Markov Finite Approximation of Frobenius-Perron Operator for Higher-Dimensional Transformations with a Special Action Matrix

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In this paper, the set of finite rank approximations of Frobenius-Perron operators is extended for higher-dimensional transformations from projections to general finite rank operators through indicating an arbitrary action matrix. The convergence is proven in the case for which the action matrix is a special type of doubly-stochastic and tridiagonal.

INTRODUCTION

Let \( I = [0, 1] \) and \( \tau : I^n \rightarrow I^n \) be a piecewise expanding transformation. For \( n = 1 \), Lasota and Yorke [1] have proven the existence of an absolutely continuous invariant measure \( \mu \) with respect to Lebesgue measure. If \( f \) is the density of \( \mu \) with respect to Lebesgue measure \( m \) on \( I^n \), then it is well-known that \( f \) is the fixed point of the Frobenius-Perron operator \( P_\tau \). Ulam conjectured that it might be possible to construct finite-dimensional operators which approximate \( P_\tau \) and whose fixed points approximate the fixed point of \( P_\tau \) [2]. In [3-6], this conjecture is proven for a class of one-dimensional piecewise expanding transformations. Recently, the above conjecture is proven for general finite rank operators through indicating an arbitrary doubly-stochastic tridiagonal action matrix [7]. In [8], Jablonski has shown that a class of piecewise \( C^2 \) transformations of the \( n \)-dimensional cube \([0,1]^n\) has an absolutely continuous invariant measure. An \( n \)-dimensional version of Ulam conjecture was proven with finite rank projection operators for Jablonski transformations (see [9]). The aim of this paper is to prove the above version of Ulam conjecture for Jablonski transformations with general finite rank operators through indicating a particular doubly-stochastic tridiagonal action matrix.

Let \( m_j \) denote Lebesgue measure on \( I \). For \( j = n \), let \( m = m_n \). The space of all Lebesgue integrable functions on \( I^n \) is denoted by \( L^1 \) and the space of all essentially bounded (with respect to Lebesgue measure) functions on \( I^n \) by \( L^\infty \). The transformation \( \tau : I^n \rightarrow I^n \) is written as:

\[ \tau(x_1, \ldots, x_n) = (\varphi_1(x_1, \ldots, x_n), \ldots, \varphi_n(x_1, \ldots, x_n)), \]

where for \( i = 1, \ldots, n \), \( \varphi_i(x_1, \ldots, x_n) \) is a function from \( I^n \) into \([0,1] \).

A measurable transformation \( \tau : I^n \rightarrow I^n \) is nonsingular if \( m(\tau^{-1}(A)) = 0 \). For nonsingular \( \tau : I^n \rightarrow I^n \), Frobenius-Perron operator \( P_\tau : L^1 \rightarrow L^1 \) is defined by the formula:

\[ \int_A P_\tau f dx = \int_{\tau^{-1}(A)} f dx, \]

where \( A \subseteq I^n \) is measurable. It follows that for \( x = (x_1, \ldots, x_n) \),

\[ P_\tau f(x) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \int_{\tau^{-1}(\prod_{i=1}^n [0, x_i])} f(y) dy. \]

The operator \( P_\tau \) has many interesting properties [10].

Suppose \( \ell \) is a positive integer. An \( \ell^n \times \ell^n \) matrix (called action matrix) will be associated to each finite rank approximation operator, which in the case of projection will be identity. The convergence of this new
scheme is proven when the action matrix is a special type of doubly-stochastic and tridiagonal.

Let $\beta = \{D_1, \ldots, D_p\}$ be a partition of $I^n$ such that $p < \infty$, i.e.,

$$
\bigcup_{j=1}^{p} D_j = I^n, \; D_j \cap D_k = \emptyset \quad \text{for} \quad j \neq k.
$$

A partition $\beta$ of $I^n$ is called rectangular if for any $1 \leq j \leq p$, $D_j$ is an $n$-dimensional rectangle.

