Scientia Iranica, Vol. 6, No. 1, pp 43–50

SCIENTIA
IRANICA

Research Note

A New Transformation Technique for Accurate Estimation of Partly-Occluded Ellipse Parameters

A. Shamaie[1], S.M. Fakhraie* and B. Benhabib[2]

One of the main problems in the area of recognition and accurate pose-estimation of objects is estimation of the five basic parameters of elliptical shapes, present in an object, based on a set of edge-point data collected from occluded images. In this paper, a new error function, a numerically-efficient transformation and a new analytical algorithm for achieving the desired prediction accuracy are introduced. Some experimental results are also provided.

INTRODUCTION

Flexibility of a robotic work-cell can significantly be enhanced via the utilization of visual sensors within the process. Much research work has been reported regarding the process of sensing the position and orientation (pose) of objects.

Identification of work-pieces or sub-assemblies, determination of the pose and estimation of the salient features are the major applications of object recognition in industrial environments [1].

Multiview feature representation based on either the characteristic views of an object or the discrete view-sphere representation as well as volumetric methods based on the exact specification of an object have been developed [2]. However, due to the need for a 3D matching process, these methods are complex and computationally expensive [1].

Furthermore, several mathematical algorithms based on the use of markers with circular geometry, for extraction of 3D coordinates from the markers' images, have already been developed [3-5]. In addition, some error functions and their geometrical interpretations have been proposed [1,6,7].

In this paper, some error functions and their geometrical interpretations are formulated. A new error function, based on a novel geometrical interpretation of an existing error function is defined for the estimation of five basic ellipse parameters. Then, a new transformation based on the distance of ellipse edge points to the center of coordinates is proposed. Moreover, an analytical algorithm is developed through which the previously defined error functions and the transformation proposed here are combined. Finally, the results of the simulated experiments are presented.

1. *Department of Electrical and Computer Engineering, University of Tehran, Tehran, I.R. Iran, 14399.*

* *Corresponding Author, Department of Electrical and Computer Engineering, University of Tehran, Tehran, I.R. Iran, 14399.*

2. *Computer Integrated Manufacturing Laboratory, Department of Mechanical and Industrial Engineering, University of Toronto, Toronto, Ontario, Canada, M5S 3G8.*

PARAMETER ESTIMATION OF ELLIPTICAL SHAPES

Problem Statement

One approach to the elliptical parameter estimation problem is using optimization techniques. From a mathematical point of view, the fitting of a conic or several conic sections to a set of data has been addressed in various papers [8,9]. Also, some applications have been addressed in other papers discussing this problem [1,6,7]. Alternative methods have been investigated for dealing with the same problem, namely: using the Hough Transformation for the detection of curves [10]; using a modified Hough Transformation for the detection of ellipses [11]; decomposing the five-dimensional Hough Transformation space into three sub-spaces based on the edge-vector-field properties of ellipses [12]; estimating the parameters of an ellipse by combining transformation, projection and optimum approximation techniques [13] and elliptical-shape parameter extraction through using the moment and Fourier descriptors [7].

Herein, it is assumed that the input data have a high degree of accuracy. This leads to the following definition of the problem:

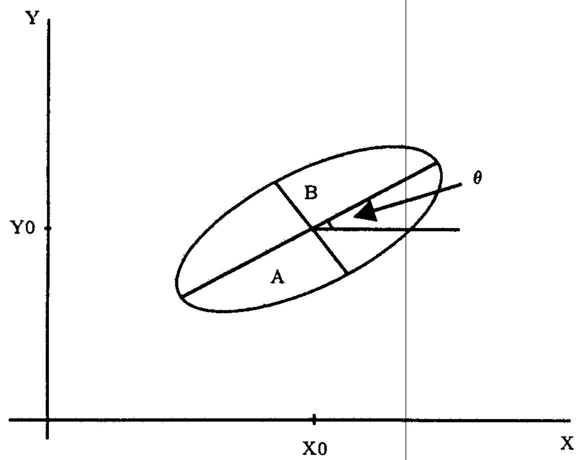


Figure 1. Five basic parameters of an ellipse.

"Given a set of 2D-image coordinates of edge points of an occluded-ellipse, it is required to determine the best ellipse that fits the points and subsequently estimate its five basic parameters (Figure 1), namely, location: (X, Y) (ellipse center), orientation: Θ (the angle between the x-axis of the computer-image frame and the major axis of the ellipse), and shape: A and B (major and minor radii, respectively)."

