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Research Note

Riemann Mapping for the Family $z^m(z^n - b), m \neq n$

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In this paper, it is shown that the connectedness locus $M(m,n)$ for the family of maps $z \mapsto z^m(z^n - b)$, $m \neq n$ is connected by constructing a Riemann map for the complement $\overline{\mathbf{C}} \setminus M(m,n)$. In particular, $\overline{\mathbf{C}} \setminus M(m,n)$ is simply connected. Therefore, the external rays of $M(m,n)$ can be defined and a combinatorial picture of the connected locus can be given. In the quadratic case, $z^2 + c$, the parameter space is based on the critical value, whereas in the case presented here, it is based on the critical point. Consequently, the definition of Riemann map is different from the quadratic case.

INTRODUCTION

In the quadratic case, Douady and Hubbard introduce the conformal map $c \mapsto \varphi_c(c)$ from the complement of Mandelbrot set onto the complement of the closed unit disk, where φ_c is the Böttcher map of $z^2 + c$ at infinity. Based on this work, Thurston gives a combinatorial picture of Mandelbrot set which is useful in explicating the dynamical behavior. However, this model cannot be used for higher degrees as the example of $F_{-1}(z) = z(z^2+1)$, $F_1(z) = z(z^2-1)$ shows (see [1]).

In this paper, the Riemann mapping from $\overline{\mathbf{C}} \setminus M(m,n)$ to $\overline{\mathbf{C}} \setminus \overline{\mathbf{D}}$ is introduced. By using this map, a combinatorial model of the connected locus of $M(m,n)$, $m \neq n$ can be given. The method follows the lines of [2] except that in [2] the argument is based on the sole critical value, whereas in the case presented here, it is necessary to keep track of several critical points.

For convenience $\frac{m+n}{m}c^n$ and $z^m(z^n - b)$ are denoted by b and by f_c. By definition:

$$M(m,n) = \{c \in \mathbf{C} : \text{The Julia set of } f_c \text{ is connected}\}$$

or equivalently:

$$M(m,n) = \{c : \text{ The sequence } \{f_c^k(\alpha)\}$$
$$\text{is bounded for all critical points } \alpha \text{ of } f_c\}.$$

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Let the simplified notation $e(t)$ for $e^{2\pi i t}$ be used. (Thus $e(k/n)$ is an n-th root of unity.) The critical points of f_c consist of the points $e(k/n)c$ with $0 \leq k \leq n-1$ together with the super-attracting fixed point $z = 0$ when $m > 1$. Note that $f_c(e(k/n)z) = e(mk/n)f_c(z)$ for any z. Hence, the orbit of every critical point of f_c is bounded if, and only if, the orbit of the critical point c is bounded.

Let c be any root of the equation $b = \frac{m+n}{m}c^n$, so that the derivative of $z^m(z^n - b)$ at c is zero. The notation f_c will be used for the pair consisting of the polynomial $z \mapsto z^m(z^n - b)$ together with the marked critical point c. (Thus f_c and $f_{e(1/n)c}$ agree as polynomials and have the same Julia set, but different designated critical points.) It will be shown later why the case $m \neq n$ is skipped.

One can see that under the n to 1 mapping:

$$L : z \longmapsto \frac{n+m}{m} z^n,$$

the image of $M(m,n)$ will be the connected locus of $F_b(z) = z^m(z^n - b)$. Since $M(m,n)$ is compact, (see Lemma 2) the connected locus of the family $F_b(z)$ will be locally connected, provided that $M(m,n)$ is locally connected. In fact, more information about the boundary of this connected locus may be obtained by studying the boundary of $M(m,n)$.

PRELIMINARIES

In this section some lemmas are proved and the tools needed for the definition of the Riemann mapping are introduced.

Lemma 1

For each m, n, there exists a constant $P = P_{m,n}$ such that:

$$c \in M(m,n) \Longleftrightarrow |f_c^k(c)|^n \leq P \qquad \forall k \geq 0.$$

Proof

Let $P = \max(2m+1\ ,\ m+n+1)$ for $(m,n) \neq (2,1)$ and $P = 4$ for $(m,n) = (2,1)$. Define:

$$W(c) = \{z \in \overline{\mathbf{C}} : |z^n| > \frac{P}{m}|c|^n\ ,\ |z^n| > P\}.$$

