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Research Note

A Fuzzy Efficient Frontier Method for Resource Allocation with Different Time Cycles

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The primary assumption, in many resource allocation problems, is that every asset has a unique length of return. However, this simple assumption may create some chaos when different investment alternatives may return in various time cycles and resources cannot be allocated at any given time. This paper presents a new extended efficient frontier problem. The new method assumes that all risky assets have different time cycles for their returns. The primary assumption is that the return for each asset is fuzzy in nature. The problem is solved and the results discussed, with some numerical examples.

INTRODUCTION

During the past few years, many different models have been proposed to solve investment problems, such as, efficient frontier, cost/benefit analysis and etc. It is believed that Markovitz [1-3] was the first person to propose a method for portfolio selection. In his implementation, he uses a modeling formulation, which is a trade-off between risk and reward, with consideration of an arbitrary ratio. However, the method suffers from a non-trivial assumption that is a constant length for all risky assets. In other words, the Markovitz model assumes that each asset has a unique time cycle, which is unrealistic for many real world problems. For example, consider a food industry chain, which supplies dairy products on a weekly schedule and the demand for such products is subject to many parameters, such as season or temperature. In this case, the suppliers may wish to know how much money must be invested for each kind of dairy product during each time cycle. The case becomes more interesting when one is interested in, simultaneously, determining the optimal asset allocation for a relatively large number of products with different time cycles. The proposed method of this paper develops the concept of investment type allocation, which has already been studied by many others. The Markovitz model is a minimization of the risk-reward problem, which ends up as a convex quadratic problem, where the quadratic term comes from a covariance matrix. The resulted problem is subject to some linear constraints, such as budget. Yong and Yamazaki [4-6] propose another method, which uses absolute deviation to measure the risk for sampling the Markovitz classic model, and substitute a minimum mean absolute deviation function with an objective one in the Markovitz model. In their modeling formulation, compared with the Markovitz model, it is not necessary to calculate the covariance matrix, which may be considered an advantage. Hanafizaseh and Seifi [7] use a semi-definite modeling formulation to study the risk-reward portfolio selection. They discuss different kinds of norm, along with the concept of robust optimization. However, all discussed models have a non-trivial assumption, where all risky assets have a unique time cycle. Sadjadi and Orugee [8] developed a method for investment problems where different assets are studied with different time cycles. They assume the return for each asset is independent and has normal distribution with a known mean and variance. The objective function is the minimization of risk and simultaneously maximizes the return for each asset. The resulted model is a quadratic convex optimization with linear constraints. In this paper, this model is extended in more realistic forms, where the parameters are assumed to be fuzzy in nature. This paper is organized as follows. First, the problem formulation is discussed when all parameters are known and certain. Then, the present modeling formulation is discussed in fuzzy form. The fuzzy models are considered, using two different triangular and trapezoid fuzzy numbers. After

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that, the implementation of the proposed method is demonstrated using some numerical examples. Finally, the concluding remarks are given to summarize the contributions of this paper.

PROBLEM STATEMENT

Suppose one is interested in allocating budget p in n differently independent alternatives. The time horizon consists of n equal periods and the investment alternatives have different time cycles. For the sake of simplicity, one assumes the ith alternative needs i period(s) for its return, where $i = 1, \cdots n$. Let x_{ij} be the amount of budget invested on alternative i in period j. Suppose P to be the amount of investment one has at the beginning of the first period, therefore, one has:

$$\sum_{i=1}^{n} x_{i0} = P. \tag{1}$$

Equation 1 is called budget constraint and plays an important rule in the proposed model. There are other constraints involved in the modeling formulation, called cash flow constraints. To understand more about this type of equation suppose, at the end of the first period, the first investment pays off the return, μ, plus the original investment and one may invest for the next $n \quad 1$ periods,

