n] = r[n]e^{-jn\omega_{\hat{k}}}$, where $\omega_{\hat{k}}$ is the central frequency of the $\hat{k}$th sub-band. The resulting signal, $d[n]$, is, then, applied to the wavelet system and the detector in Equation 6 is used with $k = 0$. The new method is termed Modified Wavelet Detector (MWD).

It must be noted that the pass-band of the low-pass filter is twice the pass-band of the band-pass filters (Figure 1c). In MWD, first, the center frequency of the band-limited signal is estimated, the signal is down-converted to the zero frequency and, then, applied to the wavelet detector. Therefore, if a signal coincides on the boundary of two adjacent band-pass filters, after down-converting, all its components almost completely will be in the pass-band of the low-pass filter (the pass-band of the low-pass filter is twice the pass-band of the band-pass filters). Therefore, an equation similar to Equation 8 is not necessary for this situation.

SIMULATION EXAMPLES AND RESULTS

In this section, two well-known and challenging examples are studied that can be solved by the authors' detectors and the performance of the new detectors are compared with the existing MDFT detector for these problems. The threshold is chosen in simulations, such that $P_{\mathrm{fa}} = 0.01$ or $P_{\mathrm{fa}} = 0.0001$; the decision statistics for 100000 or 10000000 trials are evaluated and the threshold is chosen as the first percentile or 0.0001 of the resulting data.

Example 1

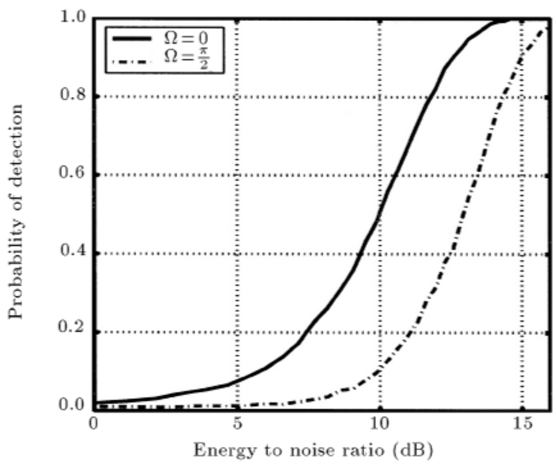
It will be shown that a sinusoid signal with unknown parameters, i.e., unknown amplitude a, frequency Ω and phase θ, $s[n] = ae^{j\theta}e^{jn\Omega}, 0 \leq n \leq N-1$ in unknown variance white Gaussian noise, can be detected using the authors' method. The spectrum, $S(e^{j\omega})$, is, as follows:

$$\begin{aligned} S(e^{j\omega}) &= ae^{j\phi} \sum_{n=0}^{N-1} e^{jn\Omega} e^{-jn\omega} \\ &= ae^{j\phi} e^{j\frac{N-1}{2}(\Omega-\omega)} \frac{\sin(\frac{N}{2}(\Omega-\omega))}{\sin(\frac{\Omega-\omega}{2})}. \end{aligned} \tag{10}$$

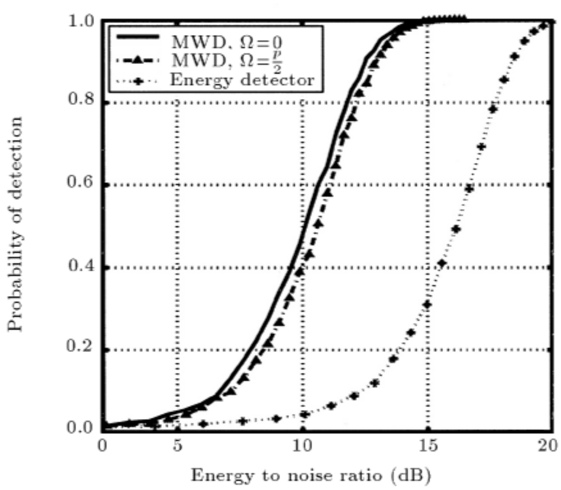
So, this signal is mostly band-limited, a result that can be detected using the proposed band-limited signal detectors. Moreover, the number of stages, q, should be selected, such that the mainlobe bandwidth of the signal is limited to one of the filters' bandwidths, i.e.:

$$\begin{cases} \frac{2\pi}{N} \leq \frac{\pi}{2^q} \Rightarrow q \leq \log_2 N - 1 & \Omega = 0 \\ 2 \times \frac{2\pi}{N} \leq \frac{\pi}{2^q} \Rightarrow q \leq \log_2 N - 2 & \Omega \neq 0 \end{cases}. \tag{11}$$