One can see that for any $z \in W(c)$,

$$\begin{aligned} |f_c(z)| &\geq |z^m|(|z^n| - \frac{m+n}{m}|c^n|) \\ &\geq |z^m|(|z^n| - \frac{m+n}{P}|z^n|) \\ &\geq |z^m||z^n|\frac{(P-m-n)}{P} > |z|^m > |z|. \end{aligned}$$

In addition, for any $z \in W(c)$, there exists $\varepsilon = \varepsilon_{m,n} > 0$ such that $|z|^n \geq P + \varepsilon$, therefore, by induction, the following can be proven:

$$|f_c^k(z)| > P^{\frac{1}{n}}(1 + \frac{\varepsilon}{P})^k\ ;\ \forall z \in W(c).$$

Hence, $|f_c^k(z)|$ tends to infinity as k tends to infinity, so the sequence $\{f_c^k(z)\}$ cannot be bounded. Now let $c \in M(m,n)$, consequently the sequence $\{f_c^k(c)\}$ is bounded. Therefore, $f_c^k(c) \notin W(c)$, $k \geq 1$ and as a result:

$$|f_c^k(c)| \leq (\frac{P}{m})^{\frac{1}{n}}|c| \quad \text{or} \quad |f_c^k(c)| \leq P^{\frac{1}{n}}. \tag{1}$$

In particular for $k = 1$,

$$\frac{n}{m}|c|^{n+m} \leq (\frac{P}{m})^{\frac{1}{n}}|c| \quad \text{or} \quad \frac{n}{m}|c^{n+m}| \leq P^{\frac{1}{n}}. \tag{2}$$

By using the following inequality:

$$(\frac{m+n+1}{m})^{\frac{1}{n}}\frac{m}{n} \leq 1\ ,\ (m+n+1)^{\frac{1}{m+n}}(\frac{m}{n})^{\frac{n}{m+n}} \leq m$$

in the case $m < n$ and the inequality:

$$(\frac{2m+1}{m})^{\frac{1}{m+n-1}}(\frac{m}{n})^{\frac{n}{m+n-1}} \leq m,$$

$$(2m+1)^{\frac{1}{m+n}}(\frac{m}{n})^{\frac{n}{m+n}} \leq m\ ,$$

in the case $m > n$ and $(m,n) \neq (2,1)$, $|c^n| \leq m$ is concluded. (Note that in the first case $P = m+n+1$ and in the second case $P = 2m+1$.)

Therefore, $|f_c^k(c)|^n \leq P\ \forall k \geq 1$ is obtained. ■

Lemma 2

$M(m,n)$ is compact.

Proof

Define $Q_1(z) = -\frac{n}{m}z^{n+m}$ and $Q_k(z) = (Q_{k-1}(z))^m (Q_{k-1}(z)^n - \frac{n+m}{m}z^n)$, so for any $c \in M(m,n)$, $Q_k(c) = f_c^k(c)$.

Let $T = T_{m,n} = \{z : |z|^n \leq P\}$, where P is as Lemma 1. So the following is obtained:

$$\begin{aligned} c \in M(m,n) &\Longleftrightarrow |f_c^k(c)|^n \leq P \\ &\Longleftrightarrow |Q_k(c)|^n \leq P \\ &\Longleftrightarrow Q_k(c) \in T \\ &\Longleftrightarrow c \in Q_k^{-1}(T). \end{aligned}$$

Therefore, $M(m,n) = \bigcap_{k\geq 1} Q_k^{-1}(T)$. Since Q_k is a polynomial and T is compact, then $M(m,n)$ is compact. ■

Now the tools needed for the definition of the Riemann mapping are introduced. In fact, it will be defined as the limit of a sequence. For simplicity, $M = M(m,n)$ is used.

Definition 1

For any $c \in \mathbf{C}$,

$$W_c(\infty) = \{z \in \overline{\mathbf{C}} : f_c^k(z) \to \infty \text{ as } k \to \infty\}\ ,$$

$$\Omega = \{(z,c) \in \overline{\mathbf{C}} \times \mathbf{C} : c \in \mathbf{C} \setminus M\ ,\ z \in W_c(\infty)\},$$

$$\Omega' = \{(z,c) \in \Omega : G_c(z) > G_c(c)\},$$

where G_c is Green function for $K(f_c)$ and $K(f_c)$ is filled-in Julia set (see [3]).