**Definition 1**

A transformation $\tau : I^n \rightarrow I^n$ is called a Jablonski transformation if it is defined on a rectangular partition of $I^n$ and is given by the formula:

$$
\tau(x_1, \ldots, x_n) = (\varphi_{ij}(x_1), \ldots, \varphi_{nj}(x_n)),
$$

where $(x_1, \ldots, x_n) \in D_j$, $1 \leq j \leq p$, $D_j = \prod_{n=1}^{a_{ij}} (b_{ij}, b_{ij})$ and $\varphi_{ij} : [a_{ij}, b_{ij}] \rightarrow [0, 1]$. If $b_{ij} = 1$ for some $i$, then $[a_{ij}, b_{ij})$ means $[a_{ij}, b_{ij}]$.

Cartesian product of the sets $A_i$ is denoted by $\prod_{i=1}^{n} A_i$, and $P_i$, the projection of $\mathbb{R}^n$ onto $\mathbb{R}^{n-1}$, is given by:

$$
P_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).
$$

Let $g : A \rightarrow \mathbb{R}$ be a function on the $n$-dimensional interval $A = \prod_{i=1}^{n} [a_i, b_i]$. For a fixed $i$, a function $V_i^A g$ with $n-1$ variables $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ is defined by the formula:

$$
V_i^A g \equiv V_i g = \sup \left\{ \sum_{k=1}^{r} g(x_1, \ldots, x_i, \ldots, x_n) - g(x_1, \ldots, x_i^{-1}, \ldots, x_n) \right\}.
$$

: $a_i = x_i^{0} < x_i^{-1} < \cdots < x_i^{r} = b_i, \quad r \in \mathbb{N}$.

For $f : A \rightarrow \mathbb{R}$, where $A = \prod_{i=1}^{n} [a_i, b_i]$, let:

$$
V_i^A f = \inf \left\{ \int_{P_i(A)} V_i g \; dm_{n-1} : g = f \right\}
$$

almost everywhere, $V_i g$ measurable,}

and $V^A f = \sup_{1 \leq i \leq n} V_i^A f$. If $V^A f < \infty$, then $f$ is a bounded variation function on $A$ and its total variation is $V^A f$.

In the next section, the finite rank operators based on an action matrix will be presented and the corresponding convergence theorems are proven.

**APPROXIMATION OF INVARIANT DENSITIES**

Let $\tau : I^n \rightarrow I^n$ be a Jablonski transformation and for any positive integer $\ell$, let $I^n$ be divided into $\ell^n$ subsets of equal measure $I_1, I_2, \ldots, I_{\ell^n}$ with:

$$
I_k = \left[ \frac{r_1 - 1}{\ell}, \frac{r_1}{\ell} \right] \times \left[ \frac{r_2 - 1}{\ell}, \frac{r_2}{\ell} \right] \times \cdots \times \left[ \frac{r_n - 1}{\ell}, \frac{r_n}{\ell} \right],
$$

for some $r_1, r_2, \ldots, r_n = 1, 2, \ldots, \ell$ and $m(I_k) = \frac{1}{\ell^n}$. Suppose $I_k$'s change according to the following quasi-code:

$$
k := 0;
$$

FOR $r_n := 1$ TO $\ell$ DO

FOR $r_{n-1} := 1$ TO $\ell$ DO

: FOR $r_1 := 1$ TO $\ell$ DO

$k := k + 1;
$$

$I_k := \left[ \frac{r_1 - 1}{\ell}, \frac{r_1}{\ell} \right] \times \cdots \times \left[ \frac{r_n - 1}{\ell}, \frac{r_n}{\ell} \right].$

For simplicity, in what follows, sometimes $\ell^n$ is denoted by $q$. Let:

$$
P_{st} = q m(I_s \cap \tau^{-1}(I_t)), \quad s, t = 1, \ldots, q.
$$

and $P_t = (P_{st})$ [9]. Suppose $\Delta_t$ is the $q$-dimensional linear subspace of $L^1$, spanned by $\{X_k\}_{k=1}^{q}$, where $X_k$ denotes the characteristic function of $I_k$. Now, suppose that $A_t = (a_{ij})$ is a given $q \times q$ doubly-stochastic and tridiagonal (DST) matrix and let $\tilde{P}_t = P_t A_t$. Note that $\tilde{P}_t$ may be considered as an operator $\tilde{P}_t = \tilde{P}_t(\tau) : \Delta_t \rightarrow \Delta_t$, given by:

$$
\tilde{P}_t(\tau) X_k = \sum_{i=1}^{q} \tilde{P}_{ki} X_i.
$$

Since the product of two stochastic matrices is stochastic, it is concluded that if $\Delta_1 = \{ \sum_{k=1}^{q} c_k X_k : c_k \geq 0 \text{ and } \sum_{k=1}^{q} c_k = 1 \}$, then $\tilde{P}_t$ maps $\Delta_1$ to a subset of $\Delta_1$; thus, there exists a fixed point $f_t \in \Delta_t$ of $\tilde{P}_t$ such that $\|f_t\|_1 = 1$ for all $\ell$ [6].

Let $\Delta = [v_1, v_p]$ be a $p$-dimensional subspace of $L^1$, spanned by $v_i \in L^1$, $i = 1, \ldots, p$. For given $u_i \in L^\infty$, $i = 1, \ldots, p$, the finite rank operator $Q_t : L^1 \rightarrow \Delta$ is defined by the tensor notation, $Q_t = \sum_{i=1}^{q} u_i \otimes v_i$, where for $f \in L^1, (u \otimes v)(f) = (f u) v$. The numbers

$$
\int_{I_n} u_j v_i = a_{ij} \quad (i, j = 1, \ldots, p)
$$

define a $p \times p$ matrix $A_t = (a_{ij})$, which is called the action matrix of the operator $Q_t$ (for notation and terminology see [11]).
Here, the choice of the functions \( v_i \) and \( u_j \) is as follows:

\[
\begin{align*}
v_i &= \chi_i, & i = 1, \ldots, q, \\
u_j &= c^i_j \chi_i + \cdots + c^i_q \chi_q, & j = 1, \ldots, q,
\end{align*}
\]

where:

\[
c^i_j = qa_j, \quad c^{i+1}_i = qa_i,
\]

\( a_j \geq 0 \) \((j = 1, \ldots, q)\) are the elements of the main diagonal of the DST matrix \( A_\ell \) and \( b_i \geq 0 \) \((i = 1, \ldots, q - 1)\) are the elements of the "diagonals" just above and below the main diagonal of the DST matrix \( A_\ell \).

The corresponding DST action matrix \( A_\ell = \text{tridiag}(a_j, b_i) \) of the operator \( Q_\ell \) satisfies:

\[
\begin{align*}
a_1 + b_1 &= 1, \\
a_{i-1} + a_i + b_i &= 1 & i &= 2, \ldots, q - 1, \\
-a_{q-1} + a_q &= 1.
\end{align*}
\]

It follows that:

\[
\begin{align*}
v_1 &= qa_1 \chi_1 + qb_1 \chi_1, \\
v_j &= q^2 b_{j-1} \chi_{j-1} + qa_j \chi_j + q b_j \chi_{j+1}, & j = 2, \ldots, q - 1, \\
v_q &= q^{q-1} b_{q-1} \chi_{q-1} + qa_q \chi_q.
\end{align*}
\]

Definition 2

For \( f \in L^1 \) and for any positive integer \( \ell \), \( Q_\ell : L^1 \rightarrow \Delta_\ell \) is defined by:

\[
Q_\ell(f) = \sum_{k=1}^{q} (u_k \otimes v_k)(f),
\]

where \( v_k = \chi_k, u_k = \sum_{j=1}^{q} c^k_j \chi_j \) \((k = 1, \ldots, q)\) and the corresponding action matrix of the operator \( Q_\ell \) is the matrix \( A_\ell \) defined as above.

Remark 1

It is not hard to prove that for any \( \ell \), the operator \( Q_\ell : L^1 \rightarrow \Delta_\ell \) is a Markov operator, and \( \tilde{P}_\ell f = Q_\ell P_\ell f \) for any \( f \in \Delta_\ell \) [6,7].