Therefore, the necessary condition for q in the above detectors to have an acceptable performance is $q \leq \log_2 N - 2$. Figure 2a shows the performance of the Wavelet Detector (WD) for this signal in terms of P_{d} versus energy-to-noise ratio (ENR= $N\frac{a^2}{\sigma^2}$). 32-length coefficients of the Daubechies wavelet are used and a signal with length $N = 64$ is processed in $q = 4$ stages. The performance is illustrated at $\Omega = 0$ and $\Omega = \frac{\pi}{2}$. The big performance loss at $\Omega = \frac{\pi}{2}$ confirms the previous discussion in the section concerning Wavelet Detector.



(a) WD



(b) MWD

Figure 2. Performance evaluation of the proposed detectors in different frequencies: The figure shows the probability of detection versus ENR. In this simulation, a sinusoid is considered with length of $N = 64$; the length of the filters is $L = 32$ and $P_{\mathrm{fa}} = 0.01$; (a) WD and the number of stages is $q = 4$, (b) MWD and the number of stages is $q = 5$ and also energy detector. It is seen that, in MWD, the performance loss is less than WD in $\frac{\pi}{2}$.

To alleviate the problems associated with WD, MWD is proposed, whose performance is illustrated for different values of $\Omega = 0, \frac{\pi}{2}$ in Figure 2b. Simulations show less performance loss in MWD compared to WD. In MWD, first, the frequency subband of the signal is estimated, down-converted to the lowest subband and, then, the detection is accomplished. So, the number of stages should satisfy the condition $q \leq \log_2 N - 1$ (see Relation 11). Therefore, in this case, one can choose $q = 5$ as the number of processing stages. This figure also illustrates the performance of the Energy Detector (ED) and shows that MWD considerably outperforms ED, even in the marginal frequencies. In Figure 3, it is seen that the ED, in contrast with the MWD, is not CFAR, i.e., the probability of a false alarm rate varies as the noise variance changes. In these simulations, the probability of false alarm is set to be $P_{\mathrm{fa}} = 0.01$ for $\sigma^2 = 1$. It is observed that P_{fa} in ED varies a lot, while, in MWD, there is a constant false alarm rate property. The performance of the proposed detectors, WD and MWD, are also compared with MDFT detectors (see the section concerning DFT Detector), in terms of the probability of detection versus ENR and the probability of detection versus frequency in Figures 4 and 5, respectively. In Figure 4, $P_{\mathrm{fa}} = 0.0001$ and the signal length is $N = 32$. Simulations show less performance loss in MWD, compared to WD and MDFT. In Figure 5, the ENR = 14.1363 dB. The

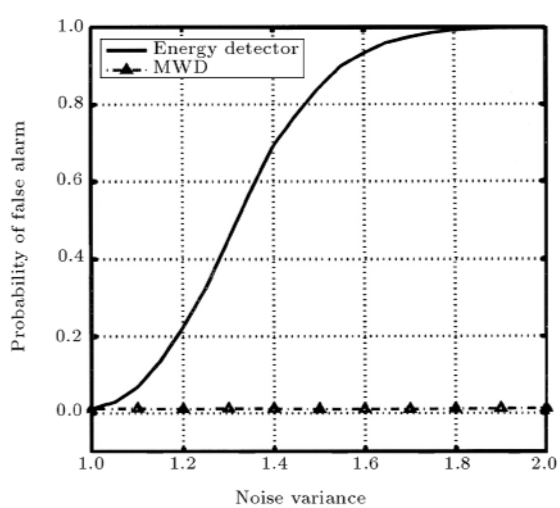


Figure 3. Probability of false alarm changes with the noise variance: In this figure, the probability of false alarm rate variations, as the noise variance changes, is evaluated. In these simulations, a sinusoid is considered with length $N = 64$; the length of the filters is $L = 32$.

