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Numerical Solution of the System of Nonlinear Fredholm Integro-Differential Equations by the Operational Tau Method with an Error Estimation

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In this paper, the operational approach to the Tau method is used for the numerical solution of a nonlinear Fredholm integro-differential equations system and nonlinear ODEs with initial or boundary conditions without linearizing. An efficient error estimation of the approximate solution is also introduced. Some examples are given to clarify the efficiency and high accuracy of the method.

INTRODUCTION

In recent years, the operational approach to the Tau method has been developed to cover the numerical solution of ODEs, PDEs and linear integro-differential equations [1-10]. Liu and Pan presented an extension of the operational approach to the Tau method for the numerical solution of a linear ODEs system with polynomial or rational polynomial coefficients, together with initial or boundary conditions [11]. Ortiz et al. have solved nonlinear ODEs and PDEs using an operational approach to the Tau method, through an iteration process defined by a sequence of linear problems with variable coefficients [1,4,5,8].

In this paper, Ortiz and Samara's operational approach to the Tau method is considered for the numerical solution of nonlinear ODEs and nonlinear Fredholm integro-differential equations system without linearizing.

Consider the following nonlinear Fredholm integro-differential equations system:

$$\sum_{j=1}^{m} D_{ij} y_j(x) \quad \lambda_i \int_a^b k_i(x,t)\varphi_i(y_1(t),\cdots,y_n(t))dt$$

$$= f_i(x),$$

$$i = 1(1)m, \qquad x \in [a,b], \tag{1}$$

with the supplementary conditions:

$$\sum_{j=1}^{m} \sum_{k=1}^{n_{d_j}} (A_{jkr} y_j^{(k \;\; 1)}(a) + B_{jkr} y_j^{(k \;\; 1)}(b)) = d_r,$$

$$r = 1(1)\omega, \tag{2}$$

where:

$$[n_{d_j} = \max_{1\leq i\leq m}\{n_{d_{ij}}\}, \qquad \omega = \sum_{j=1}^{m} n_{d_j}],$$

and:

$$\left[D_{ij} = \sum_{r=0}^{n_{d_{ij}}} p_{ijr}(x)\frac{d^r}{dx^r} = \sum_{r=0}^{n_{d_{ij}}}\sum_{s=0}^{\beta_{ijr}} p_{ijrs}x^s\frac{d^r}{dx^r}\right].$$

For $i = 1(1)m$, $f_i(x)$ and $k_i(x,t)$ are polynomials in x and in x, t, respectively and $\varphi_i(y_1(t),\cdots,y_n(t))$ are polynomials in $y_1(t),\cdots,y_n(t)$, otherwise, they can be approximated by polynomials with suitable methods and β_{ijr} is degree of $p_{ijr}(x)$.

In this paper, one assumes that, for i, $j = 1(1)m$, $\alpha_{ij} \in \mathbf{N}$, $\varphi_i(y_1(t),\cdots,y_n(t)) = \prod_{j=1}^{m} y_j^{\alpha_{ij}}(t)$, otherwise, it can be written as a sum of this form by a suitable method, for example, Taylor expansion.

The organization of this paper is as follows: First, the operational approach to the Tau method for the numerical solution of Fredholm Integro-Differential Equations (FIDEs) is explained. Then, the numerical solution of a linear FIDEs system by the Tau method is reviewed, and the operational approach to the Tau

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method is applied to a nonlinear FIDEs system. After that, the nonlinear ODEs are solved and an error function is introduced. Some numerical results are given, also, to clarify the accuracy of the method and, finally, the contains conclusions are presented. Note that, the numerical results are computed by Maple programming.

Remark 1

For $i = 1(1)m$, the following notations have been used throughout this paper:

$$\overline{x} = (1, x, x^2, \cdots)^T,$$

$$\overline{x}_a = (1, a, a^2, \cdots)^T,$$

$$\overline{x}_b = (1, b, b^2, \cdots)^T,$$

$$\overline{e}_1 = (1, 0, 0, \cdots)^T,$$

$$\overline{y}_{in} = (y_{i0}, y_{i1}, \cdots, y_{in}, 0, \cdots)^T,$$

$$\overline{f}_i = (f_{i0}, f_{i1}, \cdots, f_{in_{f_i}}, 0, \cdots)^T,$$

$$\overline{f} = (f_0, f_1, \cdots, f_{n_f}, 0, \cdots)^T,$$

$$\overline{y}_n = (y_0, y_1, \cdots, y_n, 0, \cdots)^T.$$

LINEAR AND NONLINEAR FIDES

The operational approach to the Tau method describes the reduction of given linear and nonlinear integro-differential equations (IDEs) to a linear and nonlinear algebraic equations system, based on three simple matrices:

$$\mu = \begin{pmatrix} 0 & 1 & 0 & 0 & \\ & 0 & 1 & 0 & \\ & & 0 & 1 & \vdots \\ & & & 0 & \\ & & \cdots & & \end{pmatrix},$$

$$\eta = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 2 & 0 & & \vdots \\ 0 & 0 & 3 & 0 & \\ & & \cdots & & \end{pmatrix},$$

$$\iota = \begin{pmatrix} 0 & 1 & 0 & 0 & \\ & 0 & \frac{1}{2} & 0 & \\ & & 0 & \frac{1}{3} & \vdots \\ & & & 0 & \\ & & \cdots & & \end{pmatrix}.$$