In the rest of this paper, it shall be assumed that \( A_\ell \) is a \( q \times q \) DST matrix of the form:

\[
A_\ell = \begin{bmatrix}
A & \otimes \\
\otimes & \ldots & A
\end{bmatrix},
\]

where \( A \) is an \( \ell \times \ell \) DST matrix given by:

\[
\begin{bmatrix}
a_1 & b_1 & 0 & 0 & \cdots & 0 \\
b_1 & a_2 & b_2 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\otimes & b_{\ell-2} & a_{\ell-1} & b_{\ell-1} & \cdots & \cdots \\
0 & \cdots & 0 & b_{\ell-1} & a_\ell
\end{bmatrix}
\]

In what follows it is assumed that the action matrix of the operator \( Q_\ell \) in Definition 2 is of the Form 2. Clearly,

\[
\begin{align*}
c^{i+1}_j &= qa_j, & (j = 1, \ldots, \ell; \ p = 0, \ldots, \ell^n-1 - 1), \\
c^{i+1}_i &= qa_i, & (i = 1, \ldots, \ell - 1; \ p = 0, \ldots, \ell^{n-1}-1).
\end{align*}
\]

Remark 2

It is clear that for \( f \in L^1 \) as \( \ell \to \infty \), the sequence \{\( Q_\ell f \)\} converges in \( L^1 \) to \( f \) and in particular if \( f \in \Delta_\ell \), then the sequence \{\( \tilde{P}_\ell f \)\} converges in \( L^1 \) to \( P_\ell f \). For details see [6,7].

Lemma 1

If \( f \in L^1 \), then \( V^{\ell^n} Q_\ell f \leq 3 V^{\ell^n} f \).

Proof

For any \( 1 \leq k \leq q \), let \( I_k = \prod_{i=1}^{n} [(r_i - 1)/\ell, r_i/\ell) = \prod_{i=1}^{n} J_r \), where, for \( 1 \leq i \leq n, r_i \) assumes the values \( 1, 2, \ldots, \ell \). \( I_k \) change according to the quasi-code Relation 1. For \( k = 1, \ldots, q, m(I_k) = \prod_{i=1}^{n} m(J_{r_i}) = \frac{1}{q} \). Let:

\[
Q_{\ell_1}(f) = \sum_{r_1=1}^{\ell} (u_{r_1} \otimes v_{r_1})(f),
\]

where \( v_{r_1} = \chi_{r_1}(x_1), u_{r_1} = \sum_{j=1}^{\ell} c^j_{r_1} v_j \) \((r_1 = 1, \ldots, \ell)\), \( (u_{r_1} \otimes v_{r_1})(f) = (\int f u_{r_1}(dx_1)v_{r_1}) \) and the corresponding action matrix of the operator \( Q_{\ell_1} \) is as Form 3. Hence, \( c^j_{r_1} = la_j \) and \( c^{j+1}_{r_1} = lb_j \) \((j = 1, \ldots, \ell; \nu = 1, \ldots, \ell - 1)\). For \( i = 2, \ldots, n \), the following is defined:

\[
Q_{\ell_i}(f) = \sum_{r_i=1}^{\ell} (u_{r_i} \otimes v_{r_i})(f),
\]

where \( v_{r_i} = \chi_{r_i}(x_i), u_{r_i} = \sum_{j=1}^{\ell} c^j_{r_i} v_j \) \((r_i = 1, \ldots, \ell)\), \( (u_{r_i} \otimes v_{r_i})(f) = (\int f u_{r_i}(dx_i)v_{r_i}) \) and the corresponding action matrix of the operator \( Q_{\ell_i} \) is an
$\ell \times \ell$ identity matrix. Hence, $c_{i}^j = \ell$ and $c_{j}^j = 0$ for $\nu \neq j$ ($j, \nu = 1, \ldots, \ell$). It is obtained that:

$$Q_{\ell}f(x) = Q_{\ell_1}Q_{\ell_2} \cdots Q_{\ell_r}f(x) = (\prod_{i=1}^{\ell} Q_{\ell_i})f(x).$$