Lemma 3

Ω' satisfies the following properties:

1. $(\infty, c), (f_c(c), c) \in \Omega'$,
2. $\Omega'(c) = \{z :\ (z,c) \in \Omega'\}$ is simply connected,
3. Ω' is open in $\overline{\mathbf{C}} \times \mathbf{C}$.

Proof

1. Note $G_c(f_c(c)) = (n+m)G_c(c) > G_c(c)$.
2. For $c \in \mathbf{C} \setminus M$, define:

$$\Delta^k(c) = \{z \in W_c(\infty) :\ G_c(z) > (n+m)^k G_c(c)\}.$$

It is clear that $\Delta^0(c) = \Omega'(c)$ and $\Delta^{k+1}(c) \subset \Delta^k(c)$. In addition,

$$\begin{aligned} z \in \Delta^k(c) &\Longleftrightarrow G_c(z) > (n+m)^k G_c(c) \\ &\Longleftrightarrow expG_c(z) > exp((n+m)^k G_c(c)) \\ &\Longleftrightarrow |\varphi_c(z)| > exp((m+n)^k G_c(c)), \end{aligned}$$

where φ_c is a Böttcher map (see [3]) corresponding to the super-attractive fixed point ∞. So for a large k, $\Delta^k(c)$ is a simply connected open set. Moreover, $f_c(\Delta^k(c)) = \Delta^{k+1}(c)$ and f_c is an unramified $(n+m)$-fold covering from $\Delta^k(c)$ to $\Delta^{k+1}(c)$ which has only one critical point at ∞. So for any $k \geq 0$:

$$\chi(\Delta^k(c)) + m + n - 1 = (m+n)\chi(\Delta^{k+1}(c)),$$

where χ is Euler characteristic. Therefore, $\chi(\Delta^k(c)) = 1$ for any k, in particular $\chi(\Delta^0(c)) = \chi(\Omega'(c)) = 1$.

3. Define $G(z,c)$ as $G_c(z)$. Continuity of G follows exactly as in [4], so $\Omega' = G^{-1}(]G_c(c), \infty[)$ is open. ■

Proposition 1

For $m \neq n$, the equation:

$$X^{m+n} - \frac{m+n}{m}X^m + \frac{n}{m} = 0\ , \tag{3}$$

has at least one solution in $\mathbf{C}$ which is not a critical point.

Proof

It is clear that $X = 1$ is a solution. Moreover, if $\gcd(m,n) = k$, $m = k''k$, $n = k'k$, then $X = e(tk'/n)$, $0 \leq t \leq k-1$ is also a solution. On the other hand, $X = e(ij/n)$, $0 \leq j \leq n-1$ are the critical points of multiplicity one for $h(X) = X^{m+n} - \frac{m+n}{m}X^m + \frac{n}{m}$. Then, this equation has $2k$ roots (counting with multiplicity) which are critical points $(X = e(tk'/n), 0 \leq t \leq k-1)$. Since the number of solutions is $m + n = (k'' + k')k > 2k$, then there exists $\alpha \in \mathbf{C}$ such that α is a solution but not a critical point. ■

Corollary 1

If α is a non-critical solution of Equation 3, then αc is a cocritical point for f_c. ■

Let $D(w;r) = \{z \in \overline{\mathbf{C}} : \ d(z,w) < r\ \}$, where d is the spherical metric on $\overline{\mathbf{C}}$. Define $V(c) = D(\alpha c; r_c)$, where r_c is the minimum distance αc from the critical points and $K(f_c)$. Based on the results of Mañé et al. (see [5]), the Julia set $J(f)$ varies continuously under a deformation of f through hyperbolic maps. So one can see that r_c is continuous. Let $\epsilon = \frac{1}{4}\min\{d(\alpha c, z),\ r_c - d(\alpha c, z)\}$, $z \in V(c)$. Since the maps $c \mapsto r_c$ and $c \mapsto \alpha c$ are continuous, there exist $\delta_1 < \epsilon$ and $\delta > 0$ (very small) such that:

$$d(c,c') < \delta \Longrightarrow d(\alpha c, \alpha c') < \delta_1 \Longrightarrow\ r_c - \epsilon < r_{c'} < r_c + \epsilon.$$

It is claimed that: If $d(c,c') < \delta$, then $D(z;\epsilon) \subset V(c')$. First, $4\epsilon \leq r_c - d(\alpha c, z)$, so for any $z' \in D(z;\epsilon)$:

$$\begin{aligned} d(\alpha c', z') &\leq d(\alpha c', \alpha c) + d(\alpha c, z) + d(z, z') \\ &\leq \delta_1 + d(\alpha c, z) + \epsilon \\ &\leq 2\epsilon + d(\alpha c, z) \\ &< r_c - 2\epsilon < r_{c'}. \end{aligned}$$

Therefore, $V = \bigcup_{c \in \overline{\mathbf{C}} \setminus M}(V(c) \times \{c\})$ are open sets.