$$\sum_{i=1}^{n \quad 1} x_{i1} = x_{10} + x_{10}\mu_1. \tag{2}$$

From the beginning of the second period till the beginning of period $n \quad 1$ of the planning horizon, the following hold,

$$\sum_{i=1}^{n \quad 2} x_{i2} = x_{20}(1+\mu_2) + x_{11}(1+\mu_1),$$

$$\vdots$$

$$\sum_{i=1}^{2} x_{in \quad 2} = \sum_{i=1}^{n \quad 2} (1+\mu_i) x_{i,n \quad 2 \quad i},$$

$$x_{1n \quad 1} = \sum_{i=1}^{n \quad 1} (1+\mu_i) x_{i,n \quad 1 \quad i}. \tag{3}$$

Let μ_p be the expected return of the proposed model, therefore, the return of the investment strategy is maximized at the end of the planning horizon, as follows:

$$\max \qquad \mu_p = \sum_{i=1}^{n} x_{in \quad i}(1+\mu_i). \tag{4}$$

In the next section, the model is studied when the return is fuzzy in nature.

FUZZY MODEL

Suppose $\widetilde{\mu}_i$ is fuzzy in nature. In other words, an expert explains the behavior of the return as a triangular fuzzy number, $(\mu_i^1, \mu_i^m, \mu_i^2)$. As mentioned in Equation 4, the objective function for the frontier model is the sum of $X_{i,n \quad i}(1+\mu_i)$, for every individual item $X_{i,n \quad i}(1+\mu_i)$. The objective function includes three segments. The left part of the objective function, $(1+\mu_i^m)x_{i,n \quad i}$ $(1+\mu_i^1)x_{i,n \quad i}$ must be minimized, the center part, $(1+\mu_i^m)x_{i,n \quad i}$, is maximized and the right segment, $(1+\mu_i^2)x_{i,n \quad i} \quad (1+\mu_i^m)x_{i,n \quad i}$, is maximized. Therefore, these individuals are summed in every segment and one obtains:

$$\min z_1 = (\mu_1^m \quad \mu_1^1)x_{1,n \quad 1} + (\mu_2^m \quad \mu_2^m)x_{2,n \quad 2} + \cdots + (\mu_n^m \quad \mu_n^1)x_{n,0}, \tag{5}$$

$$\max z_2 = \mu_1^m x_{1,n \quad 1} + \mu_2^m x_{2,n \quad 2} + \cdots + \mu_n^m x_{n,0}, \tag{6}$$

$$\max z_3 = (\mu_1^2 \quad \mu_1^m)x_{1,n \quad 1} + (\mu_2^2 \quad \mu_2^m)x_{2,n \quad 2} + \cdots + (\mu_n^2 \quad \mu_n^m)x_{n,0}. \tag{7}$$

In order to estimate the fuzzy coefficient associated with constraints, the following relationship is used [9,10] (suppose, for constraints in a normal state, one has $A \le b$)

$$\frac{\overline{A} + 4A^0 + A^+}{6} \le b,$$

and:

$$u_i = \frac{\mu_i^1 + 4\mu_i^m + \mu_i^2}{6}. \tag{8}$$

Therefore, the cash flow constraints are summarized, as follows,

$$\sum_{i=1}^{n \quad 2} x_{i2} = x_{20}(1+u_2) + x_{11}(1+u_1),$$

$$\vdots$$

$$\sum_{i=1}^{2} x_{in \quad 2} = \sum_{i=1}^{n \quad 2} (1+u_i) x_{i,n \quad 2 \quad i}$$

$$x_{1n \quad 1} = \sum_{i=1}^{n \quad 1} (1+u_i) x_{i,n \quad 1 \quad i}. \tag{9}$$

In order to find an efficient solution for the crisp model, one needs to solve six sub-problems. Let z_k^+ and z_k be the positive and negative objective functions with