Estimation of Parameters

Since the objective of this research is achieving a high degree of accuracy in parameter estimation of occluded ellipses, naturally, solutions have been chosen that yield the most accurate results. The reader should refer to some comparative papers regarding the accuracy of different estimation methods [7].

Previous Error Functions for Parameter Estimation

The five parameters of an ellipse, based on a set of 2D coordinates, can be estimated by defining an error function and then minimizing it. However, the accuracy of the estimation process depends on the geometrical nature of the error function.

The Error Function J_1 and Its Geometrical Nature

Let:

$$Q(X,Y) = aX^2 + bXY + cY^2 + dX + eY + f = 0 \tag{1}$$

be the general equation of an ellipse. Moreover, let (X_i, Y_i), $i = 1, ..., N$ be a set of points fitting an elliptical shape. If this set of 2D coordinates includes the edge points of an elliptical shape, it should be applicable to the general equation of the ellipse. Therefore, the quantity $Q(X_i, Y_i)$ vanishes if the point (X_i, Y_i) is on the ellipse. It is negative, if the point is inside, and positive, if the point is outside.

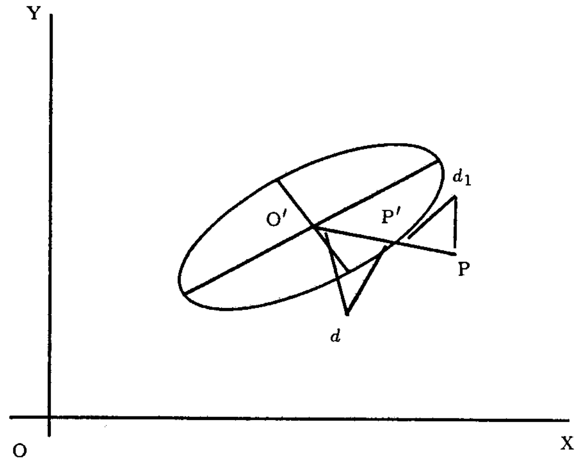


Figure 2. Distances of an edge point (P) and the cross point (P′) from the ellipse center.

The least-squares error criterion is one technique to fit an ellipse to a set of N points. In this technique, the following function should be minimized:

$$J_0 = \sum_{i=1}^{N} [Q(X_i, Y_i)]^2 . \tag{2}$$

Thus, the objective is to determine the six parameters a, b, c, d, e and f of an ellipse in a way that J_0 is minimized. This error function has been used by different researchers. Bookstein [9] proved that (with reference to Figure 2):

$$Q(X_i, Y_i) \propto [\frac{d_1}{d}(\frac{d_1}{d} + 2)] , \tag{3}$$

where (d_1/d) is the ratio of the distance point-to-conic (PP′) and center-to-conic (P′O′) boundary along the ray PO′.

The analysis of the solution method [14], which is based on the constraint $f = 1$ (implying normalization with respect to f) has been shown in [1,6]. In this paper, a review of this method is presented while discussing the new approach.

By applying the normalization with respect to f, the equation of an ellipse becomes:

$$Q(X,Y) = aX^2 + bXY + cY^2 + dX + eY + 1 . \tag{4}$$

Now, using the least-squares error criterion, the error function J_1 will be defined as:

$$J_1 = \sum_{i=1}^{N} [Q(X_i, Y_i)]^2 . \tag{5}$$

A Geometrical Interpretation for J_1

Let (X_i, Y_i), $(i = 1, ..., N)$ be a set of data points and the parameters of the optimal ellipse be

(X_0, Y_0, Θ, A, B). Also, let $(X_0, Y_0, \Theta, A', B')$ be the parameters of another ellipse that passes through the data point (X_i, Y_i). The two ellipses are concentric with the same orientation (Figure 3). An error can be defined as the difference between the areas of the two ellipses as,

$$e_i = S - S'_i \ . \tag{6}$$

It has been shown that [1] ,

$$e_i = \pi AB(1 - \frac{d_i'^2}{d_i^2}) \ , \tag{7}$$

where $d_i = \mathrm{P'O'}$, and $d'_i = \mathrm{PO'}$.