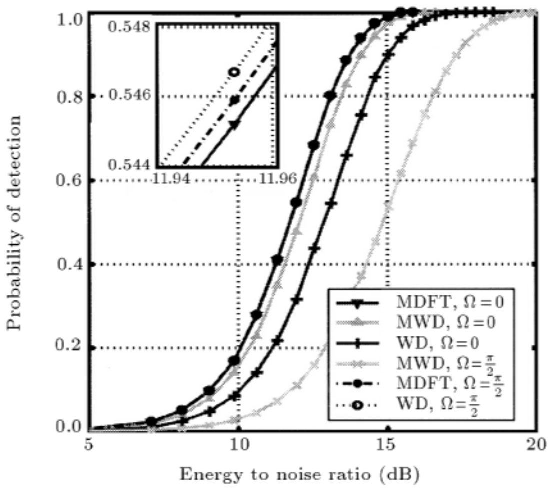


Figure 4. Performance evaluation of the proposed detectors in different frequencies: The figure shows the probability of detection for WD, MWD and MDFT versus ENR. In this simulation, a sinusoid is considered with length of $N = 32$ and $P_{\text{fa}} = 0.0001$.

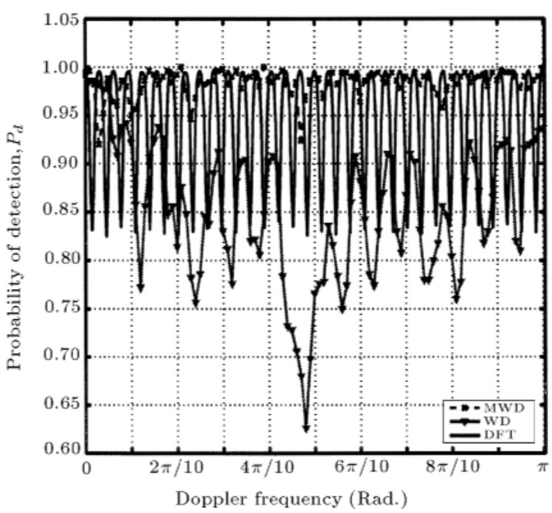


Figure 5. Performance evaluation of different methods at different frequencies: Simulations show the probability of detection versus Doppler frequency for MDFT, WD ($q = 4$) and MWD ($q = 5$). In these simulations, a sinusoid is considered with length of $N = 64$; the length of the filters is $L = 32$, ENR= 14.1364 dB and $P_{\text{fa}} = 0.01$. It is seen that performance loss is the least in MWD.

performance losses of MDFT and WD, at marginal frequencies, are evident, as discussed earlier (MDFT at multipliers of $\frac{\pi}{N}$ and WD at multipliers of $\frac{\pi}{2^q}$, especially at $\Omega = \frac{\pi}{2}$). It is clearly observable that MWD is strongly more efficient than the others.

Example 2

The second example used for the evaluation of the detector performance, is the detection of a sinusoid with variable amplitude having Rayleigh distribution. Consider the detection Problem 1 where:

$$s[n] = a_n e^{j\theta} e^{jn\Omega} P[n],$$

$$P[n] = \begin{cases} 1 & 0 \le n \le N \quad 1 \\ 0 & \text{Otherwise} \end{cases} \tag{12}$$

where the amplitudes, a_n, are the samples of an independent and identically Rayleigh-distributed random variable with unknown parameter A, θ is the unknown phase and Ω is the unknown normalized Doppler frequency. This problem is studied in [9] and, by assuming known values for Doppler frequency, noise variance and the parameter A, an approximate ALR solution for low SNRs is derived. The attempt to derive this detector with the unknown parameters results in a complicated solution and deriving an implementable closed form solution seems to be impossible. The problem is considered in the case of unknown parameters and without any claim of optimality. It is illustrated that the proposed detector is a proper solution for this complicated parameter space problem. In Appendix C, it will be shown that this signal is composed of white and approximately band-limited components. The white part can be absorbed into the additive white noise and, by testing the existence of the band-limited component, using the authors' detector, a decision is made about the existence of the signal. It is shown, in Appendix C, that the GLRT for the band-limited component rejects $\mathcal{H}_0$, if:

$$MF_{k,q}(\mathbf{z}) = \max_k \frac{\sum_{l=0}^{D_q \quad 1} |z_{k,q}[l]|^2}{\sum_{\substack{i=0\\ i\neq k}}^{2^q \quad 1} \sum_{l=0}^{D_q \quad 1} |z_{i,q}[l]|^2} > \eta_{\text{RF}}, \tag{13}$$

where η_{RF} is chosen, such that the probability of false alarm requirement is satisfied. This detector is similar to the WD in Equation 7 and so, MWD is used to overcome the mentioned problem associated with WD. The performance result of applying MDFT [8] and MWD is provided for the detection of such a signal in Figure 6. The probability of detection is calculated as A varies, where A is the parameter of the Rayleigh distribution. In this situation, the probability of detection depends on $\rho = \frac{\sum_i |b_i|^2}{\sigma^2}$, where b_i is related to a_i, via a complicated relation (see Appendix C); so, it is a R.V. and P_{d} should be calculated, as follows:

$$P_{\text{d}} = \int p(d|\rho)p(\rho)d\rho. \tag{14}$$

Even if one could find the PDF of ρ, the calculation of the above integral seems not to be easy and one can estimate it as $P_{\text{d}} = mean(p(d|\rho))$. It is also stated in

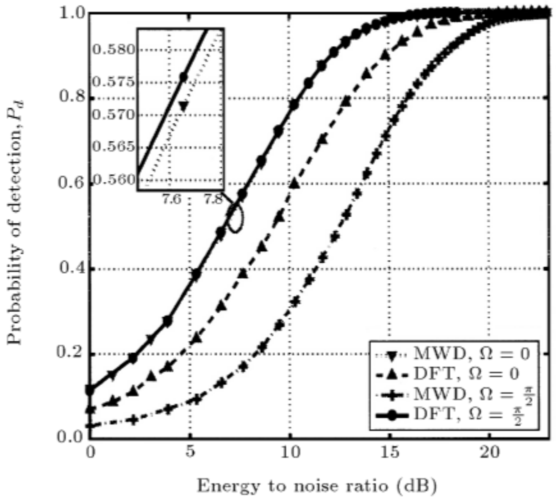


Figure 6. Performance evaluation of the detection of a Rayleigh-distributed amplitude sinusoid: Simulation shows the probability of detection versus ENR for MDFT and MWD ($q = 5$). In these simulations, a sinusoid signal is considered with Rayleigh-distributed variable amplitude and length of $N = 64$; the length of the filters is $L = 32$ and $P_{\text{fa}} = 0.01$. It is seen that, in MWD, the performance loss is less than in MDFT.

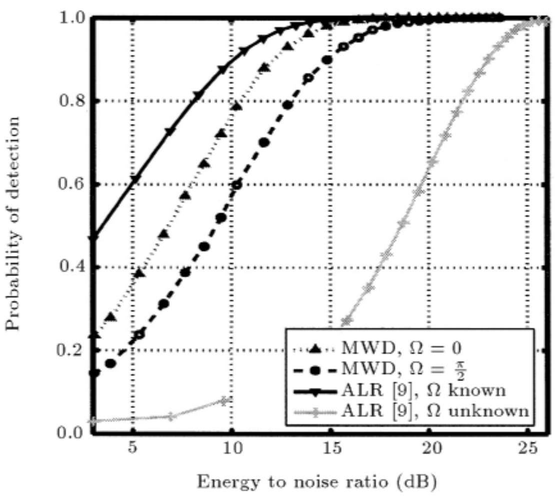


Figure 7. Performance evaluation of the detection of a Rayleigh-distributed amplitude sinusoid: Simulation shows the probability of detection versus ENR for MWD ($q = 5$) and ALR in [9]. In these simulations, a sinusoid signal is considered with Rayleigh-distributed variable amplitude and length of $N = 64$; the length of the filters is $L = 32$ and $P_{\text{fa}} = 0.01$.

this result that applying MWD is much more efficient than MDFT in this problem; e.g., MWD, in its worst condition, performs better than the MDFT detector in its worst condition (see Figure 6). In Figure 7, the ALR detector of [9], that is proposed for the known noise variance and Doppler frequency, is simulated and compared with MWD, in terms of P_{d} versus ENR. The signal length is considered to be $N = 64$ and $P_{\text{fa}} = 0.01$. It is evident that, for known Doppler frequency, this detector outperforms MWD. However, since this detector is not designed for unknown Doppler frequency, a better performance is not expected in unknown frequency situations.

