Scientia Iranica, Vol. 14, No. 6, pp 519–533
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SCIENTIA
IRANICA

Deformable Field Theory of Magnetoelastic Continua and Interactions

P. Rafinejad* **and J. Faiz**[1]

The presented deformable field theory deals with electromagnetic local forces on the basis of field energy density. In this theory, any movement, rigid or deforming, distorts the electromagnetic field continuum. This leads to novel concepts of total and local forces explicitly related to the elastic deformation gradient rather than the classical gradient of the magnetic field. It is shown how the magnetic vector potential, as the magnetic invariant variable, is associated to this deformable field continuum and is, meanwhile, reference-independent. Then, within an adiabatic virtual work, the local magnetic energy derivatives are analytically performed, converging to overall electromagnetic force and stress tensors, including Lorenz, inherent magnetization and strict magnetostriction forces.

INTRODUCTION

The Lorenz force density, $\mathbf{J}\mathrm{x}\mathbf{B}$ with ($\mathbf{B} = \mu_0\mathbf{H}$), is the magnetic basic force definition used whenever no material magnetization is present. Using the equivalent magnetic field vectors, Maxwell gave a non-material-dependent stress tensor: $T_{mn} = B_nH_m \quad \frac{1}{2}\mu_0H^2\delta_{mn}$. Although this interpretation was rather a mathematical transformation, it, nevertheless, led to a novel physical concept of total magnetic forces widely spread. However, the extrapolation of this method for the stress tensor within a magnetizable material was less evident. Following the same approach, the Lorenz force inside a magnetic isotropic material may be written as: $\mathbf{f} = (\mathbf{B}.\nabla)\mathbf{H} \quad \frac{1}{2}\mu\nabla(H^2)$, which differs from the divergence of the Maxwell tensor by ($\frac{1}{2}H^2\nabla\mu$). Woodson and Melcher [1], using the magnetic scalar potential within a non-conducting isotropic material, concluded that ($\frac{1}{2}H^2\nabla\mu$) would stand for the magnetization force, which could be divided in two parts: Inhomogeneity gradient and magnetostriction $\boldsymbol{\nabla}(\frac{1}{2}H^2\rho\partial_\rho\mu)$. The resulting stress tensor became, then: $T_{mn} = B_nH_m \quad \frac{1}{2}H^2(\mu \quad \rho\partial_\rho\mu)\delta_{mn}$. Rafinejad [2] obtained this tensor as a whole and, in a general case, with quite a different approach, which has been discussed in this paper.

Another approach was to add the magnetization force density, $\mathbf{M}.\partial_x\mu_0\mathbf{H}$ to $\mathbf{J}\mathrm{x}\mathbf{B}$, equivalent to $\mathbf{J}\mathrm{x}\mu_0\mathbf{H} + (\mathbf{M}.\boldsymbol{\nabla})\mu_0\mathbf{H}$, leading to the general Maxwell tensor with $\mathbf{B}/\mu_0 = \mathbf{M} + \mathbf{H}$. W.F. Brown [3] undertook an admirable endeavor to describe and formalize magnetoelastic behavior, in terms of total field vectors, through the matter on matter interaction method. This method has been recently used by [4]. Far later, A.C. Eringen and G.A. Maugin [5] enlarged this theory using point-like particles endowed with mass m and electric charge e. The local electromagnetic force is derived from the Lorenz force, $e\mathbf{E} + e\mathbf{v}\mathrm{x}\mathbf{B}$, integrating over all point-like particles included in a local particle, $\mathbf{E}$ being the electric field intensity and e$\mathbf{v}$ the moving charge, including free current and magnetic moment effects.

The approach of direct interaction between the point-like polarized and magnetized particles encounters difficulties when dealing with a deforming material continuum, essentially because of the finite, not infinitely small, dimension of dipoles or point-like particles. This model leads to an ill-defined concept of long-range and short-range interactions and, also, to a rather ambiguous external field definition. In fact, based on the force exerted on a single magnetic dipole, $\mathbf{m}$, by an external field converging to $(\mathbf{m}.\boldsymbol{\nabla})\mathbf{B}_0$, the local force within a magnetized body seems to be readily: $(\mathbf{M}.\boldsymbol{\nabla})\mathbf{B}_0dV$, where $\mathbf{M}dV$ is the local

*. *Corresponding Author, Department of Electrical and Computer Engineering, University of Tehran, I.R. Iran; and 5, rue du Parc, F-38180, Seyssins, France.*
1. *Department of Electrical and Computer Engineering, University of Tehran, Tehran, I.R. Iran.*

magnetization. But, in this expression, the definition of $\mathbf{B}_0$ is much less evident, leading to serious problems. In fact, according to the above-mentioned Maxwell method, $\mathbf{B}_0 = \mu_0\mathbf{H}$, which is equivalent to the Chu model, as presented in [6]. This reference surveyed the different external or exciting field interpretations and concluded that the differences were due to the magnetostriction effects. Nevertheless, later, these authors used Brown's method in [4]. W.F. Brown [3] gets rid of this difficulty by considering $\mathbf{B}_0 = \mu_0\mathbf{H}_0$ to have been issued from all sources out of the external surface of a local finite particle. Then, he "supposes" that this particle is separated from the rest of the material by a sufficiently large gap, with respect to the dipole's size. In this case, the total short-range interactions vanish. Consequently, the total force exerted on this particle would be $\int[\mathbf{J} \times \mu_0\mathbf{H}_0 + (\mathbf{M}.\boldsymbol{\nabla})\mu_0\mathbf{H}_0]dV$. Next, he replaces $\mathbf{H}_0$ by $\mathbf{H} \quad \mathbf{H}_1$, $\mathbf{H}$ being the total field intensity and $\mathbf{H}_1$, resulting from the short-range magnetic dipoles. Finally, he shows that, when the surrounding gap is reduced "almost" to zero, the remaining term, $\int \quad (\mathbf{M}.\nabla)\mu_0\mathbf{H}_1 dV$, converges to the shape dependent term, $\int_S \frac{1}{2}\mathbf{M}^2 n ds$, where s is the external surface. So, he gets the local force expression: $\int[\mathbf{J}\times\mu_0\mathbf{H}+(\mathbf{M}.\nabla)\mu_0\mathbf{H}]dV + {}_S\frac{1}{2}\mathbf{M}^2\mathbf{n}ds$. The presence of the last shape dependent term might be due to his "almost" and not infinitely small gap concept, as he predicted in Section 5.1 in [3]. The interesting generalization of this theory in [5] leads to the Amperian model [6] and, finally, to the force density, $\mathbf{J} \times x\mathbf{B} + (\mathbf{M}.\boldsymbol{\nabla})\mathbf{B}$, concluding that the fundamental field is, rather $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$. This complex theoretical demonstration starts with the atomic Lorenz force, $e\mathbf{v}\times x\mathbf{B}_0$, where the magnetic moment appears as Amperian $\boldsymbol{\nabla}x\mathbf{m}$ within $\mathbf{J} = e\mathbf{v}$, following Taylor's series expansion of either atomic charge movements or $\mathbf{B}_0$ over a point-like particle. Then, the short-range contributions are neglected following a nonrelativistic reason.

On the other hand, in these aggregate models [3,5], the extrapolation of the point-like magnetic moment, $\mathbf{m}$, to the continuum magnetization density, $\mathbf{M}$, requires jumping to the macroscopic constitutive law concept given by $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$, which is defined, rather, by the Maxwell equations. On the other hand, the force calculation, whatever the method employed, makes use of the total field vectors. So, one can assume that the force calculation by direct matter on matter interaction, in case of magnetization or polarization, requires the use of both the constitutive laws and the field vectors. So, one needs to calculate, first, these quantities. Therefore, would it not be more reasonable to get the electromagnetic force directly and uniquely from these calculated quantities? That was the idea of Maxwell, to calculate the Lorenz and magnetization forces in terms of only $\mathbf{B}$ and $\mathbf{H}$. However, the extrapolation of his method to magnetization sources remained less evident, mostly because these sources are hidden in $\mathbf{B}$ and $\mathbf{H}$, unlike the free currents, which are external. This might explain the reason for that unreasonable widespread flashback to atomic models within the classical physics of continua.

In the field approach, the magnetic continuum is only defined by the magnetic field vectors, $\mathbf{H}$ and $\mathbf{B}$. These quantities resume the overall magnetic source interactions, as ruled by the Maxwell equations. The constitutive law, $\mathbf{B} = \mu\mathbf{H}$, where permeability μ, or, generally, tensor μ, takes account of the material macroscopic observable behavior. In fact, the resulting magnetic field stands, directly, for all real electric currents ($\mathbf{J} = \sigma\mathbf{E}$, as free space electric charges and wave propagation are neglected in this paper) and, implicitly, for the material magnetization behavior. It would be unreasonable, if not a misuse, to mix up this magnetic field concept with that magnetizing equivalent dipoles and piecewise Lorenz force calculation. The only way to use the same concept is to get back to the force definition basis: The magnetic field reaction to any configuration changes from the energy point of view. Here, the question is, would the magnetic energy, $dW = \frac{1}{2}\mathbf{H}.\mathbf{B}dV$, or, generally, $(\int_0^{\mathbf{B}} \mathbf{h}.d\mathbf{b})dV$, represent the overall local magnetic field stored energy of this equivalent continuum, as discussed in this paper?