By defining $\delta i = d'_i - d_i$, Equation 7 can be rewritten in the following form:

$$e_i = \pi AB[\frac{\delta_i}{d_i}(\frac{\delta_i}{d_i} + 2)] \ . \tag{8}$$

Safaee-Rad et al. [1] have proven that the error defined by Equation 7 can be written as,

$$e_i = -\pi AB[Q(X_i, Y_i)]_{f \equiv F} \ , \tag{9}$$

where,

$$a = \frac{(A^2 \sin^2\Theta + B^2 \cos^2\Theta)}{A^2 B^2} \ ,$$

$$b = \frac{2(B^2 - A^2)\sin\Theta\cos\Theta}{A^2 B^2} \ ,$$

$$c = \frac{(A^2 \cos^2\Theta + B^2 \sin^2\Theta)}{A^2 B^2} \ ,$$

$$F = aX_0^2 + bX_0Y_0 + cY_0^2 - 1 \ ,$$

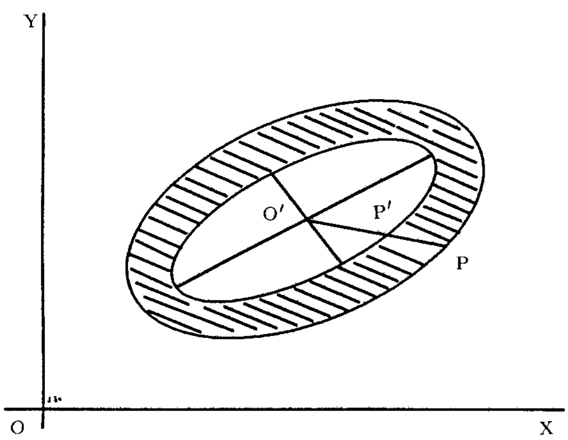


Figure 3. The area difference between two concentric ellipses.

and $Q(X_i, Y_i)$ is the equation of an ellipse, in which the constant term is normalized with respect to F. Based on Equation 9, the error function J_2 is defined as,

$$J_2 = \sum_{i=1}^{N}[\frac{1}{\pi AB}(S - S'_i)]^2 = \sum_{i=1}^{N}[\frac{e_i}{\pi AB}]^2$$

$$= \sum_{i=1}^{N}[Q(X_i, Y_i)]^2_{f \equiv F} \ . \tag{10}$$

The comparison of J_2 and J_1 demonstrates that the only difference between them is that in J_2, $f \equiv F$, while in J_1, $f \equiv 1$, which is only a constant multiplication factor. Thus, it can be concluded that the two error functions are equivalent.

Based on this interpretation of the error function, another error function, J_3, has been defined, in which the contributions of the data points to the error function are uniform.

Error Function J_3

To normalize the contribution of each data point to the error function, a weighting factor has been defined. This weighting factor is a function of the position of an individual data point. Let δ_i be the distance of a particular data point from the optimal ellipse (PP'). The amount of error due to this data point is:

$$e_1 = \pi AB[\frac{\delta_i^2 + 2d_i\delta_i}{d_i^2}] \ . \tag{11}$$

If this point was on the major axis of the optimal ellipse, with the same δ_i, the error would be:

$$e_2 = \pi AB[\frac{\delta_i^2 + 2A\delta_i}{A^2}] \ . \tag{12}$$

Through using the above two equations for a data point (X_i, Y_i), the following weighting factor can be defined:

$$w_i = \frac{e_2}{e_1} = (\frac{d_i}{A})\left[\frac{(1 + \frac{\delta_i}{2A})}{(1 + \frac{\delta_i}{2d_i})}\right] \ . \tag{13}$$

Based on this weighting factor, the error function J_3 has been presented as follows:

$$J_3 = \sum_{i=1}^{N}[w_i\frac{1}{\pi AB}(S - S'_i)]^2 = \sum_{i=1}^{N}[w_iQ(X_i, Y_i)]^2_{f \equiv F}$$

$$= \sum_{i=1}^{N}[w_iQ(X_i, Y_i)]^2_{f \equiv 1} \ . \tag{14}$$

To obtain each w_i for each individual data point, one must have an initial guess of the optimal ellipse from which δ_i and d_i can be estimated for each data point. For this purpose, using the optimal ellipse that has been obtained by minimizing the J_1 error function is proposed herein. After minimization of each of the above error functions, a solution vector $\vec{w} = (a, b, c, d, e)$ will be obtained. The five parameters of an ellipse can then be estimated using the following equations,