Linear FIDEs

Consider the linear FIDEs (the following result is quoted from [1,9,11]):

$$Dy(x) \quad \lambda \int_a^b k(x,t)y(t)dt = f(x), \quad x \in [a,b], \tag{3}$$

with the supplementary conditions:

$$\sum_{k=1}^{n_d} \left(A_{kj} y^{(k \quad 1)}(a) + B_{kj} y^{(k \quad 1)}(b)\right) = d_j,$$

$$j = 1(1)n_d, \tag{4}$$

where:

$$D = \sum_{i=0}^{n_d} p_i(x) \frac{d^i}{dx^i} = \sum_{i=0}^{n_d} \sum_{j=0}^{\beta_i} p_{ij} x^j \frac{d^i}{dx^i},$$

is the differential operator of order n_d.

Let $y_n(x) = \sum_{j=0}^n y_j x^j = \overline{y}_n{}^T \overline{x}$, then:

$$x^r y_n(x) = \sum_{j=0}^n y_j x^{j+r} = \overline{y}_n{}^T \mu^r \overline{x},$$

and:

$$y_n^{(r)}(x) = \frac{d^r}{dx^r} y_n(x) = \overline{y}_n{}^T \eta^r \overline{x}. \tag{5}$$

Theorem 1

Let $y_n(x) = \overline{y}_n{}^T \overline{x} \in C^{n_d}[a,b]$, (the space of n_d-times continuously differentiable functions on $[a,b]$) and:

$$D = \sum_{i=0}^{n_d} p_i(x) \frac{d^i}{dx^i} = \sum_{i=0}^{n_d} \sum_{j=0}^{\beta_i} p_{ij} x^j \frac{d^i}{dx^i},$$

be a linear differential operator of order n_d with polynomial coefficients, then;

$$Dy_n(x) = \overline{y}_n{}^T \Pi \overline{x}, \tag{6}$$

where:

$$\Pi = \sum_{i=0}^{n_d} \eta^i p_i(\mu) = \sum_{i=0}^{n_d} \sum_{j=0}^{\beta_i} p_{ij} \eta^i \mu^j.$$

Theorem 2

If $k(x,t) = \sum_{i=0}^n \sum_{j=0}^n k_{ij} x^i t^j$, $k_{ij} \in \mathbf{R}$ and $y(x) = \overline{y}_n{}^T \overline{x}$, then:

$$\int_a^b k(x,t) y(t) dt = \overline{y}_n{}^T \iota_f \overline{x}, \tag{7}$$

where $\iota_f = \sum_{i=0}^n \sum_{j=0}^n k_{ij} \iota_f^{ij}$ is a matrix associated uniquely with $k(x,t)$ and constants a, b, such that, $\iota_f^{ij} = \mu^j \iota (\overline{x}_b \quad \overline{x}_a) \overline{e}_1{}^T \mu^i$.

Lemma 1

If $y_n(x) = \overline{y}_n{}^T \overline{x}$, then,

$$y_n^{(k)}(a) = \overline{y}_n{}^T \eta^k \overline{x}_a,$$

$$y_n^{(k)}(b) = \overline{y}_n{}^T \eta^k \overline{x}_b. \tag{8}$$

Proof

By using Equation 5 one has:

$$y_n^{(k)}(a) = y_n^{(k)}(x)|_{x=a} = \overline{y}_n{}^T \eta^k \overline{x}|_{x=a} = \overline{y}_n{}^T \eta^k \overline{x}_a,$$

and:

$$y_n^{(k)}(b) = y_n^{(k)}(x)|_{x=b} = \overline{y}_n{}^T \eta^k \overline{x}|_{x=b} = \overline{y}_n{}^T \eta^k \overline{x}_b,$$

so, the proof is completed.□

Applying Lemma 1 for supplementary Conditions 4, one has:

$$\sum_{k=1}^{n_d} \left(A_{kj} y^{(k\ \ 1)}(a) + B_{kj} y^{(k\ \ 1)}(b) \right)$$

$$= \sum_{k=1}^{n_d} \left(A_{kj} \overline{y}_n{}^T \eta^{(k\ \ 1)} \overline{x}_a + B_{kj} \overline{y}_n{}^T \eta^{(k\ \ 1)} \overline{x}_b \right)$$

$$= \overline{y}_n{}^T \sum_{k=1}^{n_d} \left(A_{kj} \eta^{(k\ \ 1)} \overline{x}_a + B_{kj} \eta^{(k\ \ 1)} \overline{x}_b \right).$$

Let:

$$E_j = \sum_{k=1}^{n_d} \left(A_{kj} \eta^{(k\ \ 1)} \overline{x}_a + B_{kj} \eta^{(k\ \ 1)} \overline{x}_b \right),$$

thus, $E_j \in \mathbf{M}_{(n+1)\times(1)}$ and Equation 4 is converted into $\overline{y}_n{}^T E_j = d_j$, $j = 1(1)n_d$.