In chapter III, W.F. Brown [3] used the energy variational approach, using his above-mentioned force. His virtual evolution process, using the aggregate dipole model, led to a complex magnetization energy still depending on the long-range, H_0, and short-range, H_1, concepts. However, his energy time-derivative, given in chapter II, is derived from the gap concept and his above-mentioned force. In [1], the coenergy density derivative is performed on the basis of a fictitious displacement within a fixed scalar potential and permeability. Moreover, in some energy approaches, virtual evolution i.e., virtual work rate, is rather considered, including the external source contribution, heat losses, kinetics and dynamics etc.

In this paper, the magnetic local force is derived from the local energy derivative, as defined by the adiabatic virtual work method. This initially complex task is approached using the physical concepts suggested by Rafinejad [2] and fully presented in this paper under the deformable field theory.

Classical Virtual Work Method

Now, let the force definition be reviewed by the adiabatic virtual work principle, considering the example of an electromagnet. At the level of the electric circuit, the electromagnetic energy transfer is defined by the interaction between circuit current i and its magnetic flux linkage Φ fluctuation, by $\delta\Phi$. Using the reluctance

$\lambda = \Phi/i$, the system configuration is separated from the electromagnetic quantities. Hence, the stored energy under constant configuration is: $\delta w = (\Phi/\lambda)\delta\Phi$ and its conversion, during displacement $\delta\xi$ under constant flux Φ, with no electrical energy exchange (adiabatic), is resumed by:

$$\mathbf{f}.\delta\xi = \quad \delta w|_{\Phi} = \quad \int_0^{\Phi} \Phi\delta(1/\lambda)d\Phi.$$

Alternatively, by the variable change from φ to φ/λ, one gets the same expression, usually considered as the coenergy derivative:

$$\mathbf{f}.\delta\xi = \delta w' j_i = \int_0^{\Phi/\lambda} \delta\lambda(\Phi/\lambda)d(\Phi/\lambda).$$

This lumped model requires an explicit expression for the reluctance, λ, derivatives, usually performed by major simplifications not always realistic. It is even a source of erroneous extrapolations, as illustrated in the examples of Figure 1 and given in many textbooks on electromechanical transducers [7] or electrical machinery [8].

Where F_x and T_θ are the centralizing force and torque, k and k' are constants depending only on device fixed sizing parameters and B_g is the airgap induction. These wrong force or torque expressions are obtained by neglecting the airgap fringing fluxes, whereas, the centralizing forces are just due to the fringing flux! The error would have been seen, due to the fact that, under this hypothesis, the Maxwell tensor method would give a null force. Although [7] invokes this discrepancy, it concludes that the Maxwell tensor is not valid in this case. However, [8] gives, in the following paragraph, a quite different result, based on the Fourier expansion of the total flux deduced from two extreme longitudinal and transversal positions. This is a correct method, commonly used in electrical machinery. But, no instantaneous or local forces may be expected.

This lumped model fails, however, when eddy currents are to be considered or if local forces are needed and so on. Of course, in general, neither flux linkages, Φ, nor magnetic circuit $mmfs$, (Φ/λ), may be recognized explicitly in the magnetic local energy expression, $dw = (\int_0^{\mathbf{B}} \mathbf{h}.d\mathbf{b})dV$. Naturally, the question of the magnetic field position invariance and the local energy derivative excluded the generalization of this approach for magnetic stress formulation. (In this paper, invariance has the same meaning as position fixing or holding constant.)

On the other hand, there exists some basic difference between the virtual system evolution and the adiabatic virtual work principle. The latter, as implemented in this paper, can be applied identically, whatever the system complexity, such as eddy currents, hysteresis etc. The main interest of the variational method and, particularly, the virtual work principle, is due to the fact that the virtual configuration variation (virtual movement or virtual deformation) happens in a time-frozen reference. It means that the time and all time-dependent quantities are kept frozen (fixed). Consequently, $\mathbf{f}.\delta\xi$, considered here, is not equivalent to $\mathbf{f}.\mathbf{v}\delta t$ in [3]. Notice that Φ/λ, recognized as magnetomotrice force (mmf), cannot be replaced by current i, which is, in any case, fixed as materials. From this point of view, the coenergy expression, $\mathbf{f}.\delta\xi = +\delta W'$, is deduced from the identity, $\delta w|_{\Phi} + \delta w|_{\Phi/\lambda} = 0$, generalized in this paper, and not from the real complementary energy flow. Here, only the local stored energy, expressed in terms of the total field vectors, is considered getting rid of

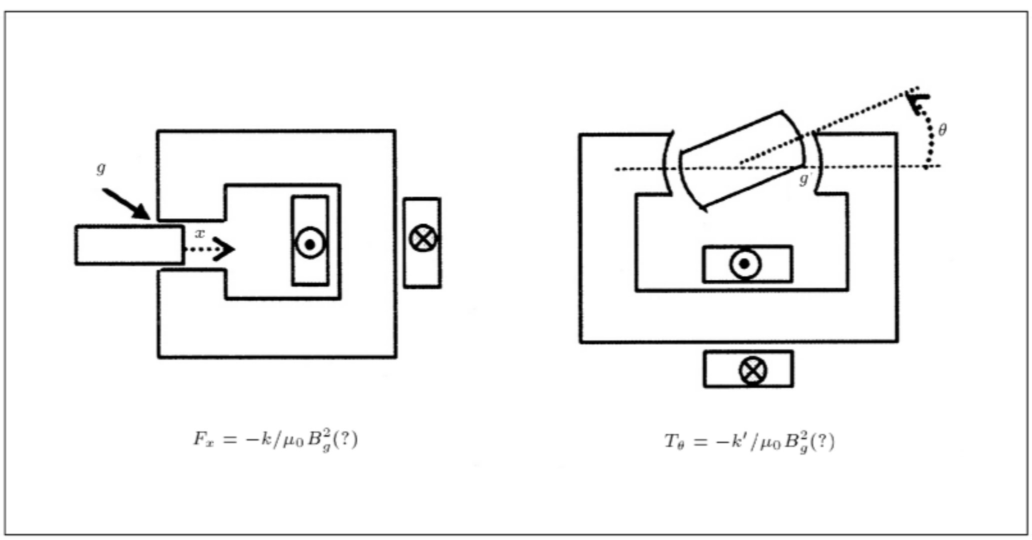

Figure 1. Wrong centralizing effort expressions.

external or internal field or short-range or long-range force concepts.

Virtual Work and Finite Element Applications

At the early stages of finite element implementation in electromagnetism [9], Rafinejad experimented on the virtual work principle, in the case of an electromagnet [10]. Later, within his Ph.D. dissertation [2], he generalized this method, developing the theoretical base and mathematical tools to achieve general stress tensor formulation. Thanks to Coulomb [11] his mathematical tool, known as the local Jacobian derivative method, has been widely used (e.g., [12-14]) in finite element method implementation, in the case of magnetic scalar potential and [15-17], in the case of edge-elements. However, his energy approach, based on the deformable field theory, has remained unknown, mostly because of the interesting practical numerical outlets, as referenced in [18] or developed in [16]. It may also be that [2] was not a widely available source for many authors. It was the same with the question of the position invariance variable that was discussed in [19,20] essentially, in the case of the **H** edge finite element approximation method. These special case discussions led to amazing conclusions regarding the nodal vector potential invariance conditions reported in [12,16] and even regarding a mistake in [21], which was rectified in [22]. It is the same with the ambiguities reported in [23], regarding the magnetic field invariance principle. This paper highlights the physical concepts preceding the above-mentioned local Jacobian derivative method, not sufficiently described in [2], with regards to [21] and some other works.

DEFORMABLE FIELD THEORY OF MAGNETOELASTIC CONTINUA

In the above-mentioned classical literature, the magnetoelastic interactions are considered by the material displacement (flow) through a fixed magnetic field, taking account of the magnetic state invariance. But, the only well-defined magnetic field is given by the total **B** and **H**, according to the Maxwell equations. In these equations, free currents, excited or induced, are external sources, whereas material magnetization, excited, permanent or remnant, is internal and implicitly present in $\mathbf{M} = \mathbf{B}/\mu_0 \quad \mathbf{H}$. So, how could it be possible to isolate a local magnetized particle from the resulting total field? Even the concept of external field is superfluous.

In fact, materials, conducting, magnetized or magnetizing, are inherently associated with the total magnetic field, before and after deformations. That is the starting point of the Rafinejad physical concept of the electromagnetic continuum. This continuum is linked to the materials but extending beyond them, as far as the energy density is significant. This can be illustrated by the famous magnetic force-lines. Imagine elastic balls, to which these elastic magnetic force-lines are solidly linked. When a ball moves, the lines are contracting or expanding all around it. That is the way in which the magnetic force is illustrated for teaching purposes in electrical machinery. Now, imagine that you pinch the ball itself, contracting the lines inside. That is the way the local force may be defined and calculated. In both cases, the lines are deformed, so, the magnetic continuum is distorted in any case. Contracting or expanding means field intensity or flux density changing. That is the way in which one looks for the magnetic induction and stored energy variation within a deforming (not flowing) magnetoelastic continua. Furthermore, this deformable continuum naturally extends to free (air) areas, which is quite a new concept. This generalized magnetic continuum is considered exactly as the material elastic continuum. These two continua: Field and material, may be coupled or not only according to **H** and **B** values. In fact, this electromagnetic continuum has no mass. Its deformation does not imply, necessarily, a material distortion, but, could be resulted from a material free movement. The coupling is due to the common configuration of the elastic and electromagnetic continua. This common configuration is limited to conducting, polarized (not considered in this paper) and magnetized corps. Another coupling factor is based on macroscopic strain/magnetization interactions and whether the property of a ferromagnetic material would change under external strains and how. This question requires more complex models based upon experimental results. However, the physical concepts and mathematical developments presented in this paper would give a suitable base for selection and interpretation of these experiments. In fact, the same questions of external/internal field and external/internal strains appear for experiment specifications and measuring interpretation, as mentioned in [24,25].