CONCLUSION

Two non-expensive and efficient detectors have been proposed for the detection of a band-limited signal with unknown parameters in unknown variance white Gaussian noise. An orthogonal, fully decimated filter-bank, based on wavelet systems, is used to convert the problem to a canonical form and, using F-test as a GLRT, a CFAR WD is proposed. There exist some problems in which the WD detector does not perform the same at different frequencies. A method is introduced, namely MWD, that enhances the performance, estimating the frequency band of the signal, down-converting and accomplishing the detection always in lowpass frequencies. It is investigated that this method is efficiently applicable to two well-known problems: Detection of a sinusoid with unknown parameters in unknown variance white Gaussian noise and the detection of a sinusoid with variable amplitude having Rayleigh distribution. There exist some solutions for the first problem and for the special cases of the later problem. The authors' main contribution is proposing a non-expensive and efficient detector for a generally modeled band-limited signal. In addition, the result of the comparison of the proposed detector performance with the existing detector (e.g. MDFT), for the second example, states that MWD presents a more efficient and reliable detector for such signals.

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APPENDIX A

Calculation of the Power Spectral Density of the Output Noise

In order to show that all the components of the output noise, $\mathbf{w}$, in Figure 1a are i.i.d., the following statements should be proven:

1. All D_q components of noise in each branch are independent with common variance σ^2;
2. Two arbitrary components of the output noise associated to two different branches are independent.

Since the input noise to the filter-bank in Figure 1a is Gaussian and the system is linear, the output noise is also Gaussian and orthogonality is sufficient for independence. The proof of the first statement is the same for both branches of the first stage. For instance it is shown that $w_{0,1}[n]$ is white:

$$\begin{aligned} &E\left\{w_{0,1}[n]w_{0,1}[m]\right\} \\ &\quad = E\left\{(n[2n]*h_0[2n])\,(n[2m]*h_0[2m])\right\} \\ &\quad = \sigma^2\sum_l h_0(2n\quad l)h_0(2m\quad l) = \sigma^2\delta(n\quad m), \end{aligned} \tag{A1}$$

where $*$ is the convolution and the above equalities come from orthogonality in wavelet structures [17] and whiteness of input noise ($n[n]$). So, because the output of each stage is the input of the next stage, the whiteness of the outputs of the first stage results in the whiteness of all branches in the qth stage. Note that the proof for $w_{1,1}[n]$ is similar, except that, this time, the orthogonal properties of $h_1[n]$ are used.

For the second statement, for different branches in one stage, two cases are considered:

- Two branches that have the same input, e.g., the branches labeled 0 and 1 at the first stage or branches 0 and 1 or 2 and 3 at the second stage;
- Two branches that have different inputs, e.g., the branches labeled 0 and 2 at the second stage.

For the first statement, assuming $q = 1$, it suffices to show that two arbitrary components of the first and the second branches are orthogonal:

$$\begin{aligned} &E\left\{w_{0,1}[n]w_{1,1}[m]\right\} \\ &\quad = E\left\{(n[2n]*h_0[2n])\,(n[2m]*h_1[2m])\right\} \\ &\quad = \sigma^2\sum_l\sum_k h_0(2n\quad l)h_1(2m\quad k)\delta[l\quad k] \\ &\quad = \sigma^2\sum_l h_0[2n\quad l]h_1[2m\quad l] = 0, \end{aligned} \tag{A2}$$

where the above equalities come from the orthogonality in the wavelet structures [17] and whiteness of input noise ($n[n]$). Bearing in mind that the noise components of each branch at each stage are i.i.d., the proof for all stages is straightforwardly similar to the proof in Equation A2. In order to show the second statement, as an example, it will be shown that $w_{0,2}[n]$ and $w_{2,2}[n]$ are orthogonal:

$$\begin{aligned} &E\left\{w_{0,2}[n]w_{2,2}[m]\right\} \\ &\quad = E\left\{(w_{0,1}[2n]*h_0[2n])\,(w_{1,1}[2m]*h_0[2m])\right\} \\ &\quad = \sum_l\sum_k h_0[2n\quad l]h_0(2m\quad k)E\left\{w_{0,1}[k]w_{1,1}[l]\right\} \\ &\quad = 0. \end{aligned} \tag{A3}$$

Since, in this proof, the orthogonal properties of the noise at the previous stage are used, the proof for each stage only requires the orthogonal properties of the previous stage and, regarding the above listed properties, this statement can also be generalized for all stages.