Define $\Omega'' = \Omega' \bigcup V$ and $\Phi^{(k)}(z,c)$ as a branch of $(f_c^k(z))^{\frac{1}{(n+m)^k}}$, for $(z,c) \in \Omega''$. It is clear that Ω'' is an open set in $\overline{\mathbf{C}} \times \mathbf{C}$ and $\Phi^{(k)}$ is locally injective since it has no critical points in $V(c) \bigcup \Omega'(c)$.

Lemma 4

The map $\Phi^{(k)}$ is well-defined and analytic in Ω''.

Proof

Define:

$$\sum = \{(z,c) \in \overline{\mathbf{C}} \times \mathbf{C} : (\frac{1}{z}, c) \in \Omega''\},$$

and for any $(z,c) \in \sum$, $H^{(k)}(z,c) = \frac{1}{z^{(m+n)^k} f_c^k(\frac{1}{z})}$.

$H^{(k)}$ is non-zero and analytic on $\sum$. In addition, $\sum(c) = \{z : (z,c) \in \sum\}$ is simply connected (Note that $\Omega''(c) = \{z : \ (z,c) \in \Omega''\} = V(c) \bigcup \Omega'(c)$ and $\Omega'(c)$ is simply connected), so $(H^{(k)}(z,c))^{\frac{1}{(m+n)^k}}$ is well-defined and analytic and for any $(z,c) \in \Omega''$,

$$\Phi^{(k)}(z,c) = (f_c^k(z))^{\frac{1}{(m+n)^k}} = \frac{z}{(H^{(k)}(\frac{1}{z}, c))^{\frac{1}{(m+n)^k}}}\ .$$

Therefore, $\Phi^{(k)}$ is well-defined and analytic. ■

Lemma 5

(a) $\Phi^{(k)}(z,c) \sim z$ near infinity (i.e., the ratio of the two sides goes to 1 as $z \to \infty$).

(b) $\Phi^{(k)}(f_c(z), c) = (\Phi^{(k+1)}(z,c))^{m+n}$.

Proof

(a) $\Phi^{(k)}(z,c) = (f_c^k(z))^{\frac{1}{(m+n)^k}} \sim z$ near infinity.

(b)
$$\begin{aligned} \Phi^{(k)}(f_c(z), c) &= (f_c^{k+1}(z))^{\frac{1}{(m+n)^k}} \\ &= ((f_c^{k+1}(z))^{\frac{1}{(m+n)^{k+1}}})^{m+n} \\ &= (\Phi^{(k+1)}(z,c))^{m+n}. \end{aligned}$$ ■

Now it is shown that the sequences of functions $\{\Phi^{(k)}\}_{k \geq 1}$ are uniformly convergent in:

$$W = \{(z,c) \in \overline{\mathbf{C}} \times \mathbf{C} : \\ c \in \mathbf{C} \setminus M, |z|^n > \frac{P}{m}|c|^n,\ |z|^n > P\},$$

where P is as described in Lemma 1.

It is clear (see proof of Lemma 1) that for any $(z,c) \in W$,

$$|z| < |f_c(z)| < \cdots < |f_c^k(z)| < \cdots$$

In addition,

$$\frac{\Phi^{(k+1)}(z,c)}{\Phi^{(k)}(z,c)} = (1 - \frac{(m+n)c^n}{m(f_c^k(z))^n})^{\frac{1}{(m+n)^{k+1}}}.$$