Electromechanical Energy Equilibrium

Let one consider the whole system of a total volume, Ω, extending, theoretically, to infinity and defined by the above-mentioned electromagnetic continuum coupled partially to the elastic continuum. The system internal energy is given, thus, by the electromagnetic and the elastic stored energies, which are in equilibrium when no external energy exchange may occur. In this deformable field theory, only free currents are external sources, according to Maxwell's equations. So, the system equilibrium is considered, ensuring that no external energy exchange happens. This condition will designate the right magnetic invariant quantity and the way in which it should be position invariant.

Now, let the electromechanical energy conservation law of the system begin, as follows:

$$\delta L = \int_{\Omega \to \infty} \delta[(w+p)d\Omega] + \mathbf{f}d\Omega.\delta\mathbf{x} = 0, \qquad (1)$$

where, w is the total magnetic field energy density, which is known to be $\frac{1}{2}\mathbf{H}.\mathbf{B}$ in a linear case or $\int_0^{\mathbf{B}} \mathbf{h}.d\mathbf{b}$ in a general case (small characters stand for historical values and capital characters for actual values at the considered instant). Here, the magnetic induction, $\mathbf{b}$, is related to the field intensity, $\mathbf{h}$, by $\mu\mathbf{h}$ with permeability μ, which may be a tensor, depending on $\mathbf{h}$ (nonlinearity), position (anisotropy) and deformation (magnetostriction). In this paper, the electrostatic forces are neglected for their very little influence in electromechanical devices and, also, for the sake of clarity of demonstration. However, they could be taken into account, in the same way, by adding the electrostatic stored energy.

δp is, in this case, the density of input energy from the external electrical sources and free currents, excited or induced. On behalf of the divergenceless property, the current densities, $\mathbf{J}$, may be considered as elementary current loops. According to Faraday's law, any energy exchange could be written as: $\delta p_e = \quad i\delta\Phi$. p_e represents, meanwhile, the potential energy of this current loop, that is, the necessary virtual work in order to remove it out of the system [26].

Using the magnetic vector potential defined by $\mathbf{B} = \boldsymbol{\nabla}\mathbf{x}\mathbf{A}$ and Stock's inverse theorem, the flux variation, $\delta\Phi$, becomes the total circulation of $\mathbf{A}$ along the elementary circuit, (c), which was illustrated in Figure 2, as $(\int_{(c)} \delta\mathbf{A}.d\mathbf{c})$. Substituting the identity $id\mathbf{c} = \mathbf{J}d\Omega$, one finally obtains:

$$\delta(pd\Omega) = \quad \mathbf{J}.\delta\mathbf{A}d\Omega. \qquad (2)$$

$\mathbf{f}$ is the searched force density responsible for any mechanical energy exchange. It should not be confused with any external force. In fact, the weak coupling method is being voluntarily employed to analyze only the magnetoelastic reactions.

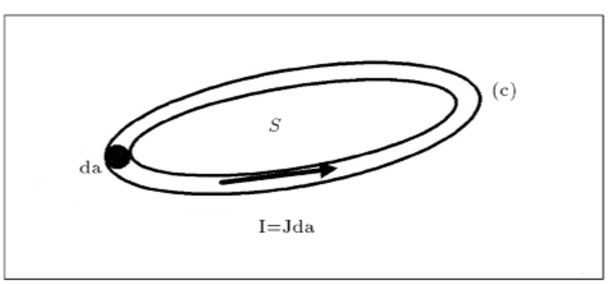

Figure 2. Elementary current loop.

$\delta\mathbf{x}$ is any arbitrary virtual local displacement or distortion.

Two independent state variables are present in this energy functional: The magnetic vector potential, $\mathbf{A}$, for the magnetic state and the displacement or configuration position, $\mathbf{x}$, for the mechanical state. Here, the mechanical state variation will be considered, assuming the solution magnetic vector potential to be known everywhere. It can be shown [2,27], that the vector potential, $\mathbf{A}$, corresponding to the Maxwell equations (displacement currents being neglected) satisfies $\delta_A L|_x = 0$ under fixed configuration. This is the variational method of the electromagnetic field resolution widely used in finite element applications [28]. In this paper, it is intended to determine the force density, $\mathbf{f}$, by the energy conservation law, $\delta_x L|_A = 0$, where δ_x means variation, with respect to material displacement (the index x will not be repeated here after) and $|_{\mathbf{A}}$ means $\mathbf{A}$ is invariant or holds constant when the position changes. The vector potential position or configuration invariance condition, given in [2] and detailed in the next section, ensures that $\delta p = 0$, meaning that no external energy exchange happens. This is known as the adiabatic virtual work principle, resumed by:

$$\int_{\Omega \to \infty} \delta(wd\Omega)|_A + \mathbf{f}.\delta\mathbf{x}d\Omega = 0, \qquad \forall \delta\mathbf{x}\text{in } \Omega. \qquad (3)$$

The way to hold invariant the vector potential, $\mathbf{A}$, may be mutually deduced from this adiabatic condition. But, in fact, the magnetic vector potential invariance is deduced from the reference-independence conditions of a vector on the basis of the deformable field theory, as detailed in the appendix. The reciprocal reasoning would not be enough due to the fact that p in Equation 2 is only defined within the conducting area.

Thus, the virtual work approach is based only on the state of the "whole" system at the considered instant, whatever its dynamic trends. Obviously, the magnetic source dynamics (eddy currents) or history (hysteresis) are implicitly present in the solution field distribution, which is supposed to be known prior to application of the virtual work principle. This would also concern the thermodynamic state of the system.

Difficulties often encountered in the magnetic local force formulation are due to the exact definition and, then, thorough separation of the state variables in Equation 3. In the equivalent magnetic moment models, the magnetization potential energy density $\mathbf{M}.\mu_0\mathbf{H}_0$, [3] variation is defined, supposing $\mathbf{M}$ flowing through $\mathbf{H}_0$ by: $\mathbf{M}.\partial_x\mathbf{H}_0$. This leads to the force expressions discussed in the introduction section. In [1], the same concept of a fictitious magnetic local source flowing through the fixed magnetic field leads to $\frac{1}{2}H^2\partial_x\mu$. In these approaches, the gradient ∂_x is resulted from a Taylor's series expansion of the magnetic field or permeability in the fixed space. In the deformable field theory, the displacement or distortion, $\delta\mathbf{x}$, affects both material and magnetic field, having the same reference in any case. Of course, from this point of view, the energy variation cannot be readily defined by $\partial_x \mathbf{H}$ or $\partial_x \mu$, requiring new theoretical concepts discussed in the next sections.

Electromagnetic Deformable Continuum

The electromagnetic field within physical materials, that is polarized, conducting and magnetized, as well as in free space, is defined by **E**, **D**, **H** and **B**. This is the basis of the deformable field theory of the magneto-elastic continua. The displacement currents, $\partial_t \mathbf{D}$, and the electric field will be neglected, as usual, for the sake of clarity. Therefore, the magnetic physical space, Ω, is continuously defined, even beyond the electromechanical coupling area: Magnetizing and current-carrying materials. This means that any position change leads to this magnetic continuum distortion, even in the vacuum. In fact, in this theory, vacuum or free fixed space has no physical sense, as far as the field energy density exists. Let this magnetic deformable continuum be called physical space, which becomes material within active materials. Naturally, the physical space is uniquely defined by the magnetic vector potential, A, which is the invariant state variable, as mentioned above. It is linked to this deforming physical space, but, should be also position invariant. At first glance, this seems rather contradictory. This is much easier to imagine with a scalar potential, that is to say: One point one potential, whatever its position. This is the physical sense of a scalar potential invariance. For a vector potential, its three components are considered as three reference independent potentials. The deformation of this continuum will change the local reference, leading to the magnetic field perception variation. For example, if two points are contracted together with their electric potentials, the field intensity will naturally increase at that point. Generally, one needs to apply the local reference transformation into an invariant common reference for mathematical operations. Thus, Taylor's series expansion of some field quantity is not used, but, rather the usual approach used in elastic continua dynamics is employed. It would seem natural to use elasto-dynamic techniques in electrodynamics.