$$X_0 = \frac{2cd - be}{b^2 - 4ac},$$

$$Y_0 = \frac{2ae - bd}{b^2 - 4ac},$$

$$\Theta = \arctan\left[\frac{(c-a) + \sqrt{(c-a)^2 + b^2}}{b}\right],$$

$$A^2 = \left[\frac{2(1-F_s)}{b^2 - 4ac}\right]\left[(c+a) + \sqrt{(c-a)^2 + b^2}\right],$$

$$B^2 = \left[\frac{2(1-F_s)}{b^2 - 4ac}\right]\left[(c+a) - \sqrt{(c-a)^2 + b^2}\right], \tag{15}$$

where:

$$F_s = \frac{bde - ae^2 - cd^2}{b^2 - 4ac} \tag{16}$$

A NEW ERROR FUNCTION

Proposed Error Function (oef)

In our newly-defined error function (oef), the contribution of each individual data point has been normalized. A weighting factor has been defined which is a function of the position of each individual data point.

Let δ_i be the distance of a particular data point to the optimal ellipse. The amount of error due to this data point is:

$$e_1 = \pi AB\left[\frac{\delta_i^2 + 2d_i\delta_i}{d_i^2}\right]$$

If the point with the same δ_i was on the minor axis of the optimal ellipse, the error would be:

$$e_2' = \pi AB\left[\frac{\delta_i^2 + 2B\delta_i}{B^2}\right] \tag{17}$$

By using Equations 16 and 17, the following new weighting factor for a data point (X_i, Y_i) is defined:

$$w_i = \frac{e_1}{e_2'} = \left(\frac{B}{d_i}\right)\left[\frac{(1 + \frac{\delta_i}{2d_i})}{(1 + \frac{\delta_i}{2B})}\right] \tag{18}$$

Based on this weighting factor, the new error function (oef) is introduced as follows:

$$oef = \sum_{i=1}^{N}[w_i \frac{1}{\pi AB}(S - S_i')]^2 = \sum_{i=1}^{N}[w_i Q(X_i, Y_i)]_{f\equiv F}^2$$

$$= \sum_{i=1}^{N}[w_i Q(X_i, Y_i)]_{f\equiv 1}^2 \tag{19}$$

This error function differs from J_3, (Equation 14) in its weighting factor, as observed by comparing Equations 13 and 18.

Classification of Different Application Examples

The occlusion of ellipses might be of different forms. In Figure 4, several types of the occluded ellipses are shown.

For each ellipse, after constructing each of the above error functions, one should minimize and extract the unknown parameters of the error function. This minimization process chooses the parameters obtained from the error function with an optimized value very close to zero. The above discussed error functions have five independent parameters and the minimization process can be performed using an appropriate optimization technique. One of the most efficient methods is the quasi-Newton technique, which has the advantages of the classical Newton method in rapidly and accurately minimizing a function and also the capability of searching for a global minimum.

One of the important parameters in function minimization through the quasi-Newton method is the ratio of the maximum to minimum eigenvalues of the Hessian matrix of the function. It has been demonstrated in [15] that if the algorithm for finding the minimum of a function converges to a local minimizer X^* (where $\nabla^2 f(X^*)$ is positive as well as definite, and $e\nu_{\max}$ and $e\nu_{\min}$ are the greatest and smallest eigenvalues of $\nabla^2 f(X^*)$), then one can show that $\{X_k\}$ satisfies:

$$\limsup_{k\to\infty} \frac{\| X_{k+1} - X^* \|}{\| X_k - X^* \|} \leq c, \quad c = \left(\frac{e\nu_{\max} - e\nu_{\min}}{e\nu_{\max} + e\nu_{\min}}\right), \tag{20}$$

in a particular weighted l_2 norm, and that the bound on "c" is tight for some starting X_0. If $\nabla^2 f(X^*)$ is suitably scaled with $e\nu_{\max} \cong e\nu_{\min}$, then "$c$" will be very small and convergence will be fast, but if $\nabla^2 f(X^*)$