If the last quantity is written as $1 + \theta^{(k)}(z,c)$, it is obtained that:

$$\theta^{(k)}(z,c) = (1 - \frac{(m+n)c^n}{m(f_c^k(z))^n})^{\frac{1}{(m+n)^{k+1}}} - 1,$$

so:

$$|\theta^{(k)}(z,c)| \leq \frac{2}{(m+n)^{k+1}} |\frac{(m+n)c^n}{m(f_c^k(z))^n}|$$

$$\leq \frac{2}{(m+n)^k} \cdot \frac{1}{P} \leq \frac{1}{(m+n)^k}.$$

Therefore, there exists analytic function Φ such that $\{\Phi^{(k)}\}_{k\geq 1}$ uniformly converges to it in W. W is an open set, so $W \cap \Omega''$ is an open set in Ω''. Moreover, $W \cap \Omega'' \neq \emptyset$. If it is proven that $\Phi^{(k)}$ is normal in Ω'', then (by the usual proof of Vitali Theorem) Φ will have an extension on Ω'' and additionally $\{\Phi^{(k)}\}$ will locally and uniformly converge to Φ in Ω''. To prove this, it is enough to show that $\Phi^{(k)}$ is uniformly bounded on the compact subsets of Ω''. Let A be a compact subset of Ω''. Since $f_c^k(z)$ tends to infinity locally and uniformly, an open neighborhood around z, $U(z)$ can be chosen for any z, such that f_c^k tends to infinity in $U(z)$ as k tends to infinity. There exists a finite number $U(z_p)$ such that $A \subset \bigcup_p U(z_p)$, so N can be chosen such that $\forall k > N$, $\forall z \in A$, $|f_c^k(z)| > 1$, or, equivalently $\forall k > N$, $\forall z \in A$, $|\Phi^{(k)}(z,c)| > 1$. By composing with the map $z \longmapsto \frac{1}{z}$, $\{\Phi^{(k)}\}$ is uniformly bounded. As a result, it will be normal.

RIEMANN MAP

Now the Riemann mapping can be presented. Define:

$$\Psi : \overline{\mathbf{C}} \setminus M \underset{c \longmapsto \Phi(\alpha c, c)}{\longrightarrow} \overline{\mathbf{C}} \setminus \overline{\mathbf{D}},$$

where α is as Proposition 1. Note that for any $c \in \overline{\mathbf{C}} \setminus M$, $(\alpha c, c) \in \Omega''$. Furthermore, for any $(z,c) \in \Omega''$,

$$\Phi(z,c) = \lim_{k\to\infty} \Phi^{(k)}(z,c) = \lim_{k\to\infty} (f_c^k(z))^{\frac{1}{(m+n)^k}} = \varphi_c(z),$$

where φ_c is Böttcher map (see [3]), so $\log|\Phi(z,c)| = \log|\varphi_c(z)| = G_c(z) > 0$ and $|\Phi(z,c)| > 1$, $\forall (z,c) \in \Omega''$, therefore, $|\Psi(c)| > 1$, $\forall c \in \overline{\mathbf{C}} \setminus M$. It will be proven that Ψ is a conformal isomorphism in several steps. Before stating the steps, note the following lemma.

Lemma 6

$|\Psi(c)| \to 1$ as $c \to \partial M$.

Proof

It is known that $G_c(z) = \lim_{k\to\infty} \frac{|f_c^k(z)|}{(m+n)^k}$, in particular, if $c \in M$, then $G_c(c) = 0$. Moreover, if $c \to \partial M$, then $G_c(c) \to 0$ (see [4]). On the other hand, $\log|\Psi(c)| = \log|\Phi(\alpha c, c)| = G_c(\alpha c) = G_c(c)$, so if $c \to \partial M$, then $\log|\Psi(c)| \to 0$ and $|\Psi(c)| \to 1$. ■

Step 1

The set $\Psi^{-1}(w)$ is finite, for any $w \in \overline{\mathbf{C}} \setminus \overline{\mathbf{D}}$.

Proof

Suppose not, then there exists an infinite distinct sequence $\{c_k\}_{k\geq 1}$, such that $\Psi(c_k) = w$. If $\{c_k\}$ has an accumulation point c, then by the Identity Theorem, it is concluded that $\Psi(c) = w$, $\forall c \in \overline{\mathbf{C}} \setminus M$. This is a contradiction. Otherwise, $c_k \to \infty$ as $k \to \infty$. Therefore $\Psi(c_k) \to \infty$, so $w = \infty$, which again is a contradiction. ■

Step 2

Ψ is surjective.

Proof

If Ψ is not surjective, then there exists $w \in \overline{\mathbf{C}} \setminus \overline{\mathbf{D}}$ such that $w \notin \Psi(\overline{\mathbf{C}} \setminus M)$.