Let Ω_0 be the initial space before a virtual displacement, considered as the common reference, and Ω the deforming physical space. After virtual displacement, any point in Ω can be located in the common reference by:

$$\mathbf{X}'(\Omega) = \mathbf{X}(\Omega_0) + \mathbf{x}(\Omega_0), \tag{4}$$

where $\mathbf{X}$ is the global coordinate vector in (Ω_0) and $\mathbf{x}$ is a virtual displacement at that point. $\mathbf{x}$ is originally zero everywhere. Using the Jacobian gradient of transformation, known to be $\mathbf{G}_{mn} = \delta_{mn} + \partial \mathbf{x}_n / \partial X_m$ and its determinant $|\mathbf{G}|$, the gradient operator, ∇', and the local volume, $d\Omega$, with respect to the physical space reference, are given in Ω_0 by:

$$\nabla'(\Omega) = \mathbf{G}\ ^{1}.\nabla(\Omega_0), \qquad d\Omega = |\mathbf{G}| d\Omega_0. \tag{5}$$

The virtual Jacobian, $\mathbf{G}$, is indeed equal to unity at the origin of a virtual displacement, $\mathbf{x} = 0$, where the force formulation is to be carried out.

Magnetic Continuum Perceptible Energy

The second principle of this approach concerns the proper definition of the local energy, as perceived by physical space. Theoretically, magnetic induction and intensity are defined within the material before and after deformation. So, the electric or magnetic potentials are associated with physical space, but, are position invariant. The scalar potential, as with any other scalar quantity, is always clearly defined in physical space, Ω. But, the physical sense of the invariance of the vector potential, $\mathbf{A}$, is a little harder to conceive, due to the fact that one is expecting a fixed vector in Ω_0, which is only a mathematical space. In the authors' deformable field theory, this invariance means that each point in the physical space conserves its potential. In other words, the potential of a given material point is independent of its position. Is this not logical? For the vector potential, this definition is enlarged to each component, $\mathbf{A}_i$, $(i = 1, 2, 3)$, in the reference independence sense. This invariance definition is also verified by ensuring the adiabatic condition, as detailed in Appendix. Moreover, this theory will be validated by the resulting stress tensors, in comparison with the Maxwell tensor and variants. According to this vector invariance definition, after a deformation, the magnetic vector potential, $\mathbf{A}'$, in the deformed physical space, can be written in the common reference, Ω_0, as:

$$\mathbf{A}'(\Omega) = \mathbf{G}\ ^{1}.\mathbf{A}(\Omega_0). \tag{6}$$

Using Equations 5 and 6, the magnetic induction, $\mathbf{B}'$, as perceived in the physical space, Ω, is given by:

$$\mathbf{B}' = \boldsymbol{\nabla}' \times \mathbf{A}'|_{\Omega} = (\mathbf{G}\ ^{1}.\boldsymbol{\nabla}) \times (\mathbf{G}\ ^{1}.\mathbf{A})|_{\Omega 0}. \tag{7}$$

Now, its variation can be developed as:

$$\delta\mathbf{B} = (\delta\mathbf{G}\ ^{1}.\boldsymbol{\nabla}) \times (\delta\mathbf{G}\ ^{1}.\mathbf{A})|_{\Omega 0}. \tag{8}$$

The local magnetic energy, $w d\Omega$, is also to be evaluated in the physical space, with $d\Omega = |G| d\Omega_0$. Consequently, the energy variation perception is defined explicitly in terms of the deformation gradient. The partial variation of the energy functional (Equation 3) becomes, now:

$$\int_{\Omega 0}[(\mathbf{H}.\delta\mathbf{B}) + w\delta|\mathbf{G}| + \int_0^{\mathbf{B}} \delta\mathbf{h}|_b.d\mathbf{b} + \mathbf{f}.\mathbf{x}]d\Omega_0 = 0. \tag{9}$$

This is the general adiabatic virtual work expression. Be aware that these Jacobian derivatives are to be carried out at $\mathbf{x} = 0$, where $\mathbf{G} = 1$ and $d\Omega = d\Omega_0$. This common reference, Ω_0, should not be confused with the eventual normalized or initial configuration at $t = 0$. Hereafter, the demonstrations are carried out with this common reference, initially defined by $\Omega|_{x=0} = \Omega_0$. Anyhow, actual elastic transformations could be either considered at this stage within a strong coupling procedure or applied to the resulting force expressions given in the next section. The first term of Equation 9 represents magnetic "dispersion", where the perceived induction variation, defined by Equation 8, can be explicitly developed using the virtual Jacobian derivative, $\delta\mathbf{G}\ ^{1}$. The second term in Equation 9 is due to the volume distortion, defined by the Jacobian determinant variation, $\delta|\mathbf{G}|$. The third term introduces the magnetization variations, which can be expressed, alternatively, in terms of the magnetic intensity, using the following identities:

$$\int_0^{\mathbf{B}} \delta\mathbf{h}|_{\mathbf{b}}.d\mathbf{b} = \delta|_{\mathbf{b}}(\mathbf{H}.\mathbf{B} \quad \int_0^{\mathbf{H}} \mathbf{b}d\mathbf{h})$$

$$= \mathbf{H}.\delta\mathbf{B}|_{\mathbf{b}} \quad \mathbf{H}.\delta\mathbf{B}|_{\mathbf{b}} \quad \int_0^{\mathbf{H}} \delta\mathbf{b}|_{\mathbf{h}}.d\mathbf{h},$$

or, finally;

$$\int_0^{\mathbf{B}} \delta\mathbf{h}|_{\mathbf{b}}.d\mathbf{b} = \quad \int_0^{\mathbf{H}} \delta\mathbf{b}|_{\mathbf{h}}.d\mathbf{h}, \tag{10}$$

or, alternatively, using $\mathbf{m} = \mathbf{b}/\mu_0 \quad \mathbf{h}$:

$$\int_0^{\mathbf{B}} \delta\mathbf{h}|_{\mathbf{b}}.d\mathbf{b} = \quad \int_0^{\mathbf{B}} \delta\mathbf{m}|_{\mathbf{b}}.d\mathbf{b} = \quad \int_0^{\mathbf{H}} \mu_0\delta\mathbf{m}|_{\mathbf{h}}.d\mathbf{h}.$$

It should be emphasized that this identity is only due to a variable change introduced here for the sake of similarity to other existing formulae. It should not be interpreted as a physical equivalence, as regards the strict magnetostriction discussed below. The same mathematical equivalence was introduced between ϕ and ϕ/λ in the magnet example, which will be generalized to $\mathbf{A}$ and $\mathbf{H}$ invariance base formulae in the last section below.

In [1], the mass conservation law is applied to conclude $\delta\mathbf{b}d\Omega = \quad \mathbf{h}\rho\partial_\rho\mu\delta d\Omega$, where ρ is the mass density. As described below, this direct permeability approach would not be adequate to deal with the magnetostrictive forces. Moreover, the magnetostriction definition, by the scalar derivative, $\partial_\rho\mu$, couldn't represent the anisotropic nature of this phenomenon. Let the permeability be replaced by magnetization, using Equation 10. The magnetization density may be also linked to the mass density and the magnetization is defined as per unit mass $\mathbf{m}' = \mathbf{m}/\rho$ [3]. Hence, the local magnetization becomes $\mathbf{m}'dm$ and:

$$\delta\mathbf{h}|_{\mathbf{b}} = \rho\partial_\rho\mathbf{m}|_{\mathbf{b}} = \mathbf{m} + \rho^2\partial_\rho\mathbf{m}'. \tag{11}$$

Using these results, Equation 9 becomes:

$$\delta(wd\Omega) = \mathbf{H}.\delta\mathbf{B}d\Omega + \frac{1}{2}B^2/\mu_0\delta\mathbf{d}\Omega + \int_0^{\mathbf{B}} \rho^2\partial_\rho\mathbf{m}'.d\mathbf{b}\delta d\Omega. \tag{12}$$

The last term seems to represent the strict magnetostriction effort. But, under constant $\mathbf{b}$ and within an adiabatic virtual work, the magnetic moments per mass $\mathbf{m}'$ would be constant, whatever the deformation. This means that the magnetic sources remain constant, which is compatible with the authors' field theory. In other words, the local global magnetization cannot change within a time-frozen virtual displacement. Therefore, the last term would rather vanish. To clarify this point of view, let the magnetization now be considered as the magnetic sources, globalized by $md\Omega = \mathbf{m}'d\mathbf{m}$, where dm is the local particle mass. Then, following the energy approach basis, the virtual work expression (Equation 9) is reconsidered, looking for the potential energy of this quantity, $\quad \delta_\xi(\mathbf{m}d\Omega.\mathbf{b})$. This term can be recognized within Equation 9, when it is rearranged as:

$$\delta(wd\Omega) = \mathbf{H}.\delta\mathbf{B}d\Omega + \frac{1}{2}B^2/\mu_0\delta d\Omega \quad \int_0^{\mathbf{B}} \delta(\mathbf{m}'dm.d\mathbf{b})|_{\mathbf{b}}. \tag{13}$$

Here, the last term represents magnetization energy exchange, which is excluded within an adiabatic virtual work:

$$\delta(\mathbf{m}'dm.d\mathbf{b})|_{\mathbf{b}} = 0. \tag{14}$$

This could be deduced from Equation 11, when $\partial_\rho\mathbf{m}' = \ 0$. It results that the last term in Equation 12, if existing, would actually be a variation of some additional magnetostriction energy, $w_{ms}(\mathbf{m}', \boldsymbol{\varepsilon})$, where $\boldsymbol{\varepsilon}$ is strain, so that:

$$w = \frac{1}{2}B^2/\mu_0 \quad \int_0^{\mathbf{B}} \mathbf{m}.d\mathbf{b} + w_{ms}(\mathbf{m}', \varepsilon). \tag{15}$$

So, δw_{ms} should be added to Equation 13. From this point of view, the permeability derivative, $\partial_\rho \mu$ [1,2], resulting from Equation 10, could not be ignored, while dealing with deformable materials as in [29]. In fact, δw_{ms} replaces the last term in Equation 12, whenever it can be explicitly measured, satisfying Equation 14, otherwise, they are equivalent, $\delta w_{ms} = \quad \delta(\mathbf{m}' dm.d\mathbf{b})|_{\mathbf{b}}$.