$k = 1, 2, 3$, associated with Equations 5 to 7. Therefore, one must solve the following problems,

$$z_1^+ = \min z_1, \quad \text{S.T. Cons.},$$

$$z_2^+ = \max z_2, \quad \text{S.T. Cons.},$$

$$z_3^+ = \max z_3, \quad \text{S.T. Cons.},$$

$$z_1 \;\; = \max z_1, \quad \text{S.T. Cons.},$$

$$z_2 \;\; = \min z_2, \quad \text{S.T. Cons.},$$

$$z_3 \;\; = \min z_3, \quad \text{S.T. Cons.} \tag{10}$$

One now solves each of the six objective functions in Equations 10, subject to the budget and cash flow constraints. The next step is to define the membership functions for each objective function.

Let λ_1, λ_2 and λ_3 be the membership functions associated with z_1 to z_3 (Fuzzy objective functions), respectively. Therefore, one has [9,10]:

$$\lambda_{z1} = \begin{cases} 1 & z_1 < z_1^+ \\ \frac{z_1 \quad z_1}{z_1 \quad z_1^+} & z_1^+ \leq z_1 \leq z_1 \end{cases}$$

$$\lambda_{z2} = \begin{cases} 1 & z_2 > z_2^+ \\ \frac{z_2 \quad z_2}{z_2^+ \quad z_2} & z_2^+ \leq z_2 \leq z_2 \end{cases}$$

$$\lambda_{z3} = \begin{cases} 1 & z_3 > z_3^+ \\ \frac{z_3 \quad z_3}{z_3^+ \quad z_3} & z_3^+ \leq z_3 \leq z_3 \end{cases} \tag{11}$$

z_1^+, z_1 , z_2^+, z_2 , z_3^+, z_3 are the optimum amount of the six programs in Equations 10. Finally, the objective function of the proposed model is summarized, as follows,

$$\max \{\min \quad \lambda_{zi}(x)\}, \quad i = 1, 2, 3.$$

This is subject to cash flow and budget constraints. The fuzzy model can be easily considered with other fuzzy numbers, such as a trapezoid. Since the modeling process of a trapezoid model is similar to a fuzzy model, only the modeling formulation and the results for the implementation of the numbers are reported in an example.

TRAPEZOID FUZZY NUMBERS

Suppose μ_i is fuzzy in nature. In other words, an expert explains the behavior of the return as a triangular fuzzy number, $(\mu_i^1,\ \mu_i^2,\ \mu_i^3,\ \mu_i^4)$. In this case, the objective function includes four segments, as follows:

$$\min z_1 = (\mu_1^2 \quad \mu_1^1)x_{1,n \quad 1} + (\mu_2^2 \quad \mu_2^1)x_{2,n \quad 2}$$
$$+ \cdots + (\mu_n^2 \quad \mu_n^1)x_{n,0},$$

$$\max z_2 = (\mu_1^3 \quad \mu_1^2)x_{1,n \quad 1} + (\mu_2^3 \quad \mu_2^2)x_{2,n \quad 2}$$
$$+ \cdots + (\mu_n^3 \quad a_n^2)x_{n,0},$$

$$\max z_3 = \mu_1^3 x_{1,n \quad 1} + \mu_2^3 x_{2,n \quad 2} + \cdots + \mu_n^3 x_{n,0},$$

$$\max z_4 = (\mu_1^4 \quad \mu_1^3)x_{1,n \quad 1} + (\mu_2^4 \quad \mu_2^3)x_{2,n \quad 2}$$
$$+ \cdots + (\mu_n^4 \quad a_n^3)x_{n,0}, \qquad \text{S.T.},$$

$$\frac{A_1 + 2(A_2 + A_3) + A_4}{6} X \leq b. \tag{12}$$

NUMERICAL EXAMPLE

In this section, an example is provided to present the implementation of the proposed models with triangular and trapezoid fuzzy numbers.