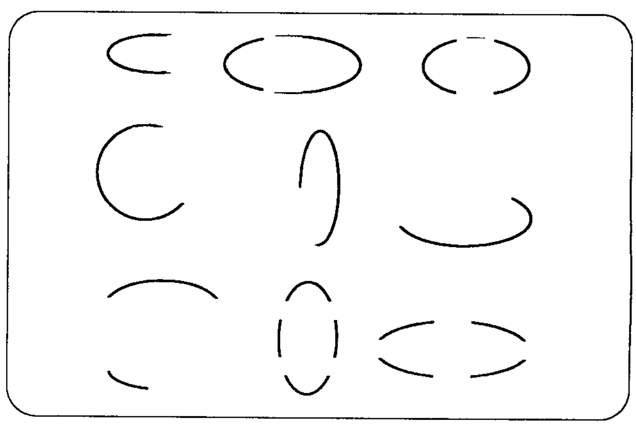


Figure 4. Some of the possible forms of partly-occluded ellipses.

is poorly scaled, then "c" will be almost 1 and the convergence may occur very slowly.

Therefore, by reducing the above discussed ratio, the optimization process will work more efficiently. In this paper, this ratio is referred to as the maximum-to-minimum eigenvalue ratio (MMER) for simplicity.

In the previous section, two different error functions were discussed in which the contribution of each edge point has been considered in construction of the error functions. In these two error functions (J_3 and "oef"), the existence of weighting factors, which are always smaller than 1, will greatly affect the coefficients of the variables which consequently will change the difference between the maximum and minimum eigenvalues of the Hessian matrix of the error functions J_3 and "oef". Based on the definitions of the error functions and the difference between their weighting factors, in various cases of occlusion, MMER for error function "oef" is different from MMER for J_3. In some cases, MMER for "oef" is smaller than MMER for J_3, implying that the minimization process of the "oef" error function converges faster and more accurate results are obtained in comparison to J_3 error function. However, in some other cases, the reverse situation occurs.

Based on a comparison between MMER of J_3 and "oef" error functions, a new algorithm is developed for the rapid and accurate extraction of the parameters of a partly-occluded ellipse.

A NUMERICALLY-EFFICIENT TRANSFORMATION

By simplifying the error function J_1, based on its parameters and coefficients, the following is obtained:

$$\begin{aligned} J_1 = {} & (\sum_i X_i^4)a^2 + (\sum_i X_i^2Y_i^2)b^2 + (\sum_i Y_i^4)c^2 \\ & + (\sum_i X_i^2)d^2 + (\sum_i Y_i^2)e^2 + 2[(\sum_i X_i^3Y_i)ab \\ & + (\sum_i X_i^2Y_i^2)ac + (\sum_i X_i^3)ad + (\sum_i X_i^2Y_i)ae \\ & + (\sum_i X_i^2)a + (\sum_i X_iY_i^3)bc + (\sum_i X_i^2Y_i)bd \\ & + (\sum_i X_iY_i^2)be + (\sum_i X_iY_i)b + (\sum_i X_iY_i^2)cd \\ & + (\sum_i Y_i^3)ce + (\sum_i Y_i^2)c + (\sum_i X_iY_i)de \\ & + (\sum_i X_i)d + (\sum_i Y_i)e + \frac{1}{2}n] \ (i \text{ from } 1 \text{ to } n) \ , \end{aligned} \tag{21}$$

In the above equation, there is a symmetrical relation between X and Y coordinates of edge points constructing the coefficients of the error function. If there is a large difference between the values of X and Y coordinates of edge points, as can be seen from the parameteric form of J_1, the differences between the elements of Hessian matrix of error function J_1 become significant. Let,

$$H = \begin{bmatrix} 2\sum_i X_i^4 & \sum_i X_i^3Y_i & \sum_i X_i^2Y_i^2 & \sum_i X_i^3 & \sum_i X_i^2Y_i \\ \sum_i X_i^3Y_i & 2\sum_i X_i^2Y_i^2 & \sum_i X_iY_i^3 & \sum_i X_i^2Y_i & \sum_i X_iY_i^2 \\ \sum_i X_i^2Y_i^2 & \sum_i X_iY_i^3 & 2\sum_i Y_i^4 & \sum_i X_iY_i^2 & \sum_i Y_i^3 \\ \sum_i X_i^3 & \sum_i X_i^2Y_i & \sum_i X_iY_i^2 & 2\sum_i X_i^2 & \sum_i X_iY_i \\ \sum_i X_i^2Y_i & \sum_i X_iY_i^2 & \sum_i Y_i^3 & \sum_i X_iY_i & 2\sum_i Y_i^2 \end{bmatrix} \tag{22}$$