Let this discussion be postponed to the magnetostriction section below and the energy approach be continued resuming $\delta w_{ms} d\Omega$ by $\quad \partial_\rho w_{ms} \delta d\Omega$. Now, using Equation 8, Equation 13 becomes:

$$\delta(wd\Omega)|_{\mathbf{x}\to 0} = \mathbf{H}.[(\delta\mathbf{G}\ \ ^{1}.\boldsymbol{\nabla})\times\mathbf{A} + \boldsymbol{\nabla}\times(\delta\mathbf{G}\ \ ^{1}.\mathbf{A})]d\Omega + (\frac{1}{2}B^2/\mu_0 \quad \partial_\rho w_{ms})\delta|\mathbf{G}|d\Omega. \qquad (16)$$

This is the first result of the deformable field theory of magnetoelastic interactions. It can be seen that the displacement function is thoroughly separated from the magnetic field within the local magnetic stored energy derivative. At this point, Equation 16 is already a suitable formula for strong magnetoelastic coupled approaches. However, the authors' analysis is continued to highlight the interest of the extended deformable magnetic continuum concept, giving a new physical sense to the total force tensor expression. Then, the local force and, finally, the stress tensors, including Maxwell tensors, will be obtained, step by step, following the same analysis within deformable and generally magnetized and conducting materials.

To formalize the local force, $\mathbf{f}$, formulation from Equation 9, one needs to analyze the nature of the deformation, in order to develop the Jacobian derivatives, $\delta\mathbf{G}\ \ ^{1}$ and $\delta|\mathbf{G}|$. In order to approach, exactly, a local force, the arbitrary virtual displacement is precise by a point-like infinitesimal displacement vector, ξ_i, at that point. Then, an arbitrary function is introduced to define the local continuous deformation. This is the subject of the third proposal methodology.

Local Jacobian Derivative Method

According to the deformable field theory, any material movement, deforming or not, leads to a magnetic continuum distortion. It is obvious that, if the virtual displacement takes place in an uncoupled area (air or non-magnetizing and current-free), the virtual energy variation would be zero. More generally, the nature of the force density depends on the nature of this displacement. If one is looking for a local force, the virtual displacement should be applied at a local point. For a global force or torque exerted on a rigid body, the virtual displacement distorts all the surrounding area. In order to ensure the physical space continuity under a virtual point-like or global displacement, ξ_i, an arbitrary function, $\beta(\Omega_0)$, is introduced, which defines the overall local continuous distortion in Equation 4 as a function of a single position displacement:

$$\mathbf{x}(\Omega_0) = \beta(\Omega_0)\xi_i. \qquad (17)$$

It is understood that this elastic arbitrary deformation distribution will add, more or less, to the contribution of the surrounding distorted area and, so, will give some mean value of the local force. But, this area can be infinitely reduced, converging to the force density, as discussed below. Contrary to the point-like magnetic moment approach in [3], here, the statistical smoothing [5] is being already operated within the constitutive law. Now, the Jacobian and its derivatives can be expressed, in terms of β, as follows:

$$\delta\mathbf{G}^{\ 1}_{\mathbf{x}\to 0} = \quad \delta\boldsymbol{\xi}\boldsymbol{\nabla}\beta, \qquad (18)$$

$$\delta|\mathbf{G}|_{\mathbf{x}\to 0} = \boldsymbol{\nabla}\beta.\delta\boldsymbol{\xi}. \qquad (19)$$

The details of these mathematical developments are given in [2,7,16]. Note that this definition involves no approximation by the fact that the local displacement is arbitrary. But, the nature of the searching force depends on the nature of β, as discussed below.

MAGNETIC FORCE AND STRESS TENSOR

Initially, a rigid body free movement is used, just in order to illustrate how the magnetoelastic extended deformable continuum theory may lead to the Maxwell stress tensor, before converging to the local stress tensor within a deformable magnetizable material.

Global Force

Imagine a movable rigid magnetizable body, S, with or without electric currents, moving or stationary and surrounded by other similar bodies.

It is intended to determine the force exerting on this body at a given time. Let field, current, position and velocity, be kept frozen at that precise moment and let S be moved by a virtual arbitrary global displacement, $\boldsymbol{\xi}_s$. This displacement will distort the surrounding field or physical space, here, supposedly uncoupled. Within this uncoupled area, one is free to define how far this deformation might extend.

The moving surface, S, may be the external surface of the body or any surface in the surrounding space. Let surface, S' be any limiting surface, beyond which no distortion may occur, as shown in Figure 3. S' may be chosen everywhere, as far as it does not include any other coupled magnetic material or otherwise, as near as possible to S. The continuous virtual deformation within $\Delta\Omega$ may be defined by a known

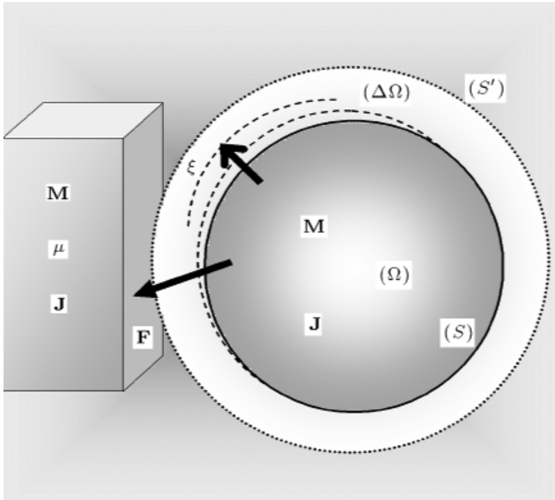

Figure 3. Deforming area around a moving rigid body.

arbitrary function, $\beta(\Omega_0)$, according to Equation 17 and satisfying the following conditions:

$$\beta(X_i) = 1 | i \in (S \text{ and inside volume } \Omega),$$

$$\beta(X_j) = 0 | j \in (S' \text{ and outside } S'). \tag{20}$$

β and $\boldsymbol{\nabla}\beta$ are continuously defined and are non-zero in $(\Delta\Omega)$.

Now, Equation 9 with Equation 8 defines the global force, $\mathbf{F}$, as:

$$\begin{aligned}\mathbf{F}.\delta\boldsymbol{\xi}_s|_{\mathbf{x}=0} &= \int_{\Omega+\Delta\Omega} \beta\mathbf{f}.\delta\boldsymbol{\xi}_s d\Omega \\ &= \int_{\Omega+\Delta\Omega} [\mathbf{H}.\delta\mathbf{B} \\ &\quad + (\frac{1}{2}B^2/\mu_0 \quad \partial_\rho w_{ms})\delta|\mathbf{G}|]d\Omega. \end{aligned} \tag{21}$$

This is the adiabatic virtual work expression of the global force, $\mathbf{F}$. By introducing a known space function, β, the Jacobian derivatives, with respect to $\boldsymbol{\xi}_s$, can be performed in terms of β, according to Equations 18 and 19. The magnetic induction variation, $\delta\mathbf{B}$, given by Equation 8 becomes:

$$\delta\mathbf{B} = [\boldsymbol{\nabla}\beta(\quad \delta\boldsymbol{\xi}.\boldsymbol{\nabla})] \times \mathbf{A} + \boldsymbol{\nabla} \times [(\quad \delta\boldsymbol{\xi}.\mathbf{A})\boldsymbol{\nabla}\beta]. \tag{22}$$

Rearranging and expanding Equation 22 by using $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ and the vector identities $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\beta = 0$ and $\boldsymbol{\nabla}(\delta\boldsymbol{\xi}.\mathbf{A}) = (\delta\boldsymbol{\xi}.\boldsymbol{\nabla})\mathbf{A} + \delta\boldsymbol{\xi} \times (\boldsymbol{\nabla} \times \mathbf{A})$, one can write:

$$\begin{aligned}\delta\mathbf{B} &= \quad (\delta\boldsymbol{\xi} \times \mathbf{B}) \times \boldsymbol{\nabla}\beta \\ &= (\mathbf{B}.\boldsymbol{\nabla}\beta)\delta\boldsymbol{\xi} \quad (\delta\boldsymbol{\xi}.\boldsymbol{\nabla}\beta)\mathbf{B}. \end{aligned} \tag{23}$$