For a triangular fuzzy number, consider the following information:

$$n = 4, \quad P = 1000, \quad \alpha = 0.05, \quad i = 1, 2, 3, 4$$

$$\tilde{\mu}_1 = (0.1, 0.12, 0.13), \quad \tilde{\mu}_2 = (0.22, 0.24, 0.29),$$

$$\tilde{\mu}_3 = (0.35, 0.42, 0.44), \quad \tilde{\mu}_4 = (0.53, 0.55, 0.6),$$

and α is return.

A triangular fuzzy problem formulation is applied in the previous section to find an optimum solution to this example, which is, as follows:

$$\min Z_1 = 0.02x_{1,3} + 0.02x_{2,2} + 0.06x_{3,1} + 0.02x_{4,0},$$

$$\max Z_2 = 1.12x_{1,3} + 1.24x_{2,2} + 1.41x_{3,1} + 1.55x_{4,0},$$

$$\max Z_3 = 0.01x_{1,3} + 0.05x_{2,2} + 0.03x_{3,1} + 0.05x_{4,0}. \tag{13}$$

subject to:

$$x_{1,0} + x_{2,0} + x_{3,0} + x_{4,0} = 1000,$$

$$1.118x_{1,0} \quad x_{1,1} \quad x_{2,1} \quad x_{3,1} = 0,$$

$$1.118x_{1,1} + 1.245x_{2,0} \quad x_{1,2} \quad x_{2,2} = 0,$$

$$1.118x_{1,2} + 1.245x_{2,1} + 1.405x_{3,0} \quad x_{1,3} = 0.$$

The optimal solutions for z_i^+, z_i , $i = 1,2,3$ are summarized as the following:

$$Z_1^+ = \min Z_1 = 20, \qquad Z_1 = \max Z_1 = 67.08,$$

$$Z_2^+ = \max Z_2 = 1576.38, \quad Z_2 = \min Z_2 = 1543.8,$$

$$Z_3^+ = \max Z_3 = 62.5, \qquad Z_3 = \max Z_3 = 13.92.$$

Therefore, the final model is formulated, as follows:

$$\max \lambda$$

S.T.

$$0.000425x_{1,3} + 0.00042x_{2,2} + 0.00127x_{3,1} + 0.000425x_{4,0} + \lambda \leq 1.425,$$

$$0.034x_{1,3} + 0.038x_{2,2} + 0.043x_{3,1} + 0.0476x_{4,0} \quad \lambda \geq 47.38,$$

$$0.000206x_{1,3} + 0.00103x_{2,2} + 0.00062x_{3,1} + 0.00103x_{4,0} \quad \lambda \geq 0.29,$$

$$x_{1,0} + x_{2,0} + x_{3,0} + x_{4,0} = 1000,$$

$$1.118x_{1,0} \quad x_{1,1} \quad x_{2,1} \quad x_{3,1} = 0,$$

$$1.118x_{1,1} + 1.245x_{2,0} \quad x_{1,2} \quad x_{2,2} = 0,$$

$$1.118x_{1,2} + 1.245x_{2,1} + 1.405x_{3,0} \quad x_{1,3} = 0,$$

$$x_{ij} >= 0. \tag{14}$$

Table 1 summarizes the results of the implementation of the proposed fuzzy model using two different fuzzy numbers, triangular and trapezoid. For a trapezoid example, μ_1 to μ_4 are set, as follows:

$$\tilde{\mu}_1 = (0.1, 0.11, 0.12, 0.14),$$

$$\tilde{\mu}_2 = (0.24, 0.25, 0.255, 0.26),$$

$$\tilde{\mu}_3 = (0.33, 0.345, 0.355, 0.37),$$

$$\tilde{\mu}_4 = (0.51, 0.516, 0.52, 0.54).$$

Therefore:

$$\min z_1 = 0.01x_{1,3} + 0.01x_{2,2} + 0.015x_{3,1} + 0.006x_{4,0},$$

$$\max z_2 = 0.01x_{1,3} + 0.005x_{2,2} + 0.01x_{3,1} + 0.004x_{4,0},$$

$$\max z_3 = 1.12x_{1,3} + 1.255x_{2,2} + 1.355x_{3,1} + 1.52x_{4,0},$$

$$\max z_4 = 0.02x_{1,3} + 0.005x_{2,2} + 0.015x_{3,1} + 0.02x_{4,0}. \tag{15}$$

Table 1. Combined report for fuzzy triangular model.