Now, by using Equations 6 through 8, the integro-differential Equation 3 and supplementary Conditions 4 reduce to the following algebraic equations system:

$$\begin{cases} \overline{y}_n{}^T \Pi_f = \overline{f}^T \\ \overline{y}_n{}^T E = \overline{d}^T \end{cases} \tag{9}$$

with $\Pi_f = \Pi \quad \lambda \iota_f$ and:

$$E = (E_1, E_2, \cdots, E_{n_d}) \in \mathbf{M}_{(n+1)\times(n_d)}. \tag{10}$$

By setting $G = (E, \Pi_f)$ and $\overline{g}^T = (\overline{d}^T, \overline{f}^T)$, Equation 9 can be written as $\overline{y}_n{}^T G = \overline{g}^T$. To obtain $y_n(x)$, the system of equations $\overline{y}_n{}^T G_n = \overline{g}_n{}^T$ must be solved for the unknown coefficients, $y_0, y_1, \cdots, y_n$, where G_n is the matrix defined by considering the first $(n+1)$ rows and columns of G, and $\overline{g}_n$ is the vector defined by considering the first $(n+1)$ elements of the vector, $\overline{g}$.

Nonlinear FIDEs

Consider the nonlinear Fredholm integro-differential equation:

$$Dy(x) \quad \lambda \int_a^b k(x,t)\varphi(y(t))dt = f(x), x \in [a,b], \tag{11}$$

with the supplementary Conditions 4, where D is defined as before and $\varphi(y(t))$ is a polynomial in $y(t)$, otherwise, it is approximated by a polynomial with suitable methods. In this paper, $\varphi(y(t)) = y^m(t), m \in \mathbf{N}$, is considered, since other types of $\varphi(y(t))$ can be reduced to the sum of this form.

Theorem 3

If $u(x) = \sum_{j=0}^m u_j x^j = \overline{u}^T \overline{x}$ and $v(x) = \sum_{i=0}^m v_i x^i = \overline{v}^T \overline{x}$, then, $u(x)v(x) = \overline{u}^T v(\mu)\overline{x}$, where $v(\mu) = \sum_{i=0}^n v_i \mu^i$.

Proof

See [9].□

Lemma 2

Let $y(x) = \overline{y}_n{}^T \overline{x}$, then, $y^m(x) = \overline{y}_n{}^T y^{m\ \ 1}(\mu)\overline{x}$.

Proof

Let $u(x) = \overline{y}_n{}^T \overline{x}$ and $v(x) = y^{m\ \ 1}(x)$ and Theorem 3 be applied.□

Theorem 4

If $y_n(x) = \sum_{i=0}^n y_{in} x^i = \overline{y}_n{}^T \overline{x}$ and $k(x,t) = \sum_{i=0}^n \sum_{j=0}^n k_{ij} x^i t^j$, then,

$$\int_a^b k(x,t) y_n^m(t)dt = \overline{y}_n{}^T \iota_{fm} \overline{x}, \tag{12}$$

where $\iota_{fm} = y_n^{m\ \ 1}(\mu)\iota_f$ and ι_f is the same matrix as introduced in Theorem 2.

Proof

Use Lemma 2 and Theorem 2 for $y(x) = \overline{y}_n{}^T y^{m\ \ 1}(\mu)\overline{x}$. □

If Equations 6, 10 and 12 are used for:

$$Dy(x) \quad \lambda \int_a^b k(x,t) y^m(t)dt = f(x), \quad x \in [a,b],$$

$$m \in \mathbf{N},$$

with supplementary Conditions 4, the following nonlinear system will be obtained:

$$\begin{cases} \overline{y}_n{}^T \Pi_f = \overline{f}^T, \\ \overline{y}_n{}^T E = \overline{d}^T, \end{cases} \tag{13}$$

where $\Pi_f = \Pi \quad \lambda \iota_{fm}$ and ι_{fm} is the same matrix as introduced in Theorem 4 and setting $G = (E, \Pi_f)$

and $\overline{g}^T = (\overline{d}^T, \overline{f}^T)$, one has $\overline{y}^T G = \overline{g}^T$ instead of Equation 13, which is a nonlinear algebraic equations system, because G contains unknown elements of the vector $\overline{y}_n$. To find $y_n(x)$, the nonlinear equations system $\overline{y}_n{}^T G_n = \overline{g}_n{}^T$ must be solved, where G_n and g_n are defined by considering the $(n+1)\times(n+1)$ leading submatrix of G and the first $(n+1)$ elements of the vector $\overline{g}$, respectively.

LINEAR FIDES SYSTEM

Consider the system of linear FIDEs:

$$\sum_{j=1}^{m}\left(D_{ij}y_j(x) \quad \lambda_{ij}\int_a^b k_{ij}(x,t)y_j(t)dt\right) = f_i(x),$$

$$x \in [a,b], \qquad i = 1(1)m, \tag{14}$$

with the supplementary Conditions 2, where D_{ij}, $i,j = 1(1)m$, are the same as that in Equation 1, $f_i(x) = \sum_{j=0}^{n_{f_i}} f_{ij}x^j = \overline{f}_i{}^T\overline{x}$. Now, applying Equations 6 through 8, for Equations 14 and 2, they are converted into a system of linear algebraic equations. Let $y_{jn}(x) = \overline{y}_{jn}{}^T\overline{x}$ $j = 1(1)m$, be the Tau approximates and let Π_{ij}, $i,j = 1(1)m$ denote the matrices associated with D_{ij}, $i,j = 1(1)m$ by Theorem 1 and $\iota_{f_{ij}}$, $i,j = 1(1)m$ be the matrices associated with $k_{ij}(x,t), i,j = 1(1)m$ by Theorem 2. Then, one has:

$$\sum_{j=1}^{m}\left(D_{ij}y_{jn}(x) \quad \lambda_{ij}\int_a^b k_{ij}(x,t)y_{jn}(t)dt\right)$$

$$= \sum_{j=1}^{m} \ \overline{y}_{jn}{}^T\Pi_{ij}\overline{x} \quad \lambda_{ij}\overline{y}_{jn}{}^T\iota_{f_{ij}}\overline{x})$$

$$= \sum_{j=1}^{m}\overline{y}_{jn}{}^T \ \Pi_{ij} \quad \lambda_{ij}\iota_{f_{ij}})\,\overline{x}$$

$$= \sum_{j=1}^{m}\overline{y}_{jn}{}^T\overline{\Pi}_{ij}\overline{x}, \quad i = 1(1)m,$$

where:

$$\overline{\Pi}_{ij} = \Pi_{ij} \quad \lambda_{ij}\iota_{f_{ij}}. \tag{15}$$

Applying Lemma 1 to supplementary conditions:

$$\sum_{j=1}^{m}\sum_{k=1}^{n_{d_j}}\left(A_{jkr}y_j^{(k \quad 1)}(a) + B_{jkr}y_j^{(k \quad 1)}(b)\right) = d_r,$$

$$r = 1(1)\omega,$$

yields:

$$\sum_{j=1}^{m}\sum_{k=1}^{n_{d_j}} \ A_{jkr}\overline{y}_{jn}{}^T\eta^{k \quad 1}\overline{x}_a + B_{jkr}\overline{y}_{jn}{}^T\eta^{k \quad 1}\overline{x}_b)$$

$$= \sum_{j=1}^{m}\overline{y}_{jn}{}^T\sum_{k=1}^{n_{d_j}} \ A_{jkr}\eta^{k \quad 1}\overline{x}_a + B_{jkr}\eta^{k \quad 1}\overline{x}_b) = d_r.$$

Assume that:

$$\sum_{k=1}^{n_{d_j}} \ A_{jkr}\eta^{k \quad 1}\overline{x}_a + B_{jkr}\eta^{k \quad 1}\overline{x}_b) = E_{rj} \in \mathbf{M}_{(n+1)\times(1)},$$

then, one has:

$$\sum_{j=1}^{m}\overline{y}_{jn}{}^T E_{rj} = d_r, r = 1(1)\omega,$$

and:

$$\sum_{j=1}^{m}\overline{y}_{jn}{}^T E_j = \overline{d}^T,$$

with:

$$E_j = (E_{1j}, E_{2j}, E_{2j}, \cdots, E_{\omega j}),$$

and:

$$\overline{d}^T = (d_1, d_2, d_3, \cdots, d_\omega).$$

Let $\overline{y}_M{}^T = (\overline{y}_{1n}{}^T, \overline{y}_{2n}{}^T, \overline{y}_{3n}{}^T, \cdots, \overline{y}_{mn}{}^T) \in \mathbf{R}^M$, $M = m(n+1)$, where $\overline{y}_{jn}{}^T$, $j = 1(1)m$ are the coefficients vectors of $y_{jn}(x)$ in a standard basis. The problem of determining $\overline{y}_M{}^T$ can be formulated as the linear equations system [11]:

$$\overline{y}_M{}^T\mathbf{G} = \overline{S}_M{}^T, \tag{16}$$

with:

$$\mathbf{G} = \begin{pmatrix} E_1 & Q_{11} & Q_{21} & \cdots & Q_{m1} \\ E_2 & Q_{12} & Q_{22} & \cdots & Q_{m2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ E_m & Q_{1m} & Q_{2m} & \cdots & Q_{mm} \end{pmatrix}$$

$$\in \mathbf{M}_{m(n+1)\times m(n+1)},$$

where $Q_{ij} = (\overline{\Pi}_{ij})_{(n+1)(n \quad n_{d_i}+1)}$ is the restriction of $\overline{\Pi}_{ij}$ (as defined in Equation 15) to its first $(n+1)$ rows and $(n \quad n_{d_i}+1)$ columns and:

$$\overline{S}_M{}^T = (\overline{d}^T, \overline{f}_{1n_{d_1}}{}^T, \overline{f}_{2n_{d_2}}{}^T, \cdots, \overline{f}_{mn_{d_m}}{}^T) \in \mathbf{R}^M,$$

where $\overline{f}_{in_{d_i}}{}^T$ is the restriction of $\overline{f}_i{}^T$ to its first $(n \quad n_{d_i}+1)$ components. By solving Equation 16, one obtains $y_{jn}(x)$ for $j = 1(1)m$.