It follows that (see [6,7]):

$$V_{\ell_i}^n Q_{\ell_i}f = V_{\ell_i}^n Q_{\ell_i}((\prod_{j=1}^{\ell} Q_{\ell_j})f) = V_{\ell_i}^n Q_{\ell_i}(\prod_{j=1, j \neq i}^{\ell} Q_{\ell_j})f \leq \left( \begin{array}{c} V_{\ell_i}^n (\prod_{j=1, j \neq i}^{\ell} Q_{\ell_j})f \\ 3V_{\ell_i}^n (\prod_{j=1, j \neq i}^{\ell} Q_{\ell_j})f \end{array} \right) \text{ if } i \neq 1,$$

$$\leq V_{\ell_i}^n Q_{\ell_i}((\prod_{j=1, j \neq i}^{\ell} Q_{\ell_j})f) \text{ if } i = 1.$$

Now it is shown that for any $i$,

$$\int_{f_{\ell_i}} V_{\ell_i}^n (\prod_{j=1, j \neq i}^{\ell} Q_{\ell_j})f(\prod_{j=1, j \neq i}^{\ell} dx_j) \leq \int_{f_{\ell_i}} V_{\ell_i}^n f(\prod_{j=1, j \neq i}^{\ell} dx_j). \tag{4}$$

For $i = 1$, the Relation 4 was proven in Lemma 6 of [9]. If $i \neq 1$, then for $t = 1, 2, \ldots, \ell^{-n-1}$ let $I_{t}^{n-1} = \prod_{j=1, j \neq i}^{\ell} J_{r_j}$, where $r_j$ assume the values $1, 2, \ldots, \ell$. $I_{t}^{n-1}$ change according to the quasi-code Relation 1 when $k$ is replaced by $t$, $l_k$ by $I_{t}^{n-1}$ and the loop for $r_i$ is eliminated. It is clear that for $t = 1, 2, \ldots, \ell^{-n-1}$, $m(I_{t}^{n-1}) = \prod_{j=1, j \neq i}^{\ell} m(J_{r_j}) = \ell^{-n-1}$. For simplification, notation $f(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, x_i, \ldots, x_n)$ is denoted by $f_i^{k, k-1}$, $\prod_{j=1, j \neq i}^{\ell} dx_j$ by $\prod_{j=1, j \neq i}^{\ell} dx_j$ and $\ell^{-n-1}$ by $M$. It is obtained that:

$$\prod_{j=1, j \neq i}^{\ell} Q_{\ell_j}f(x_1, \ldots, x_i, \ldots, x_n) = \sum_{t=1}^{M} (u_t \otimes v_t)(f),$$

where:

$$v_t = \chi_{I_{t-1}^{n-1}}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \equiv \chi_{i},$$

$$u_t = \sum_{\nu=1}^{M} c_{\nu}^i v_{\nu} \quad (t = 1, \ldots, M),$$

$$(u_t \otimes v_t)(f) = \int_{f_{\ell_i}} f u_t \, dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n v_t,$$

and the corresponding action matrix of the operator $\prod_{j=1, j \neq i}^{\ell} Q_{\ell_j}$ is similar to Form 2, but the number of iterations of the matrix $A$ defined by Form 3 equals $\ell^{-n-2}$. Hence:

$$c_{i+1, j+1}^{p+1} = M a_{i} \quad (j = 1, \ldots, \ell; \quad p = 0, \ldots, \ell^{-n-2} - 1),$$

$$c_{i+1, j+1}^{p+1} = c_{i+1, j+1}^{p+1} = M b_{i} \quad (\nu = 1, \ldots, \ell - 1; \quad \nu = 0, \ldots, \ell^{n-2} - 1).$$

For any $0 = x_0 < x_1 < \cdots < x_{i-1} < x_i = 1$,

$$\sum_{k=1}^{r} \left( \sum_{j=1, j \neq i}^{\ell} Q_{\ell_j}f(x_1, \ldots, x_i, \ldots, x_n) \right) \leq \sum_{k=1}^{r} \left( \sum_{j=1, j \neq i}^{\ell} f^{k, k-1}_i \right).$$