Decision Variable	Solution Value	
	Triangular	**Trapezoid**
λ (objective)	0.4791	0.5750
$X_{1,0}$	502.1632	422.94
$X_{2,0}$	0	0
$X_{3,0}$	123.90	161.34
$X_{4,0}$	373.93	415.71
$X_{1,1}$	0	0
$X_{2,1}$	0	472
$X_{3,1}$	561.41	0
$X_{1,2}$	0	0
$X_{2,2}$	0	0
$X_{1,3}$	174.08	808.76

ST:

$$x_{1,0} + x_{2,0} + x_{3,0} + x_{4,0} = 1000,$$

$$0.116x_{1,0} \quad x_{1,1} \quad x_{2,1} \quad x_{3,1} = 0,$$

$$0.116x_{1,1} + 0.252x_{2,0} \quad x_{1,2} \quad x_{2,2} = 0,$$

$$0.116x_{1,2} + 0.252x_{2,1} + 0.35x_{3,0} \quad x_{1,3} = 0,$$

$$x_{ij} >= 0.$$

CONCLUSION

A new model is presented for an extended resource allocation problem, where each alternative has a different life cycle and the parameters are assumed to be fuzzy in nature. The fuzzy models are considered using two different triangular and trapezoid fuzzy numbers. Fuzzy linear optimization has been used to solve the resulted fuzzy models. The implementation of the proposed model is demonstrated using some numerical examples and the results have been discussed. The modeling of the proposed method can be easily extended to other continuous fuzzy numbers, such as Gaussian, etc. Another extension of this model is to choose the cash flow constraints as a chance constraint optimization. Such models can be formulated as robust optimization and this could lead to work on recent advances in semi-definite programming. These topics are left as open research areas for other interested researchers.

REFERENCES

1. Jones, P.C., *Portfolio Management*, Translator: Shahalizadeh, Tehran University (2001).
2. Markowitz, H. "Portfolio selection", *Journal of Finance*, **7**, pp 77-9 (1952).

3. Papahristodoalou, C. and Dotzauer, E. "'Optimal portfolios using linear programming models", *Journal of the Operational Research Society*, **55**, pp 1169-1177 (2004).

4. Cai, X., Teo, K.L., Yang, X.Q. and Zhou, X.Y. "Minimax portfolio optimization: An empirical numerical study", *Operational Research Society*, **55**, pp 65-72 (2004).

5. Konno, H. and Yamazaki, H. "Mean-absolute deviation portfolio optimization models and its application to Tokyo stock market", *Management Science*, **37**, pp 519-531 (1991).

6. Young, M.R. "A minimax portfolio selection rule with linear programming solution", *Mngt. Sci.*, **44**, pp 673-683 (1998).

7. Hanafizadeh, P. and Seifi, A. "A unified model for robust optimization of linear programs with uncertain parameters", *Operational Research*, **16**, pp 25-45 (2004).

8. Sadjadi, S.J. and Orougee, M., *An Efficient Frontier Method for Resource Allocation with Different Time Cycles*, Technology and Science University (2005).

9. Thole, V., Zimmermann, H.J. and Zysno, P. "On the suitability of minimum and product operators for the intersection of fuzzy sets", *FSS*, **2**, pp 167-180 (1979).

10. Zimmermann, H.J., *Fuzzy Sets, Decision Making, and Expert System*, Kluwer Acadamic Publishers, Boston (1987).