NONLINEAR FIDES SYSTEM

Consider the nonlinear FIDEs system:

$$\sum_{j=1}^{m} D_{ij} y_j(x) \quad \lambda_i \int_a^b k_i(x,t) \prod_{j=1}^{m} y_j^{\alpha_{ij}}(t)dt = f_i(x),$$

$$x \in [a,b], \quad i = 1(1)m, \tag{17}$$

with the supplementary conditions:

$$\sum_{j=1}^{m} \sum_{k=1}^{n_{d_j}} \left(A_{jkr} y_j^{(k \quad 1)}(a) + B_{jkr} y_j^{(k \quad 1)}(b) \right) = d_r,$$

$$r = 1(1)\omega,$$

where:

$$n_{d_j} = \max_{1 \le i \le m} \{n_{d_{ij}}\}, \qquad \omega = \sum_{j=1}^{m} n_{d_j},$$

$$D_{ij} = \sum_{r=0}^{n_{d_{ij}}} p_{ijr}(x) \frac{d^r}{dx^r} = \sum_{r=0}^{n_{d_{ij}}} \sum_{s=0}^{\beta_{ijr}} p_{ijrs} x^s \frac{d^r}{dx^r},$$

for $i,j = 1(1)m, n_{d_{ij}}$ is the order of operator $D_{ij}, f_i(x), k_i(x,t)$ are algebraic polynomials in x and x,t, respectively, and λ, a, b are given constants.

Lemma 3

Let $y_i(x) = \overline{y}_{in}{}^T \overline{x}, i = 1(1)m$ then:

$$\prod_{i=1}^{m} y_i^{r_i}(x) = \overline{y}_{1n}{}^T y_1^{r_1 \quad 1}(\mu) \prod_{i=2}^{m} y_i^{r_i}(\mu)\overline{x}, \quad r_i \in \mathbf{N}.$$

Proof

One has:

$$\prod_{i=1}^{m} y_i^{r_i}(x) = y_1(x) y_1^{r_1 \quad 1}(x) \prod_{i=2}^{m} y_i^{r_i}(x),$$

if one sets:

$$v(x) = y_1^{r_1 \quad 1}(x) \prod_{i=2}^{m} y_i^{r_i}(x), \quad u(x) = y_1(x),$$

then, by using Theorem 3, the proof is completed.□

Theorem 5

If $k(x,t) = \sum_{i=0}^{n} \sum_{j=0}^{n} k_{ij} x^i t^j$ with $k_{ij} \in \mathbf{R}, i,j = 1(1)m$ and $y_j(x) = \overline{y}_{jn}{}^T \overline{x}, j = 1(1)m$, then:

$$\int_a^b k(x,t) \prod_{i=s}^{m} y_i^{r_i}(t)dt = \overline{y}_{sn}{}^T \iota_{fs} \overline{x},$$

where:

$$\iota_{fs} = y_s^{r_s \quad 1}(\mu) \prod_{i=s+1}^{m} y_i^{r_i}(\mu) \sum_{i=0}^{n} \sum_{j=0}^{n} k_{ij} \mu^j \iota(\overline{x}_b \quad \overline{x}_a) \overline{e}_1{}^T \mu^i,$$

is a matrix associated uniquely with $k(x,t)$.

Proof

Using Theorem 2 with $y(x) = \sum_{i=s}^{m} y_i^{r_i}(x) = \overline{y}_{sn}{}^T y_s^{r_s \quad 1}(\mu) \prod_{i=s+1}^{m} y_i^{r_i}(\mu)\overline{x}$ and setting $\overline{y}_n{}^T = \overline{y}_{sn}{}^T y_s^{r_s \quad 1}(\mu) \prod_{i=s+1}^{m} y_i^{r_i}(\mu)$, the proof is completed.□

Let $y_{jn}(x) = \overline{y}_{jn}{}^T \overline{x}$, $j = 1(1)m$ be the Tau approximations, then for $i = 1(1)m, 1 \le x \le b$, one has:

$$\sum_{j=1}^{m} D_{ij} y_{jn}(x) \quad \lambda_i \int_a^b k_i(x,t) \prod_{r=s}^{m} y_{rn}^{\alpha_{ir}}(t)dt$$

$$= \sum_{j=1}^{m} \overline{y}_{jn}{}^T \Pi_{ij} \overline{x} \quad \lambda_i \overline{y}_{sn}{}^T \iota_{f_s} \overline{x}$$

$$= \sum_{j=1}^{m} \overline{y}_{jn}{}^T \overline{\Pi}_{ij} \overline{x},$$

where:

$$\overline{\Pi}_{ij} = \begin{cases} \Pi_{ij} & j \ne s \\ \Pi_{is} \quad \lambda_{is} \iota_{f_s} & j = s, \end{cases} \tag{18}$$

$f_i(x) = \sum_{j=0}^{n_{f_i}} f_{ij} x^j = \overline{f}_i{}^T \overline{x}$ and $s = \min\{1, \cdots, m\}$ for which $\alpha_{is} \ne 0$.

In the same way as done in the previous section, one has $\sum_{j=1}^{m} \overline{y}_{jn}{}^T E_j = \overline{d}^T$ instead of Equation 2. For determining the Tau approximations $y_{jn}(x) = \overline{y}_{jn}{}^T \overline{x}, j = 1(1)m$, Equations 17 and 2 are converted into the system of nonlinear algebraic equations:

$$\overline{y}_M{}^T \mathbf{G} = \overline{S}_M{}^T, \tag{19}$$

where $\overline{y}_M{}^T$ and $\overline{S}_M{}^T$ are the same notations as used in Equation 16 and:

$$\mathbf{G} = \begin{pmatrix} E_1 & Q_{11} & Q_{21} & \cdots & Q_{m1} \\ E_2 & Q_{12} & Q_{22} & \cdots & Q_{m2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ E_m & Q_{1m} & Q_{2m} & \cdots & Q_{mm} \end{pmatrix}$$

$$\in \mathbf{M}_{m(n+1) \times m(n+1)},$$

with $Q_{ij} = (\overline{\Pi}_{ij})_{(n+1)(n \quad n_{d_i}+1)}$, which is the restriction of $\overline{\Pi}_{ij}$ to its first $(n+1)$ rows and $(n \quad n_{d_i}+1)$ columns. It should be noted that, in Equation 18, the elements of $\overline{\Pi}_{is}$ contain the unknown coefficients of $y_{jn}(x), j = 1(1)m$. By solving the nonlinear system (Equation 19), one finds the unknown coefficients of $y_{jn}(x)$ for $j = 1(1)m$.