Recall that if $f, g \in L^1$ and $f = g$ a.e., then $Q_{\ell}f = Q_{\ell}g$ a.e. Also, the measurability of $V_{\ell_i}^n g$ implies the
measurability of $V^* Q_{fg}$. Now,
\[ V^* Q_{fg} = \inf \left\{ \int_{I^n} V^* h(\prod_{j=1, j\neq i}^n dx_j) : h = Q_{fg} \text{ a.e., } V^* h \text{ measurable} \right\} \]
\[ \leq \inf \left\{ \int_{I^n} V^* Q_{fg}(\prod_{j=1, j\neq i}^n dx_j) : g = f \right\} \]
\[ \text{a.e., } V^* g \text{ measurable} \]
\[ \leq 3 \inf \left\{ \int_{I^n} V^* (\prod_{j=1, j\neq i}^n Q_{gj}) g(\prod_{j=1, j\neq i}^n dx_j) : g = f \right\} \]
\[ \text{a.e., } V^* g \text{ measurable} \]
\[ \leq 3 \inf \left\{ \int_{I^n} V^* g(\prod_{j=1, j\neq i}^n dx_j) : g = f \right\} \]
\[ \text{a.e., } V^* g \text{ measurable} \]
\[ = 3V^* f. \]

Therefore,
\[ V^* Q_{fg} = \max_{1 \leq i \leq n} V^* Q_{fg} \leq 3 \max_{1 \leq i \leq n} V^* f = 3V^* f. \]

The following result is established in [8].

**Theorem 1**

Let $\tau$ be a Jablonski transformation, where:
\[ \tau(x) = (\varphi_{i1}(x_1), \ldots, \varphi_{in}(x_n)), x \in D_j. \]

If $\lambda = \inf_{i,j} \inf_{[a_{ij}, b_{ij}]} \|\varphi_{ij}\| > 2$, then for any $f \in L^1$:
\[ V^* P_{\tau f} \leq K_r \|f\| + \alpha V^* f, \]
where $K_r$ is a constant depending on $\tau$ and $\alpha = 2^{1-\frac{1}{2}} < 1$.

**Lemma 2**

Let $\tau$ be a Jablonski transformation,
\[ \tau(x) = (\varphi_{i1}(x_1), \ldots, \varphi_{in}(x_n)), x \in D_j. \]

and $f_\ell \in \Delta_\ell$ be any fixed point of $P_\ell(\tau)$ with $\|f_\ell\| = 1$.

If:
\[ \lambda = \inf_{i,j} \inf_{[a_{ij}, b_{ij}]} \|\varphi_{ij}\| > 6, \]

then the sequence $\{V^* f_\ell\}_{\ell=1}^\infty$ is bounded.

**Proof**

By Remark 1, $f_\ell = \hat{P}_\ell f_\ell = Q_{f_\ell} P_{\tau} f_\ell$ for all $\ell$. Hence, by Lemma 1 and Theorem 1, it is obtained that:
\[ V^* f_\ell = V^* Q_{f_\ell} P_{\tau} f_\ell \leq 3V^* f_\ell \]
\[ \leq 3(K_r \|f_\ell\| + \alpha V^* f_\ell) = 3K_r + 3\alpha V^* f_\ell, \]
where $K_r > 0$ and $0 < \alpha < \frac{1}{2}$. Since $V^* f_\ell < \infty$,
\[ V^* f_\ell \leq 3K_r/(1 - 3\alpha). \]

The following self-adjoint property of $Q_{\ell}$, which was not needed in [6], plays a vital role in the sequel.

**Lemma 3**

For any $f \in L^1$, $\ell = 1, 2, \ldots,$ and measurable subset $A$ of $I^n$
\[ \int_{I^n} \chi_A Q_{\ell} f \, dx = \int_{I^n} f \, Q_{\ell} \chi_A \, dx. \]