be Hessian matrix of error function J_1. Referring to Gerschgorin theorem [15], a large difference between X and Y parameters causes a great difference between the related elements of $\mathbf{H}$ matrix, which itself results in a greater difference between the center of regions where the eigenvalues might exist. As a result, a larger value for MMER is obtained. This means that the ratio of the maximum to minimum eigenvalues of Hessian matrix of error function J_1 becomes large and diverges from 1. As discussed in the previous section, this leads to a low speed or failure in convergence.

If all of the edge points move in the direction of one axis, namely, the mean value of the coordinates of points in that direction is larger than the other, the difference between the maximum and minimum eigenvalues of the Hessian matrix of J_1 will be reduced, and the optimization process will converge faster and more accurately than before. Based on this point of view, a Transfer-to-Center transformation has been proposed that strongly affects the accuracy and speed of the optimization process.

In this transformation, first the mean values of X and Y coordinates of the edge points are obtained and, then, if there is a large difference between these two values, a constant will be subtracted from the X or Y coordinate (whichever has a larger mean value) of all edge points (e.g., if the mean of Y coordinates is larger than the mean of X coordinates all the edge points in the direction of Y axis are moved closer to the center). This constant value can be equal to the difference between the above two mean values. Also, if these two values are far from the center point $(0, 0)$, two constant values are subtracted from the coordinates of edge points. These constants can be close to the minimum value of each set of coordinates.

Details of this transformation are discussed in a related algorithm in the next section.

ANALYTICAL PARAMETER-EXTRACTION ALGORITHM

In the proposed new algorithm, more accurate results are obtained using the Transfer-to-Center transformation and a combination of three different error functions (J_1, J_3 and "oef"). In this method, comparing MMER for J_3 and "oef" error functions determines the way the algorithm is completed.

The proposed algorithm is shown below:

1. Transfer-to-Center Transformation:
   a) Determine the mean value of the X coordinate of all edge points (μ_x).
   b) Determine the mean value of the Y coordinate of all edge points (μ_y).
   c) Compute $\mu = |\mu_x - \mu_y|$
   d) d1) If ($\mu_x >> \mu_y$), Then: move all of the edge points in the direction of X axis closer to the center by subtracting μ from X coordinates of edge points, else:
      d2) If ($\mu_x << \mu_y$), Then: Move all of the edge points in the direction of Y axis closer to the center by subtracting μ from Y coordinates of edge points.
   e) Compute the mean values again.
   f) If the mean values are far from the center of coordinates (0, 0), (e.g. μ_x, μ_y > 100), Then:
      f1) Obtain the minimum values of the X coordinates (m_x).
      f2) Obtain the minimum values of the Y coordinates (m_y).
      f3) Subtract m_x from the X and m_y from the Y coordinates of edge points.
2. Use the coordinates of the edge points of the occluded ellipse to construct the J_1 error function.
3. Minimize the error function J_1 and obtain an initial estimation of the ellipse parameters.
4. By using the results of Step 3, construct the error functions J_3 and "oef".
5. Compute the minimum and maximum eigenvalues ($ev_{\min}$ and $ev_{\max}$) of the Hessian matrix of both J_3 and "oef" error functions.
6. If ($ev_{\max}/ev_{\min}$) for J_3 is less than that of the "oef", Then:
   6.1 Minimize the error function J_3 and extract the optimal ellipse parameters. Else:
   6.2 Minimize the error function "oef" and extract the optimal ellipse parameters.
7. Transfer the X and Y coordinates of the center of the extracted ellipse to its main position if necessary (reverse of the Transfer-to-Center transformation).

End of the algorithm.

EXPERIMENTAL RESULTS

Based on the above analytical algorithm, several experiments are performed in which real images of ellipses taken by a (512 × 480) CCD camera, as shown in Figure 5, are used. As a determination of the satisfactory level of fit, there are different techniques as used in [1], or the MSE value of error function.