Using this result, together with the Jacobian determinant derivative (Equation 19), and, also, by the fact that $\boldsymbol{\xi}_s$ is arbitrary, the integrands in Equation 21 give the mean force density contribution at any point in $\Omega + \Delta\Omega$, as follows:

$$\begin{aligned}\beta\mathbf{f} = & \quad (\mathbf{B}.\boldsymbol{\nabla}\beta)\mathbf{H} + \frac{1}{2}(\mathbf{H}.\mathbf{B} \quad \mathbf{M}.\mathbf{B})\boldsymbol{\nabla}\beta \\ & + \partial_\rho w_{ms}\boldsymbol{\nabla}\beta. \end{aligned} \tag{24}$$

This mean local contribution may be real if the area is coupled. If one is searching for forces acting on a rigid body, one has $\beta = 1$ everywhere inside the body, implying that $\boldsymbol{\nabla}\beta = 0$ and, so, the force density is identically zero. So, the integration in Equation 21, using Equation 24, is to be carried out only in $\Delta\Omega$, where $\mathbf{M} = 0$ and the last term vanishes:

$$\mathbf{F} = \int_{\Delta\Omega} [\quad (\mathbf{B}.\boldsymbol{\nabla}\beta)\mathbf{H} + \frac{1}{2}\mathbf{H}.\mathbf{B}\boldsymbol{\nabla}\beta]d\Omega. \tag{25}$$

This expression is the first result of the Rafinejad local Jacobian derivative method. In this expression, the usual surface traction integral is replaced by volume integration, including, directly, the field quantities, $\mathbf{B}$ and $\mathbf{H}$. In this form, it is suitable for numerical applications, which has been being widely referenced in [11]. It can be seen that the total force is not the integral of local forces but just a total perceived reaction, giving no other information. Of course, the integrand in Equation 25 may be a non-zero local force, but, it results from a non-material, so, is a non-feasible displacement.

Magnetic Stress Tensor

$\beta\mathbf{f}$, in Equation 24, can be written in terms of a stress tensor, as follows:

$$\beta\mathbf{f} = \quad T.\boldsymbol{\nabla}\beta, \tag{26}$$

where $\mathbf{T}$ is given by:

$$\mathbf{T} = \mathbf{BH} \quad \frac{1}{2}\mathbf{H}.\mathbf{BI}. \tag{27}$$

Taking into account that $\Delta\Omega$ is a free area, $\mathbf{B} = \mu_0\mathbf{H}$, and, so, Equation 27 is the well-known Maxwell tensor, obtained by an energy approach. See the next sections for more details of this tensor within a magnetized material. Using divergence identities, Equation 26 can also be written as:

$$\beta\mathbf{f} = \quad \boldsymbol{\nabla}.(\beta\mathbf{T}) + \beta\boldsymbol{\nabla}.\mathbf{T}. \tag{28}$$

One can now apply Gauss's theorem to write the total force expression (Equation 25), as:

$$\mathbf{F} = \int_{(S)} \mathbf{T}.\mathbf{n}ds + \int_{\Delta\Omega} \beta\boldsymbol{\nabla}.\mathbf{T}d\Omega, \tag{29}$$

where $\mathbf{n}$ is the outward-directed normal vector on the surface, S. Since $\beta = 0$ on S', the surface integral on S' is omitted. For the total force on a rigid body, the second term vanishes, if $\Delta\Omega$ is an empty area where $\boldsymbol{\nabla}.\mathbf{T} = 0$, or, otherwise, when $S' \to S$, so, $\Delta\Omega \to 0$. In these cases, Equation 29 leads to the well-known Maxwell tensor integration over a moving body surface. It results that Equation 25 gives the exact total force, whatever S', provided that the area, $\Delta\Omega$, is magnetizing/current free and whatever β is satisfying Equation 20. Obviously, Equation 25 could be inversely obtained by starting from the right side of Equation 26, using the Maxwell stress tensor and profiting its numerical advantages, as in [18,27].

Force Density

In the authors' theory of magnetic elastic continuum, a local distortion acts in the same way as an elastic body deformation. A magnetic particle moves by contracting and expanding its surrounding area and is not freely surrounded by airgaps [4]. Let the surface, S, in Figure 3, be reduced now toward zero around point i within a compressible magnetized and conducting material, Ω, as shown in Figure 4. S' is again a fixed surface, as small as possible, limiting the local distortion.

The mean force density can be deduced from Equation 24, as follows:

$$\beta\mathbf{f} = \quad (B.\boldsymbol{\nabla}\beta)\mathbf{H} + \frac{1}{2}(\mathbf{H}.\mathbf{B} \quad \mathbf{M}.\mathbf{B})\boldsymbol{\nabla}\beta + \partial_\rho w_{ms}\boldsymbol{\nabla}\beta. \tag{30}$$

Or, in the tensor form :

$$\beta\mathbf{f} = \quad \mathbf{T}'.\boldsymbol{\nabla}\beta,$$

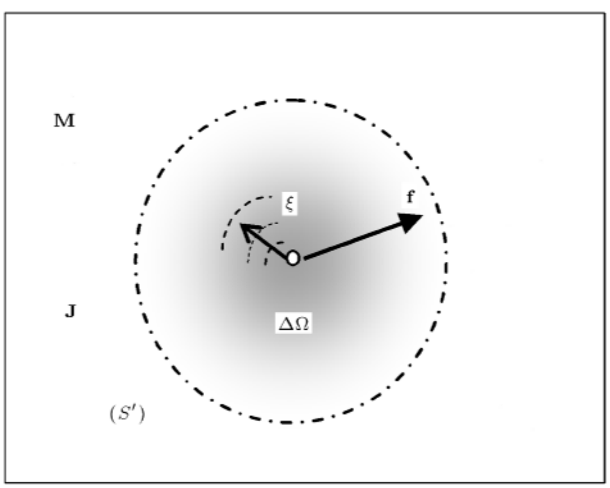

Figure 4. Deforming area around a local displacement.

where:

$$\mathbf{T}' = \mathbf{T}^{\mathbf{M}} + \mathbf{T}^{\mathbf{ms}},$$

$$\mathbf{T}^{\mathbf{M}} = \mathbf{BH} \quad \frac{1}{2}\mu_0\mathbf{H}^2\mathbf{1} + \frac{1}{2}\mu_0\mathbf{M}^2\mathbf{1}, \tag{31}$$

$$\mathbf{T}^{\mathbf{ms}} = \quad \partial_\rho w_{ms}\mathbf{1}.$$

In conducting, but non-magnetizable materials, $\mathbf{T}' = \mathbf{T}$ (Equation 27). Here, β may be considered as an interpolating function, similar to Equation 20:

$$\beta(Xi) = 1,$$

$$\beta(X_j) = 0|j\in(S' \text{ and outside } S'). \tag{32}$$

β and $\boldsymbol{\nabla}\beta$ being continuously defined and non-zero in $(\Delta\Omega)$.

At point i, where $\beta = 1$, the right-hand side of Equation 30 gives the mean force density expression. As mentioned above, this mean force is to be integrated within an infinitesimal volume, $\Delta\Omega$, around point i, which is as small as possible without being strictly a null volume.

When S' approaches point $i(\Delta\Omega \to 0)$, the mean force density (Equation 30) converges to the exact force density at that point where $\beta = 1$:

$$\mathbf{f} = (1/\Omega')\int_{\Omega'}\beta\mathbf{f}d\Omega = \boldsymbol{\nabla}.(\mathbf{T}')$$

$$\text{when } \Delta\Omega \to 0. \tag{33}$$

S' having been reduced to zero, the surface integral in Equation 29 vanishes. Consequently, T' is the overall electromagnetic stress tensor, where $\mathbf{B} = \mu_0(\mathbf{M} + \mathbf{H})$. It results that Equation 30 gives the mean value of the force density converging smoothly to the divergence of a stress tensor, whatever β is satisfying Equation 32.

Substituting $\partial_\rho w_{ms} = \quad \frac{1}{2}(\mathbf{H}.\mathbf{B} \quad \mathbf{M}.\mathbf{B}) + \int_0^{\mathbf{H}}(\mu \quad \partial_\rho\mu)\mathbf{h}.d\mathbf{h}$, which can be deduced from Equations 13 and 16, the total tensor, $\mathbf{T}'$ (Equations 31), gives the tensor obtained by [1,2]. It can be seen that, in the inherent magnetic stress tensor, $\mathbf{T}^{\mathbf{M}}$ (Equation 31), the history dependent term automatically vanishes. This expression is used in [30], referring to [5] and mentioned in [6], based on the Amperian model, which supposes $\mathbf{J} \times \mathbf{B} + (\mathbf{M}.\boldsymbol{\nabla})\mathbf{B} \equiv \boldsymbol{\nabla}.\mathbf{T}^{\mathbf{M}}$. It is comparable to Brown's formula, as presented in [4], where the last term appears in a shape-dependent form, $\int \frac{1}{2}\mu_0\mathbf{M}_n^2\mathbf{n}ds$. Relevant to the permeability approach of the magnetostiction, this term may be called the inherent magnetostriction, implicitly included in $\partial_\rho\mu$.