**Proof**

\[ \int_{I^n} \chi_A(x) Q_{\ell} f(x) \, dx \]
\[ = \int_{I^n} \chi_A(x) \left( \sum_{k=1}^q \int_{I^n} f(y) \sum_{j=1}^q c_j^k \chi_j(x)dy \right) \chi_k(x) \, dx \]
\[ = \sum_{k=1}^q \int_{I^n} f(y) \sum_{j=1}^q c_j^k \chi_j(x)dy \int_{I^n} \chi_A(x) \chi_k(x) \, dx \]
\[ = \sum_{k=1}^q \int_{I^n} f(y) c_k^\ell \chi_k(x) \, dx \]
\[ = \sum_{k=1}^q \int_{I^n} f(y) c_k^\ell \chi_k(x) \, dx \]
\[ = \int_{I^n} f(x) \sum_{k=1}^q \int_{I^n} \chi_k(y) \sum_{j=1}^q c_j^k \chi_k(x)dy \chi_j(x) \, dx \]
\[ = \int_{I^n} f(x) Q_{\ell} \chi_A(x) \, dx. \]

**Theorem 2**

Let $\tau$ be a nonsingular Jablonski transformation with partition $\{D_1, \ldots, D_q\}$ and
\[ \lambda = \inf_{i,j} \inf_{[a_{ij}, b_{ij}]} \|\varphi_{ij}\| > 6. \]

Suppose $P_{\tau}$ has a unique fixed point. Then for any positive integer $\ell$, $P_{\tau}(\lambda)$ has a fixed point $f_\ell$ in $\Delta_\ell$ with $\|f_\ell\| = 1$ and the sequence $\{f_\ell\}$ converges weakly to the fixed point of $P_{\tau}$.

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Proof
From Lemma 3 of [8] and Lemma 2, it is known that the set \( \{ f_\ell \}_{\ell=1}^\infty \) is weakly and relatively compact in \( L^1 \). Let \( \{ f_\ell \} \) be a weakly convergent subsequence of \( \{ f_\ell \}_{\ell=1}^\infty \) and let \( f = \lim_{j \to \infty} f_\ell \) weakly. Then, for any \( g \in L^\infty \),
\[
\left| \int_{I^n} g(f - P_\tau f) \, dx \right| \leq \int_{I^n} |g(f - f_\ell)| \, dx
\]
\[
+ \int_{I^n} |g(f_\ell - Q_\ell P_\tau f_\ell)| \, dx
\]
\[
+ \int_{I^n} |g(Q_\ell P_\tau f_\ell - P_\tau f)| \, dx. \tag{5}
\]

The first term approaches zero since \( f_\ell \) converges weakly to \( f \) as \( j \to \infty \). From Remark 1, \( Q_\ell P_\tau f_\ell = \tilde{P}_\ell f_\ell = f_\ell \), hence, the second term is identically zero.

Now the last term is considered. Because of the weak continuity of \( P_\tau \), \( P_\tau f_\ell \) converges weakly to \( P_\tau f \) as \( j \to \infty \). It will be proven that \( Q_\ell P_\tau f_\ell \) converges weakly to \( P_\tau f \) as \( j \to \infty \). It is enough to show that for any measurable subset \( A \) of \( I^n \), the following is obtained:
\[
\lim_{j \to \infty} \int_{I^n} \chi_A Q_\ell h_\ell \, dx = \int_{I^n} \chi_A h \, dx,
\]
where \( h_\ell = P_\tau f_\ell \) and \( h = P_\tau f \).

From Corollary IV.8.11 in [12, p 294],
\[
\int_{E} h_\ell(x) \, dx \to 0 \quad \text{as} \quad m(E) \to 0
\]
uniformly in \( j \). Because \( \|h_\ell\| = 1 \) and \( h_\ell \geq 0 \), from Theorem 7.5.3 in [13, p 296], \( h_\ell \)'s are uniformly integrable, i.e.,
\[
\int_{\{h_\ell \geq K\}} |h_\ell| \, dx \to 0 \quad \text{as} \quad K \to \infty,
\]
uniformly in \( j \). Therefore, for any \( \epsilon > 0 \), there exists \( K > 0 \) such that for all \( j \):
\[
2 \int_{\{h_\ell \geq K\}} |h_\ell| \, dx < \epsilon.
\]