The Mean Square Error (MSE) function is evaluated by using the extracted parameters from each of the error functions and substituting them in $\sum_i[Q(X_i, Y_i)]^2$ as a determination of the satisfactory level of fit. Therefore, an estimation that yields a result which makes the MSE value smaller than the other is preferred.

A set of tables are provided herein to illustrate the results of these experiments. Each ellipse is occluded by a surface whose border lines are specified under the table. In these tables, the first five columns show the basic parameters of the ellipse and the last three columns demonstrate the number of iterations, the value of MSE function by substituting the results of that row in the specified error function and MMER of the employed error function, respectively.

In Table 1a, the original parameters of an ellipse are illustrated. In Table 1b, the extracted parameters using J_1 and improvement of extraction by using "oef" have been described. In this table, the error function "oef" has employed the results of J_1 error function's optimization as its initial values and has extracted the

Table 1a. Original ellipse parameters (Figure 5, Picture 1).

X	Y	Θ	A	B
453.20	217.37	0.0247	54.73	18.09

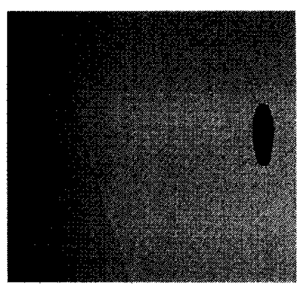
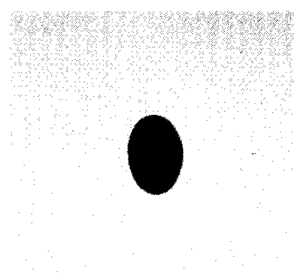

1

2

Figure 5. The pictures in this figure have been used here in the experiments. In these pictures the top-left corner is (0, 0) and the bottom-right corner is (512, 480).

Table 1b. Comparison of the results extracted by using J_1 and oef.

Error Function	X	Y	Θ	A	B	MSE Value
J_1	444.7315	214.3169	-0.1701	57.9433	29.4295	$8.272\ e-4$
oef	453.3591	218.1202	0.0118	51.9467	18.3358	$4.426\ e-5$

Occluded by: $(x \geq 441)$ and $(x \leq 465)$; Percent of occlusion: %70.64

parameters closer to the original ellipse parameters. This illustrated the effect of normalization of each data point contribution to the error function.

In Table 2, the original parameters of another ellipse are shown. In the next two tables, two different occluded forms of this ellipse have been employed and their parameters have been extracted using the above algorithm. In Table 3, MMER for the error function J_3 is smaller than the error function "oef". Therefore, the MSE function has a smaller value and the results obtained by using J_3 are more accurate than those of "oef". In Table 4, MMER for error function "oef" has a smaller value than that of J_3. Therefore, the MSE function, by using these results, has a smaller value than that of J_3 and the results are more accurate in this case.

In Tables 5a to 5c, the effect of Transfer-to-Center

Table 2. Original ellipse parameters (Figure 5, Picture 2).

X	Y	Θ	A	B
197.01	175.63	-0.0072	69.60	47.33

(TTC) transformation has been shown. Because of the availability of the original ellipse parameters, MSE column of the tables has been eliminated. By comparing the extracted parameters in Tables 5b and 5c, it can be seen that by applying TTC transformation, MMER value of the Hessian matrix of the error function has been reduced and the results are much closer to the original parameters.

SUMMARY

For the estimation of the parameters of a partly-occluded ellipse, a new weighting factor and a new error function have been defined. Also, the effects of the ratio of maximum to minimum eigenvalues of Hessian matrix (MMER) for each of J_3 and "oef" error functions on the accuracy and the convergence time of the optimization process have been discussed. By which to combine error functions and the proposed transformation, a new analytical algorithm for the extraction of parameters of a partly-occluded ellipse has also been developed. Finally, the results of some experiments have been discussed.

Table 3. Comparison of the results obtained by using J_3 and oef.

Error Function	X	Y	Θ	A	B	# Iteration	MSE Value	MMER Value
J_3	197.6643	176.4844	-0.0071	69.5732	47.1642	323	$5.783\ e-4$	$2.311\ e+8$
oef	197.6206	177.2977	0.0980	65.0408	49.5816	1001	$5.400\ e-3$	$2.546\ e+8$

Occluded by: $(x \geq 197)$ and $(x \leq 207)$; Percent of occlusion: % 6.5

Table 4. Comparison of the results obtained by using J_3 and oef.