APPENDIX B

Length Calculations

In this appendix, the lengths of the filters, $h'_i[n]$, and outputs $y_i[n]$ of the structure in Figure 1b, will be derived, assuming that the input, $s[n]$, is of length N and the wavelet filters, $h_0[n]$ and $h_1[n]$, are of length L. The wavelet structure in Figure 1a is equivalent to the structure in Figure 1b [16]. For the last branch, (lowpass filter) in Figure 1b, one has:

$$H'_0(z) = H_0(z)H_0(z^2)\cdots H_0(z^{2^q}), \tag{B1}$$

where $H_0(z)$ and $H'_0(z)$ are z-transformations of $h_0[n]$ and $h'_0[n]$, respectively. Let L_i be the length of $H_0(z^{2^i})$ and L_q^{total} be the length of $h'_0[n]$ (this length is the same for $h'_i[n], i = 1, \cdots, 2^q - 1$). Note that, if $H(z)$ is a filter of order L, then $H(z^2)$ is of order $2L - 1$ [16] and one has $L_0 = L, L_i = 2L_{i-1} - 1 = 2^{i-1}L - 2^{i-1} + 1, i = 1, \cdots, q-1$. Therefore, the length of all the filters, $h'_i[n], i = 0, \cdots, 2^q - 1$, is:

$$L_q^{\text{total}} = \sum_{i=0}^{q-1} L_i - (q-1) = (2^q - 1)(L-1) + 1. \tag{B2}$$

It is simply derived from Figure 1b that if the length of the output, $y_i[n], i = 0. \cdots, 2^q - 1$, is D_q, one has:

$$D_q = \left\lceil \frac{N + L_q^{\text{total}} - 1}{2^q} \right\rceil. \tag{B3}$$

APPENDIX C

Derivation of a Detector for a Sinusoid with Unknown Phase and Frequency and Rayleigh-Distributed Variable Amplitude

In this appendix, an efficient detector is derived for a sinusoid with unknown phase, frequency and Rayleigh-distributed variable amplitude, introduced previously, in unknown variance white Gaussian noise, using the authors' band-limited detectors. It is shown that this signal is composed of a white component that can be absorbed in white noise and an approximately band-limited component that can be used for the detection of the signal. The correlation function of the stochastic process, $s[n]$, defined in Equation 12, is:

$$\begin{aligned} R_s[m,n] &= E\left\{s[n+m]s^*[n]\right\} \\ &= E\left\{a_{n+m}a_n^*\right\} e^{jm\Omega} P[n+m]P[n] \\ &= \begin{cases} 2A^2 P[n] & m = 0 \\ \frac{\pi}{2}A^2 e^{jm\Omega}P[n+m]P[n] & m \neq 0 \end{cases} \end{aligned} \tag{C1}$$

where $E\{a_n^2\} = 2A^2$ and $E\{a_n\} = A\sqrt{\frac{\pi}{2}}$ are the second moment and the expected value of the Rayleigh-distributed variable, a_n, respectively. Since this process is not stationary, averaging $R_s[m,n]$ over n results in the following correlation function:

$$\begin{aligned} R_s[m] &= \frac{1}{N}\sum_{n=0}^{N-1} R_s[m,n] \\ &= \begin{cases} 2A^2 & m = 0 \\ \frac{\pi}{2}\frac{A^2}{N} e^{jm\Omega} \sum\limits_{n=0}^{N-1} P[m+n] & m \neq 0 \end{cases} \\ &= \left(2A^2 - \frac{\pi}{2}A^2\right)\delta[m] \\ &\quad + \frac{\pi}{2}A^2\left(1 - \frac{|m|}{N}\right) e^{jm\Omega} T[m]. \end{aligned} \tag{C2}$$

Note that:

$$\sum_{n=0}^{N-1} P[m+n] = (N - |m|)T_m,$$

$$T[m] = \begin{cases} 1 & |m| \leq N - 1 \\ 0 & \text{otherwise} \end{cases}. \tag{C3}$$