It should be noted that, without consideration of the magnetostriction effects, resumed by $\partial_\rho\mu$, the force density would be undefined within a magnetizing material, only because the material deformations would be implicitly ignored. In this case, only the Maxwell tensor (Equation 27) can be used in free areas for total forces or within a conducting, $J \neq 0$, but, non-magnetizing, $\mathbf{M} = 0$, material for the force density. In fact, the magnetic sources are included in local energy as media property. So, the energy exchange cannot be calculated without reconsidering the magnetization elastic behavior, as discussed below. To better understand this conclusion, consider a moving rigid structure embodying movable magnetic rigid parts. The Maxwell tensor cannot be used to evaluate the total force on this structure, because the global rigidity condition fails. In this case, the force calculation should be carried out separately for all movable bodies.

MAGNETOSTRICTION

The magnetization potential energy conservation (Equation 14) led to the above-mentioned inherent magnetization force in Equation 31, whereas, strict magnetostriction, as appeared in Equation 12, should rather be due to some additional magnetostriction energy, $\delta w_{ms}(\mathbf{m}', \boldsymbol{\varepsilon})|_m$, already introduced in Equation 15. Of course, this definition resulted from the authors' energy approach, but could be compared to the overall free energy analysis in [3], or to the magnetostriction energy in [4].

Strict Magnetostriction

In the permeability approach [1,2] this energy is implicitly considered within the magnetization stored energy. In Equation 15, the strict variation of the local magnetization, $\mathbf{m}'dm$, was explicitly considered in function of strain $\boldsymbol{\varepsilon}$, under constant $\mathbf{B}$, which can be resumed by some function $\boldsymbol{\kappa}(\boldsymbol{\varepsilon})$, such that $\mathbf{m}'dm.\mathbf{b} = \boldsymbol{\kappa}(\boldsymbol{\varepsilon})\mathbf{m}.\mathbf{b}$. On the other hand, within an infinitesimal virtual displacement, the whole history would not change and one may resume the variation of w_{ms} in Equation 15, as $\delta w_{ms} = \mathbf{M}.\mathbf{B}\delta(\boldsymbol{\kappa})$. Finally, $\boldsymbol{\kappa}$ may be linearized in function $\boldsymbol{\varepsilon}$ by some constant tensor, $\boldsymbol{\alpha}$, such as:

$$\delta_{\mathbf{x}} w_{ms} = \alpha_{xx}\mathbf{M}.\mathbf{B}\delta\varepsilon^{xx},$$

or, using the Jacobian method:

$$\delta_{\mathbf{x}} w_{ms} = \alpha_{xx}\mathbf{M}.\mathbf{B}\partial_x\beta\delta\xi_{\mathbf{x}}, \tag{34}$$

where ε^{xx} is the strain in the $x(1,2,3)$ direction and α_{xx} is a coefficient generally, depending on direction x. In a general case, three couples of α_{xx} should be measured, according to the anisotropy axis. α is, meanwhile, a function of the material saturation state, so, a function of M^2.

In particular cases of isotropic materials, α_{xx} may be defined, according to the directions of $\mathbf{B}$ and its transverse, α_b and α_q, respectively. Hence, the above expression may be written as:

$$\delta w_{ms} = (\alpha_b \quad \alpha_q)(\mathbf{M}.\boldsymbol{\nabla}\beta)(\mathbf{B}.\delta\boldsymbol{\xi}) + \mathbf{M}.\mathbf{B}\alpha_q(\boldsymbol{\nabla}\beta.\delta\boldsymbol{\xi}). \tag{35}$$

This gives the mean local strict magnetostrictive force and the overall magnetic tensor stress (Equations 31) become:

$$\mathbf{T}' = \mathbf{BH} \quad \frac{1}{2}\mathbf{H}.\mathbf{B1} + (\alpha_b \quad \alpha_q)\mathbf{MB} + (\alpha_q + \frac{1}{2})\mathbf{M}.\mathbf{B1}. \tag{36}$$

In cases, of highly magnetizable materials, the total stress tensor is very close to:

$$\mathbf{T}' = \mu_0(\alpha_b \quad \alpha_q)\mathbf{MM} + \mu_0(\alpha_q + \frac{1}{2})\mathbf{M}^2. \tag{37}$$

Vandevelde et al. [24], using [4], give the same expression, in cases of homogeneous soft iron rings and employ polynomial expressions for the coefficients, in terms of M^2. Delaere et al. [25] use the thermal stress analogy to define the overall magnetostriction stress, that is, the free strain in a function of the applied magnetic field. These experimental results confirm the existence of a magnetostriction stress of a B^2 order. Mohammed et al. [31] employ, directly, the reluctivity method for $\delta\mathbf{H}|_{\mathbf{B}}$ in Equation 9, but as a function of stress. However, these inherent and strict magnetostriction effects require further experimental investigations by reciprocal measuring of the external stress effects.

COENERGY AND ENERGY EQUIVALENCE

The force is identically equal to the coenergy derivative, when mmf is held constant. In the local virtual work, mmf should be replaced by $\mathbf{H}$. On the basis of the same energy approach (Equation 3), it is shown that the coenergy virtual variation responds to the commonly admitted identity:

$$\delta(wd\Omega)|_{\mathbf{A}} + \delta(w'd\Omega)|_{\mathbf{H}} = 0, \tag{38}$$

where $w + w' = \mathbf{H}.\mathbf{B}$ or $w' = \int_0^{\mathbf{H}} \mathbf{b}.d\mathbf{h}$. It means that, whatever the overall system configuration, one may apply the energy or coenergy derivative, provided that one holds constant, correctly, the state field vector, $\mathbf{A}$ or $\mathbf{H}$, according to Equation 6.

By substituting Equation 38 in Equation 3, the force coenergy expression can be written, as follows:

$$\int_{\Omega 0} [\delta(w'|\mathbf{G}|)|_{\mathbf{H}} \quad \mathbf{f}.\delta\mathbf{x}]d\Omega_0 = 0. \tag{39}$$

And, as in Equation 9, the coenergy, w', variation becomes :

$$\delta(w'|\mathbf{G}|) = (\mathbf{B}.\delta\mathbf{H} + w'\delta|\mathbf{G}| + \int_0^{\mathbf{H}} \delta\mathbf{b}.d\mathbf{h}). \qquad (40)$$

The third term introduces the magnetization variations, which can be expressed alternatively in terms of the magnetic induction using Identity 10. The **H** invariance is defined exactly as for **A** through its three components. This could also be resulted from the magnetic scalar potential [2] when possible. Using this field intensity invariance condition, according to Equation 6, one can write:

$$\delta(w'|\mathbf{G}|)=\mathbf{B}.(\delta\mathbf{G}\quad{}^{1}\mathbf{H})+\int_0^{\mathbf{H}}(\mathbf{b}\quad\mu_0\rho\partial_\rho\mathbf{m}|_{\mathbf{h}}).d\mathbf{h}\delta|\mathbf{G}|. \qquad (41)$$

This is the most useful local force expression, particularly for teaching purposes. Notice that the energy and coenergy definitions are distinctly defined only in integral forms. Otherwise, the derivation of $(\frac{1}{2}\mathbf{H}.\mathbf{B})$ may unexpectedly lead to zero, as in [23]. Using the Jacobian derivatives (Equations 18 and 19), Equation 41 is written as:

$$\delta(w'd\Omega) = [\quad(\mathbf{B}.\boldsymbol{\nabla}\beta)\mathbf{H} + w'_m\boldsymbol{\nabla}\beta].\delta\boldsymbol{\xi}d\Omega, \qquad (42)$$

where by using Equation 10 $w'_m = \mathbf{H}.\mathbf{B} \quad \int_0^{\mathbf{B}}(\mathbf{h} + \rho\partial_\rho\mathbf{m}|_{\mathbf{b}}).d\mathbf{b}$. Equation 42, together with the w_{ms} definition in Equations 14 and 15, leads exactly to the right-hand side of Equation 30, obtained by the energy derivative. It is concluded that Identity 38 was justified and that the energy and coenergy variational formulations are equivalent in any case, but, with different invariance conditions. The field quantities to be held constant for the virtual work are, respectively, the magnetic potential vector, **A**, or the field intensity, **H**. In special cases of the edge finite element method, these vectors are approximated by their circulation along element edges. These finite scalar variables are automatically held constant, thanks to the double Jacobian transformations, as described in Appendix.

However, notice that this mathematical alternative equivalent field invariance should not be extrapolated to the deformable field theory and energy concept developed in this paper. Obviously, $\delta(\mathbf{m}'dm.\mu_0 d\mathbf{h})|_{\mathbf{h}} = 0$ is not equivalent to $\delta(\mathbf{m}'dm.d\mathbf{b})|_{\mathbf{b}} = 0$ (Equation 13), or, $\rho\partial_\rho\mathbf{m}|_{\mathbf{b}} = \mathbf{m} + \rho^2\partial_\rho\mathbf{m}'|_{\mathbf{b}}$ (Equation 11) is not equivalent to $\rho\partial_\rho\mathbf{m}|_{\mathbf{h}} = \mathbf{m} + \rho^2\partial_\rho\mathbf{m}'|_{\mathbf{h}}$. They are as different as $\mu_0\mathbf{H}$ and **B**, regarding the exciting or external field discussed in the introduction. In this deformable field theory and magnetostrictive analysis, the magnetization potential energy appeared as $\frac{1}{2}\mathbf{M}.\mathbf{B}d\Omega$. This confirms that the total magnetic field intensity in a fictitious atomic free space would rather be $\mathbf{H_t} = \mathbf{B}/\mu_0$. Suppose that one intends to remove an infinitesimal local magnetic material, $\mathbf{M}d\Omega$, keeping the vector potential constant. The local magnetic energy will change from $\frac{1}{2}\mathbf{H}.\mathbf{B}d\Omega$ to $\frac{1}{2}\mathbf{B}.\mathbf{B}/\mu_0 d\Omega$, so, the removing work is $\frac{1}{2}\mathbf{M}.\mathbf{B}d\Omega$. Now, if one supposes that the magnetic field, **H**, is constant, for example, with the magnetic scalar potential invariant, the magnetic energy will be reduced to $\frac{1}{2}\mu_0\mathbf{H}.\mathbf{H}d\Omega$, which is not compatible with the system equilibrium condition. This is compatible to Brown's method [3], considering the total field as $B/\mu_0 = H_0 + M + H_1$. This is also the conclusion of [5].