NONLINEAR ODES

In this section, the nonlinear differential equation $f(x,y,y',y'') = 0$ is considered, where f is an analytic function in terms of y, y' and y''. Therefore, the equation can be written as:

$$f(x, y, y', y'') = \sum_{i=0}^{r} p_i(x) y^{n_i} (y')^{m_i} (y'')^{q_i} = 0,$$

where $r, n_i, m_i, q_i \in \mathbf{N} \bigcup \{0\}$ and $p_i(x)$ is an analytic function in terms of x. Using Theorem 3 and Lemma 3, one can write:

$$f(x, y, y', y'') = \sum_{i=0}^{r} \overline{y}_n{}^T (y(\mu))^{n_i \quad 1} (y'(\mu))^{m_i}$$

$$\times (y''(\mu))^{q_i} p_i(\mu) \overline{x} = 0^T \overline{x},$$

or:

$$\overline{y}_n{}^T \sum_{i=0}^{r} (y(\mu))^{n_i \quad 1} (y'(\mu))^{m_i} (y''(\mu))^{q_i} p_i(\mu) = 0^T, \qquad (20)$$

where $0^T = (0 \cdots, 0)_{(1)(n+1)}$. Equation 20 and the supplementary Conditions 4 form a system of nonlinear algebraic equations. Each equation of this system is a polynomial, in terms of unknown elements of vector $\overline{y}_n$.

ESTIMATION OF ERROR FUNCTION

In this section, an error function is obtained for the approximate solution of Equations 2 and 17. Let $e_{jn}(x) = y_j(x) \quad y_{jn}(x)$, $j = 1(1)m$ be called the error function of Tau approximation $y_{jn}(x)$ to $y_j(x)$, where $y_j(x)$, $j = 1(1)m$ is the exact solution. Substituting $y_j(x) = e_{jn}(x) + y_{jn}(x)$, $j = 1(1)m$ in Equations 2 and 17, for $x \in [a, b]$, $i = 1(1)m$ and $s \in \mathbf{N}$, they can be written as:

$$\sum_{j=1}^{m} D_{ij}(y_{jn}(x) + e_{jn}(x)) \quad \lambda_i \int_a^b k_i(x, t) \prod_{j=s}^{m} (y_{jn}(t)$$

$$+ e_{jn}(t))^{\alpha_{ij}}(t) dt = f_i(x), \qquad (21)$$

and:

$$\sum_{j=1}^{m} \sum_{k=1}^{n_{d_j}} \Bigg(A_{jkr}(y_{jn}(a) + e_{jn}(a))^{(k \quad 1)}$$

$$+ B_{rjk}\,(y_{jn}(b) + e_{jn}(b))^{(k \quad 1)} \Bigg) = d_r,$$

$$r = 1(1)\omega.$$

By using $(y_{jn}(t) + e_{jn}(t))^p = \sum_{k=0}^{p} \binom{k}{p} y_{jn}^{p \quad k}(t) e_{jn}^k(t)$ in Equation 21 and because of satisfying $y_{jn}(x)$ in Equation 2, for $x \in 5[a, b]$ and $i = 1(1)m$, one has:

$$\sum_{j=1}^{m} D_{ij} e_{jn}(x) \quad \lambda_i \int_a^b k_i(x, t) \varphi_i \Big(e_{sn}(t), e_{(s+1)n}(t),$$

$$\cdots, e_{mn}(t) \Big) dt = \quad H_{in}(x),$$

and:

$$\sum_{j=1}^{m} \sum_{k=1}^{n_{d_j}} \Big(A_{jkr} e^{(k \quad 1)}(a) + B_{jkr} e_{jn}^{(k \quad 1)}(b) \Big) = 0,$$

$$r = 1(1)\omega,$$

where:

$$H_{in}(x) = \sum_{j=1}^{m} D_{ij} y_{jn}(x) \quad \lambda_i \int_a^b k_i(x, t) \prod_{j=s}^{m} y_{jn}^{\alpha_{ij}}(t) dt$$

$$f_i(x), i = 1(1)m,$$

are the perturbation terms associated with $y_{jn}(x), j = 1(1)m$ and:

$$\varphi_i(e_{sn}(t), e_{(s+1)n}(t), \cdots, e_{mn}(t))$$

$$= \varphi_{1is} \prod_{j=s+1}^{m} (y_{jn}^{\alpha_{ij}}(t) + \varphi_{1ij})$$

$$+ \sum_{r=s}^{m \quad 2} \Bigg(\prod_{j=s}^{r} y_{jn}^{\alpha_{ij}}(t)) \varphi_{1i(r+1)} \prod_{j=r+2}^{m} (y_{jn}^{\alpha_{ij}}(t) + \varphi_{1ij}) \Bigg)$$