So, its spectrum is $S_s(e^{j\omega}) = A^2(2 - \frac{\pi}{2}) + F(e^{j\omega})$, i.e., the signal, $s[n]$, is composed of a white process that appears as a constant in the spectrum and an approximately band-limited signal with the spectrum, $F(e^{j\omega})$ whose absolute value is proportional to $\left(\frac{\sin(\frac{N}{2}(\Omega-\omega))}{\sin(\frac{\Omega-\omega}{2})}\right)^2$. The mainlobe bandwidth of the band-limited component is similar to that of the signal in Equation 10. Therefore, the proposed detector can be employed for the GLR detection of the non-white part. Applying the signal to the filterbank results in testing $\mathcal{H}_1 : \mathbf{z} = \mathbf{y} + \mathbf{v}$ against $\mathcal{H}_0 : \mathbf{z} = \mathbf{w}$, where $\mathbf{v} = \mathbf{w} + \mathbf{u}$ is white and assumed to be Gaussian, $\mathbf{w}$ is the output noise and $\mathbf{u}$ is the white process with the variance, $A^2(2 - \frac{\pi}{2})$. So, $\mathbf{v}$ is white, Gaussian with the variance $\sigma_v^2 = \sigma^2 + A^2(2 - \frac{\pi}{2})$ and $\mathbf{y}$ is the approximately band-limited part of $\mathbf{s}$ in Equation 12. As discussed earlier, in the case where Ω is in the kth subband, $\mathbf{y}$ has the following form:

$$\mathbf{y} = \left[\underbrace{0,\cdots,0}_{kD_q}, y_{k,q}[0],\cdots,y_{k,q}[D_q-1], \underbrace{0,\cdots,0}_{(2^q-k-1)D_q}\right]^T . \tag{C4}$$

The GLR test will be derived for the new problem. The PDF of the wavelet output signal under the hypotheses, $\mathcal{H}_1$ and $\mathcal{H}_0$, are ($M = 2^q D_q$):

$$f(\mathbf{z};\mathcal{H}_1) = \frac{1}{(2\pi)^M \sigma_v^{2M}} \exp\left(-\frac{\sum_{l=0}^{D_q-1} |z_{k,q}[l] - y_{k,q}[l]|^2}{\sigma_v^2}\right)$$
$$\exp\left(-\frac{\sum_{\substack{i=0\\ i\neq k}}^{2^q-1}\sum_{l=0}^{D_q-1} |z_{i,q}[l]|^2}{\sigma_v^2}\right),$$

$$f(\mathbf{z};\mathcal{H}_0) = \frac{1}{(2\pi)^M \sigma^{2M}} \exp\left(-\frac{\sum_{i=0}^{2^q-1}\sum_{l=0}^{D_q-1} |z_{i,q}[l]|^2}{\sigma^2}\right). \tag{C5}$$

The ML estimates of the unknown parameters are obtained, maximizing the density functions:

$$\widehat{y_{k,q}}[l] = z_{k,q}[l], \quad l = 0,\cdots,D_q-1$$

$$\widehat{\sigma_v^2} = \frac{1}{M}\sum_{\substack{i=0\\ i\neq k}}^{2^q-1}\sum_{l=0}^{D_q-1} |z_{i,q}[l]|^2 ,$$

$$\widehat{\sigma^2} = \frac{1}{M}\sum_{i=0}^{2^q-1}\sum_{l=0}^{D_q-1} |z_{i,q}[l]|^2 . \tag{C6}$$

Substituting the ML estimates of the unknown parameters into a likelihood ratio results in:

$$\left.\frac{f(\mathbf{z};\mathcal{H}_1)}{f(\mathbf{z};\mathcal{H}_0)}\right|_{(26)} = \max_k \left(\frac{\sum_{i=0}^{2^q-1}\sum_{l=0}^{D_q-1} |z_{i,q}[l]|^2}{\sum_{\substack{i=0\\ i\neq k}}^{2^q-1}\sum_{l=0}^{D_q-1} |z_{i,q}[l]|^2}\right)^M , \tag{C7}$$

and, since t^M is an increasing function of t, the detector with the rejection region in Equation 13 is obtained.