CONCLUSIONS

Two well-known classical concepts: Electromagnetic stored energy and virtual work, are employed to deal with one electromechanical local force in a general case. It is shown how the deformable field theory on the basis of Maxwell's equations, would overcome the mathematical complexity of this basic force definition. This theory led to a novel energy variation, in terms of the deformation gradient (Jacobian of transformation). This first theoretical result is already suitable for strong coupling approaches. Then, introducing an arbitrary deformation distribution function, this result was analytically developed, leading to the overall stress tensor expressions. The Maxwell tensor is the first result.

The deformable field theory can be resumed by the electromagnetic field and the active material considered as two deformable (elastic) continua, solidly linked point-to-point together. In cases of magnetoelasticity, the free currents, excited or induced, are the external sources. But, the magnetization, permanent or excited, is implicitly present through the stored field energy. To each point of this electromagnetic continuum is solidly associated an electric scalar and a magnetic vector. Moreover, within an adiabatic virtual displacement, the three components of the vector potential are reference-independent, excluding energy exchange, via the free currents.

In this theory, the electromagnetic deformable continuum extends even over free areas and there is no vacuum. By this novel concept, a moving rigid body deforms the surrounding uncoupled area and a local deformation is due to an infinitely small particle displacement. The magnetic flux density and the magnetic energy are to be evaluated in this extended elastic field continuum. Using the local reference transformation, these changing quantities were written in terms of the Jacobian derivatives. That is the way the independent state variables were separated within the local energy partial derivatives. Hence, the magnetic field variation appeared in terms of the deformation gradient, similar to other elastic movements, whereas, in the other approaches, the field intensity gradient, $\partial_x \mathbf{H}$, appears, involving singularity at the boundaries [3].

It is shown that the surface integral of the Maxwell tensor is not the integration of local forces but gives, in cases of strictly rigid bodies only, the total force. In this case, no meaning should be expected for surface traction densities, nor for the way this total force is exerted on the magnetizing body.

This approach avoids indeterminations [5,6] or, at least, the complexity [3,5] encountered in matter on matter interaction approaches. The presented theory and methodologies may be applied identically to electro-elastic or eletromagneto-elastic.

It is shown that $\mathbf{H}$ could be identically considered as the magnetic position invariant quantity, even within the conducting areas. However, this energy/coenergy equivalence is, rather, a mathematical variable change. Within a magnetizable material, only $\mathbf{B}$ appears as the fundamental magnetic field quantity, through the potential energy of magnetic moments.

The magnetization inherent behavior appears while searching for the magnetization potential energy conservation, whereas, the strict magnetostriction is related to the additional specific energy to be defined experimentally. However, the aim of this paper was to put forward the theoretical basis and mathematical tools, in order to orient future investigations in this area.

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APPENDIX

The following notations are used:

$\partial_{\mathbf{x}} \equiv \partial/\partial\mathbf{x}$ derivative operator,

$\delta..|_{\mathbf{b}} \equiv$ variation with $\mathbf{b}$ invariant,

$\mathbf{C} \equiv$ vector (C_1, C_2, C_3),

$\boldsymbol{\nabla} \equiv$ nabla $(\partial_{\mathbf{x}1}, \partial_{\mathbf{x}2}, \partial_{\mathbf{x}3})$,

$\mathbf{DC} \equiv$ tensor $C_m D_n$,

$\mathbf{DC}.\boldsymbol{\xi} = (\mathbf{D}.\boldsymbol{\xi})\mathbf{C}$,

$$\mathbf{1} \equiv \boldsymbol{\delta}_{mn} \text{ (Kronecker delta)} = \text{unity tensor}. \tag{A1}$$

Invariance Condition of the Magnetic Vector Potential

A material dependent vector, such as a distance $d\mathbf{l}$, changes to $d\mathbf{l}'$ in the fixed reference, as:

$$d\mathbf{l}'_i\mathbf{u}_i = d\mathbf{l}_i\mathbf{u}'_i, \qquad i = (1, 2, 3), \tag{A2}$$

where $\mathbf{u}$ and $\mathbf{u}'$ are the local unit vectors of fixed and material references, defined as $\mathbf{u}' = \mathbf{G}\mathbf{u}$. So, Equation A2 can be written as:

$$d\mathbf{l}' = \mathbf{G}d\mathbf{l}. \tag{A3}$$

For a reference independent vector, the absolute measures of the components remain constant before and after deformations:

$$\mathbf{A'}_{\mathbf{i}}\mathbf{u'}_i = \mathbf{A}_i\mathbf{u}_i, \qquad i = (1, 2, 3). \tag{A4}$$

Hence, the material perception, $\mathbf{A}'$, can be written in the fixed common reference as $\mathbf{A}' = \mathbf{G}\ ^{1}\mathbf{A}$ (Equation 6). In case of the magnetic vector potential, this invariance definition ensures the adiabatic condition. In fact, the energy exchange of an elementary current loop, i, (Figure 2) using Stock's inverse theorem, can be written as:

$$i\delta\Phi = i\int_S \mathbf{B}.\mathbf{n}ds = \mathbf{J}.da\int_c \mathbf{A}.d\mathbf{c}. \tag{A5}$$

The circulation, $\mathbf{A}.d\mathbf{c}$, is automatically held constant, because of the double Jacobian transformations in the Equation A2 and Equation 6:

$$\mathbf{A}.d\mathbf{c}|_{\Omega o} \equiv \mathbf{A}'.d\mathbf{c}'|_{\Omega} \equiv (\mathbf{G}\ ^{1}\mathbf{A})^{\mathbf{T}}.(\mathbf{G}^{\mathbf{T}}d\mathbf{c})|_{\Omega o}. \tag{A6}$$

In the time-frozen virtual work method, the current, i, defined by a given number of electrons and their velocity, does not change during a virtual displacement. Consequently, the magnetic flux, $\int_s \mathbf{B}.\mathbf{n}ds$, also holds constant. Obviously, one could start with this condition and deduce the vector invariance condition (Equation 6), but then, how could it be generalized to current-free areas? Anyhow, historically, it was defined in this order for the first time in [2].

Furthermore, this vector invariance condition (Equation 6) also applies to the magnetic field intensity in the case of a coenergy approach, leading to $\mathbf{H}' = \mathbf{G}\ ^{1}\mathbf{H}$. Consequently, the circulation, $\int_c \mathbf{H}.d\mathbf{c}$, known as the magnetomotrice force, (mmf), is automatically held constant, according to Equation A6. As for $\mathbf{A}$, one could start with this condition. Historically, Rafinejad [2] deduced this condition from $(\mathbf{H} = \ \nabla\Psi)$, using Equation 5, where Ψ is the magnetic scalar potential. One can conclude that the vector invariance, according to Equation 6, has the same nature as the nabla ∇ transformation (Equation 5). From this point of view, the magnetic induction, $\mathbf{B}$, according to Equation 7, is also held constant by both nabla $\boldsymbol{\nabla}$ and $\mathbf{A}$. These three Correlations 6, 7 and A6 are a perfect demonstration of the uniqueness of the magnetic field invariance condition in variational methods.

In fact, the magnetic flux linkage or magnetomotrice force has no local definition other than the vector potential $\mathbf{A}$ or the field intensity $\mathbf{H}$. Of course, in 2-D cases, the **z** component, $\mathbf{A}_z$, could be considered identically as the magnetic flux [31]. But, in general cases, even in the cases of the edge finite elements, the actual magnetic variable is either **H** [20] or **A** [15], which, for some other reasons, is approximated by scalar finite variables, $a* = \mathbf{A}.\Delta\mathbf{l}$ or $h* = \mathbf{H}.\Delta\mathbf{l}$, equivalent to the field circulation over a Finite Element edge. Otherwise, Equation 6 applies in the same way [16]. In this case, the vector direction of **A** is represented by the gradient of a space function. Amazingly, Kameria [16] concluded that the nodal vector potential finite element could not be used for applying the virtual work principle. This could be explained by the fact that the authors could not make a relation between gradient transformation and vector invariance ($\mathbf{G}\ {}^{1}\mathbf{A} \rightarrow \mathbf{G}\ {}^{1}\boldsymbol{\nabla}\alpha a*$). One does not necessarily need $\boldsymbol{\nabla}\alpha a*$ to write $\mathbf{G}\ {}^{1}\mathbf{A}$. This confirms that the mathematical (p-form) analysis given in [19] did not clarify the invariance mechanism of **A** or **H**.