(a) If $w \in \overline{\Psi(\overline{\mathbf{C}} \setminus M)}$, then there exists a sequence $\{w_k\} \subset \Psi(\overline{\mathbf{C}} \setminus M)$ and $\{c_k\} \subset \overline{\mathbf{C}} \setminus M$ such that $\Psi(c_k) = w_k \to w$ as $k \to \infty$. By Lemma 6, if $c_k \to \partial M$, then $\Psi(c_k) \to \partial\mathbf{D}$. So w must belong to $\partial\mathbf{D}$, which is a contradiction. And if $c_k \to c \in \overline{\mathbf{C}} \setminus M$ as $k \to \infty$, then $\Psi(c_k) \to \Psi(c)$, so $\Psi(c) = w$, again a contradiction.

(b) $w \notin \overline{\Psi(\overline{\mathbf{C}} \setminus M)}$. In this case there exists an open neighborhood N around w in $\overline{\mathbf{C}} \setminus \overline{\mathbf{D}}$ such that $N \cap \Psi(\overline{\mathbf{C}} \setminus M) = \emptyset$. Let:

$$\mathcal{A} = \{N : N \cap \Psi(\overline{\mathbf{C}} \setminus M) = \emptyset,$$
$$N \text{ is an open neighborhood around } w \text{ in } \overline{\mathbf{C}} \setminus \overline{\mathbf{D}}\}$$

and $\mathcal{N} = \bigcup_{N\in\mathcal{A}} N$.

It is clear that for any $N \in \mathcal{A}$, $N \subseteq \mathcal{N}$. For any $w' \in \partial\mathcal{N} \setminus \mathcal{N}$, $w' \in \overline{\Psi(\overline{\mathbf{C}} \setminus M)}$. If not, then there exists some $w' \in \partial\mathcal{N} \setminus \mathcal{N}$, such that $w' \notin \overline{\Psi(\overline{\mathbf{C}} \setminus M)}$, then there exists N' around w' such that $N' \bigcap \Psi(\overline{\mathbf{C}} \setminus M) = \emptyset$. So $N' \bigcup \mathcal{N}$ is an open neighborhood around w. As a result $N' \bigcup \mathcal{N} \in \mathcal{A}$, $N' \bigcup \mathcal{N} \subset \mathcal{N}$, $N' \subset \mathcal{N}$ and $w' \in \mathcal{N}$, a contradiction. Hence, $w' \in \overline{\Psi(\overline{\mathbf{C}} \setminus M)}$, for any $w' \in \partial\mathcal{N} \setminus \mathcal{N}$. From the proof of part (a), it is concluded that for any $w' \in \partial\mathcal{N} \setminus \mathcal{N}$, $w' \in \partial\mathbf{D}$. Therefore, $\partial\mathcal{N} \subset \partial\mathbf{D}$ and, as a result, $\mathcal{N} = \overline{\mathbf{C}} \setminus \overline{\mathbf{D}}$ and $\Psi(\overline{\mathbf{C}} \setminus M) = \emptyset$, a contradiction again. ■

Step 3

Ψ is a conformal isomorphism.

Proof

Ψ is the composition of the following two maps:

$$c \overset{g}{\longmapsto} (\alpha c, c) \overset{\Phi}{\longmapsto} \Phi(\alpha c, c).$$

It is clear that $g : \overline{\mathbf{C}} \setminus M \longmapsto g(\overline{\mathbf{C}} \setminus M)$ is bijection and by Corollary 1, $\frac{\partial \Phi}{\partial z} \neq 0$, therefore, Ψ is locally injective and as a result, by Lemma 6, it is a (k-fold) covering space.

On the other hand, near infinity,

$$\Psi(c) = \Phi(\alpha c, c) \sim c(\frac{-n}{m})^{\frac{1}{n+m}},$$

so Ψ is a conformal isomorphism. ■

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REFERENCES

1. Ahmadi, D. and Rabii, M. "A combinatorial model for the dynamics of the family $z(z^n - b)$", *Proc. of the 26th AIMC*, Kerman, Iran, pp 3-11 (1995).
2. Beardon, A. F. *Iteration of Rational Function*, Springer-Verlag (1991).
3. Milnor, J. "Dynamics in one complex variable: Introductory lectures", Stony Brook, IMS Preprint (1990).
4. Carleson, L. and Gamelin, T.W., *Complex Dynamics*, Springer-Verlag (1992).
5. Mañé, R., Sad, P. and Sullivan, D. " On the dynamics of rational maps", *Ann. Sci. Es. Norm. Sup. Paris*, **16**(4), pp 193-217 (1983).