$$+ \prod_{j=s}^{m \quad 1} y_{jn}^{\alpha_{ij}}(t)) \varphi_{1im},$$

with:

$$\varphi_{1ij} = \sum_{p=1}^{\alpha_{ij}} \binom{\alpha_{ij}}{p} y_{jn}^{\alpha_{ij} \quad p}(t) e_{jn}^p(t),$$

$$j = s(1)m, \qquad i = 1(1)\omega.$$

One proceeds to find approximations $e_{jn,N}(x)$ to the error functions, $e_{jn}(x)$, for $j = 1(1)m$ and $N \in \mathbf{N}$, in the same way as done before for solving problem in Equations 17 and 2. With problems in Equations 17 and 2, the Tau problem:

$$\sum_{j=1}^{m} D_{ij} e_{jn}(x) \quad \lambda_i \int_a^b k_i(x, t) \varphi_i \Big(e_{sn}(t), e_{(s+1)n}(t), \cdots,$$

$$e_{mn}(t) \Big) dt = \quad H_{in}(x),$$

was associated for $i = 1(1)m, a \leq x \leq b$, with the supplementary conditions:

$$\sum_{j=1}^{m}\sum_{k=1}^{n_{d_j}}\left(A_{jkr}e^{(k-1)}(a) + B_{jkr}e_{jn}^{(k-1)}(b)\right) = 0,$$

$$r = 1(1)\omega,$$

which defines $e_{jn,N}(x)$ for $j = 1(1)m$, where N is the degree of error polynomial $e_{jn}(x)$.

NUMERICAL EXAMPLES

In this section, the efficiency of the presented method is shown by some numerical results. Numerical results for Examples 1 to 3, were reported in Tables 1 and 2. In these tables, the terms y_iTau, y_iExact, $e(y_i)$ and Est. $e(y_i)$ stand for Tau approximations of $y_i(x)$, exact solution, $y_i(x)$, their absolute error and estimation error of y_i for $i = 1, 2$. It should be noted that, in the following examples, $N = n + 2$ has been used.

Example 1

Consider the following system of nonlinear FIDEs with the exact solutions, $y_1(x) = x - x^2$ and $y_2(x) = 4 - 2x$.

$$y_1''(x) + x^2 y_1'(x) + 3y_1(x) + y_2'(x) - 4y_2(x) - \int_{-1}^{1}(3y_1^2(t) + 2xy_1(t)y_2(t))dt = f_1(x),$$

$$y_1'(x) - 2y_1(x) + y_2''(x) - 2y_2'(x) + y_2(x) - \int_{-1}^{1}((t-x)y_1(t) + 5t^2y_2^2(t))dt = f_2(x),$$

with the supplementary conditions:

$$y_1(-1) + y_1'(-1) = 1, \qquad y_1(1) + y_1'(1) = -1,$$

$$y_2(-1) + y_2'(-1) = 4, \qquad y_2(1) + y_2'(1) = 0.$$

where $f_1(x) = -\frac{116}{5} + 19x - 2x^2 - 2x^3$ and $f_2(x) = -53 - \frac{20}{3}x + 2x^2$. For the numerical results, see Table 1.

Table 1. Numerical results of Example 1.

x	y_1Tau	y_1Exact	$e(y_1)$	Est$e(y_1)$	y_2Tau	y_2Exact	$e(y_2)$	Est$e(y_2)$
$n = 5$								
-1.0	-2.00	-2.00	0	0	6.00	6.00	1.00e-09	1.03e-09
-0.8	-1.44	-1.44	1.00e-09	0.02e-08	5.60	5.60	1.00e-09	1.21e-09
-0.6	-0.96	-0.96	4.00e-10	4.07e-10	5.20	5.20	1.00e-09	1.07e-09
-0.4	-0.56	-0.56	1.00e-10	1.13e-10	4.80	4.80	0	0
-0.2	-0.24	-0.24	2.00e-10	2.01e-10	4.40	4.40	0	0
0.0	0.00	0.00	5.86e-10	5.89e-10	4.00	4.00	0	0
0.2	0.16	0.16	1.00e-09	1.02e-09	3.60	3.60	0	0
0.4	0.24	0.24	1.30e-09	1.33e-09	3.20	3.20	0	0
0.6	0.24	0.24	1.40e-09	1.46e-09	2.80	2.80	1.00e-09	1.02e-09
0.8	0.16	0.16	1.40e-09	1.44e-09	2.40	2.40	1.00e-09	1.04e-09
1.0	0.00	0	1.23e-09	1.29e-09	2.00	2.00	1.00e-09	1.26e-09
$n = 10$								
-1.0	-2.00	-2.00	0	0	6.00	6.00	1.00e-09	1.01e-09
-0.8	-1.44	-1.44	0	0	5.60	5.60	1.00e-09	1.03e-09
-0.6	-0.96	-0.96	2.00e-10	2.17e-10	5.20	5.20	1.00e-09	1.04e-09
-0.4	-0.56	-0.56	2.00e-10	2.10e-10	4.80	4.80	0	0
-0.2	-0.24	-0.24	2.00e-10	2.02e-10	4.40	4.40	0	0
0.0	0.00	0.00	1.52e-10	1.53e-10	4.00	4.00	0	0
0.2	0.16	0.16	2.00e-10	2.01e-10	3.60	3.60	0	0
0.4	0.24	0.24	2.00e-10	2.00e-10	3.20	3.20	0	0
0.6	0.24	0.24	2.00e-10	2.00e-10	2.80	2.80	1.00e-09	1.02e-09
0.8	0.16	0.16	1.00e-10	7.20e-10	2.40	2.40	1.00e-09	1.00e-09
1.0	0.00	0	1.31e-10	1.31e-10	2.00	2.00	1.00e-09	1.00e-09