Error Function	X	Y	Θ	A	B	# Iterations	MSE Value	MMER Value
J_3	198.4940	176.9712	0.0244	70.6668	45.1649	1003	$1500\ e-3$	$3.799\ e+8$
oef	196.7609	175.9705	-0.0041	69.3955	47.0972	167	$1.773\ e-4$	$3.368\ e+8$

Occluded by: $(y \geq 140)$ and $(y \leq 200)$; Percent of occlusion: % 55

Table 5a. The ellipse parameters.

X	Y	Θ	A	B
99.40	118.67	1.56	86.72	21.69

Occluded by: ($x \geq 45$)

Table 5b. The results of extraction without Transfer-to-Center transformation.

Error Function	X	Y	Θ	A	B	MMER value
J_1	62.2041	118.6465	1.5583	48.1576	17.2618	5.5612e8

Table 5c. The results of extraction using Transfer-to-Center transformation ($y = y - 90$).

Error Function	X	Y	Θ	A	B	MMER value
J_1	99.7732	118.4420	1.5607	86.5725	21.2182	5.5072e4

REFERENCES

1. Safaee-Rad, R., Smith, K.C. and Benhabib, B. "Accurate estimation of elliptical shape parameters from a grey-level image", *IEEE Int'l. Conference on Pattern Recognition*, **2**, Atlantic City, NJ, USA, pp 20-26 (June 1990).
2. Basel, P.J. and Jane, R.C. "Three-dimensional object recognition", *Computing Surveys*, **17**(1), pp 75-145 (Mar. 1985).
3. Zhou, Z., Smith, K.C., Benhabib, B. and Safaee-Rad, R. "Morphological skeleton transforms for determining position and orientation of pre-marked objects", *IEEE Proc., PacificRim Conference on Communications, Computers and Signal Processing*, Victoria, B.C., pp 301-305 (June 1989).
4. Safaee-Rad, R., Benhabib, B., Smith, K.C. and Zhou, Z. "Pre-marking methods for 3D object recognition", *IEEE Int'l. Conference on Systems, Man and Cybernetics*, Boston, MA (Nov. 1989).
5. Safaee-Rad, R., Smith, K.C., Benhabib, B. and Tchoukanov, I. "A new technique for 3D-location estimation of pre-marked objects for an active vision system", *IEEE Int'l. Conference on Systems, Man and Cybernetics* (Nov. 1990).
6. Safaee-Rad, R., Tchoukanov, I., Benhabib, B. and Smith, K.C. "Accurate parameter estimation of quadratic curves from grey-level images", *Computer Vision, Graphics and Image Processing: Image Understanding*, **54**(2), pp 259-274 (1991).
7. Safaee-Rad, R., Smith, K.C., Benhabib, B. and Tchoukanov, I. "Application of moment and Fourier descriptors to the accurate estimation of elliptical-shape parameters", *Pattern Recognition Letters 13*, Elsevier Science Publishers, pp 497-508 (Jul. 1992).
8. Gnanadesikan, R. *Methods for Statistical Data Analysis of Multivariate Observations*, John Wiley & Sons (1977).
9. Bookstein, F.L. "Fitting conic sections to scattered data", *Computer Graphics and Image Processing*, **9**, pp 59-71 (1979).
10. Duda, R.O. and Hart, P.E. "Use of the Hough transformation to detect lines and curves in pictures", *Communications of the ACM*, **15**(1), pp 11-15 (Jan. 1972).
11. Tsuji, S. and Matsumoto, F. "Detection of ellipses by a modified Hough transformation", *IEEE Trans. Computers*, **C-27**(8), pp 777-781 (1978).
12. Tsukune, H. and Goto, K. "Extracting elliptical figures from an edge vector field", *IEEE Conference on Computer Vision and Pattern Recognition*, pp 138-141 (Jan. 1983).
13. Wang, R., Hansen, A.R. and Riseman, E.M. "Fast extraction of ellipses", *Proc. 9th Int'l Conference on Pattern Recognition*, **1**, pp 508-510 (1988).
14. Albano, A. "Representation of digitized contours in terms of conic arcs and straight-lines segments", *Computer Graphics and Image Processing*, **3**, pp 23-33 (1974).
15. Dennis, J.E. and Schnabel, R.B. *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Prentice-Hall Inc. (1983).