Hence,
\[
\left| \int_{I^n} h_\ell(Q_\ell \chi_A - \chi_A) \, dx \right|
\]
\[
\leq \int_{I^n} |h_\ell||Q_\ell \chi_A - \chi_A| \, dx
\]
\[
= \int_{\{h_\ell \geq K\}} |h_\ell||Q_\ell \chi_A - \chi_A| \, dx
\]
\[
+ \int_{\{h_\ell < K\}} |h_\ell||Q_\ell \chi_A - \chi_A| \, dx
\]
\[
\leq 2 \int_{\{h_\ell \geq K\}} |h_\ell| \, dx + K \int_{\{h_\ell < K\}} |Q_\ell \chi_A - \chi_A| \, dx
\]
\[
\leq 2 \int_{\{h_\ell \geq K\}} |h_\ell| \, dx + K \int_{\{h_\ell < K\}} \chi_A \, dx.
\]
The first term is less than \( \epsilon \) and by Remark 2 the second term approaches zero as \( j \to \infty \). Thus:
\[
\lim_{j \to \infty} \int_{I^n} h_\ell(Q_\ell \chi_A - \chi_A) \, dx = 0.
\]

By Lemma 3,
\[
\lim_{j \to \infty} \int_{I^n} \chi_A Q_\ell h_\ell \, dx = \lim_{j \to \infty} \int_{I^n} h_\ell Q_\ell \chi_A \, dx
\]
\[
= \lim_{j \to \infty} \int_{I^n} h_\ell(Q_\ell \chi_A - \chi_A) \, dx
\]
\[
+ \lim_{j \to \infty} \int_{I^n} h_\ell \chi_A \, dx
\]
\[
= \int_{I^n} \chi_A \, dx.
\]

This means that the last term in Relation 5 approaches zero.

Therefore, it is established that for any \( g \in L^\infty \),
\[
\int_{I^n} g(x)(f(x) - P_\tau f(x)) \, dx = 0.
\]
It follows that \( P_\tau f(x) = f(x) \) almost everywhere. Therefore, any weakly convergent subsequence of \( \{ f_\ell \} \) converges weakly to a unique fixed point of \( P_\tau \). Hence, \( f_\ell \to f \) weakly as \( \ell \to \infty \).

Corollary 1
If the fixed point of \( P_\tau \) is not unique in Theorem 2, then any weak limit point of \( \{ f_\ell \}_{\ell=1}^\infty \) is a fixed point of \( P_\tau \).
Theorem 3
Let $\tau$ be a nonsingular Jablonski transformation with $\lambda = \inf_{i,j}(\inf_{[a_{i,j}, b_{i,j}]} |\psi_{i,j}|) > 1$. Suppose $P_\tau$ has a unique fixed point. For an integer $k$ such that $\lambda^k > 6$, let $\phi = \tau^k$ and $f_k$ be a fixed point of $P_{\tau}(\phi)$. Define:
\[ g_k = \frac{1}{k} \sum_{j=0}^{k-1} P_{\tau^j} f_k. \]
Then, $\{g_k\}$ converges weakly to the fixed point of $P_\tau$.

Proof
Since $P_\tau$ is a weakly continuous operator [10, p 43], Theorem 2 implies that $g_k \to g = \frac{1}{k} \sum_{j=0}^{k-1} P_{\tau^j} f$ weakly as $k \to \infty$. Therefore,
\[ P_\tau g = \frac{1}{k} \sum_{j=1}^{k} P_{\tau^j} f = \frac{1}{k} \sum_{j=0}^{k-1} P_{\tau^j} f = g, \]
where $f$ is the fixed point of $P_\phi = P_{\tau^k}$, i.e., $P_{\tau^k} f = f$.

Corollary 2
If the fixed point of $P_\tau$ is not unique in Theorem 3, then any weak limit point $f$ of $\{f_k\}_{k=0}^\infty$ is a fixed point of $P_\phi$ and $g = \frac{1}{k} \sum_{j=0}^{k-1} P_{\tau^j} f$ is a fixed point of $P_\tau$. If $f_k \to f$ weakly as $k \to \infty$, then $g_k = \frac{1}{k} \sum_{j=0}^{k-1} P_{\tau^j} f_k \to g$ weakly as $k \to \infty$.

References