Example 2

Consider the nonlinear FIDEs system:

$$y_1(x) + 3xy_2(x) \quad \int_0^{\frac{1}{2}} (xty_1^2(t) + t^2y_2^3(t))dt$$

$$= \frac{1}{9} + \frac{3}{4}x + 3x^3,$$

$$x^2y_1(x) \quad y_2(x) \quad \int_0^{\frac{1}{2}} (ty_1^3(t) \quad xy_2(t))^2dt$$

$$= \frac{1}{9} + \frac{2}{7}x \quad \frac{6}{5}x^2 + x^3,$$

with exact solution $y_1(x) = x$ and $y_2(x) = x^2$. Table 2 presents the numerical results.

Example 3

Consider the nonlinear ODE:

$$xy'^2 \quad 2yy' + x = 0,$$

with the supplementary condition $y(0) = \frac{1}{2}$ and exact solution $y(x) = \frac{1}{2}(x^2 + 1)$. For $n = 4$, the presented method gives the system of nonlinear equations:

$$\begin{cases} y_0 = 1/2 \\ 2y_0y_1 = 0 \\ 4y_0y_2 \quad y_1^2 = \quad 1 \\ 6y_0y_3 \quad 2y_1y_2 = 0 \\ 8y_0y_4 \quad 2y_1y_3 = 0 \end{cases},$$

which has the solution $\{y_0 = y_2 = 0.5, y_1 = y_3 = y_4 = 0\}$ and leads to $y_n(x) = 0.5+0.5x^2$ and this is the exact solution. For $n = 7$, one has the system of nonlinear equations:

$$\begin{cases} y_0 = 1/2 \\ 2y_0y_1 = 0 \\ 6y_0y_3 \quad 2y_1y_2 = 0 \\ 4y_0y_2 \quad y_1^2 = \quad 1 \\ 8y_0y_4 \quad 2y_1y_3 = 0 \\ 10y_0y_5 \quad 2y_1y_4 + 2y_3y_2 = 0 \\ 12y_0y_6 \quad 2y_1y_5 + 3y_3^2 + 4y_4y_2 = 0 \\ 14y_0y_7 \quad 2y_1y_6 + 10y_3y_4 + 6y_5y_2 = 0 \end{cases}$$

and its solution is $\{y_0 = y_2 = 0.5, y_1 = y_3 = y_4 = y_5 = y_6 = y_7 = 0\}$, which leads to the exact solution.

CONCLUSION

Nonlinear FIDEs systems are usually difficult to solve analytically, therefore, one needs to find an approximate solution. It has been shown that the operational approach to the Tau method is a suitable method of high accuracy for these problems.

The advantages of this method are, as follows:

1. It solves Nonlinear FIDEs systems and nonlinear ODEs without linearization;
2. It gives an error estimator as a polynomial and improves accuracy by increasing n reasonably.

In Tables 1 and 2, one can see that the accuracy of the Tau method at the end points of the intervals is less than the others. The authors will try to improve this in the future.

Table 2. Numerical results of Example 2.

x	y_1 Tau	y_1 Exact	$e(y_1)$	Este(y_1)	y_2 Tau	y_2 Exact	$e(y_2)$	Este(y_2)
$n = 5$								
0.0	0.00	0	2.170e-06	2.175e-06	0.00	0	2.170e-07	2.174e-07
0.1	0.10	0.10	1.797e-06	1.801e-06	0.01	0.01	3.765e-06	3.782e-06
0.2	0.20	0.20	3.352e-05	3.382e-05	0.04	0.04	1.192e-06	1.199e-06
0.3	0.30	0.30	4.990e-05	4.997e-05	0.09	0.09	3.512e-06	3.529e-06
0.4	0.39	0.40	6.818e-05	6.851e-05	0.16	0.16	7.563e-06	7.913e-06
0.5	0.49	0.50	8.945e-04	8.973e-04	0.25	0.25	1.432e-05	0.920e-04
$n = 10$								
0.0	0.00	0	2.170e-09	2.200e-09	0.00	0	2.170e-10	2.300e-10
0.1	0.10	0.10	2.101e-09	2.152e-09	0.01	0.01	7.494e-09	7.498e-09
0.2	0.20	0.20	6.997e-08	6.997e-08	0.04	0.04	2.294e-09	2.311e-09
0.3	0.30	0.30	1.828e-08	1.860e-08	0.09	0.09	4.275e-09	4.278e-09
0.4	0.40	0.40	4.837e-08	4.851e-08	0.16	0.16	5.634e-09	5.642e-09
0.5	0.50	0.50	7.004e-07	7.302e-07	0.25	0.25	5.808e-08	7.116e-08

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