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Phragmen-Lindelöf Type Theorem for a Class of Quasi-Linear Fourth Order Parabolic Equations

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In this paper, a Phragmen-Lindelöf type theorem for a class of quasi-linear fourth order parabolic equations is proved.

INTRODUCTION

In this paper, the spatial behavior of solutions of initial-boundary value problems for a class of quasi-linear parabolic equations is considered. More precisely, a Phragmen-Lindelöf type theorem is proved for a class of parabolic equations of the form,

$$u_t + \Delta^2 u - \Delta f(u) = 0.$$

Under a growth condition on nonlinearity f, the growth rate of the nontrivial solutions of an initial-boundary value problem for the above equation in unbounded cylindrical domains with homogeneous boundary conditions is established.

Phragmen-Lindelöf type theorems for some classes of nonlinear elliptic and parabolic equations have been obtained previously [1-13].

The results established here mainly follow the ideas in [3,5,7], in which Phragmen-Lindelöf type theorems for some semi-linear fourth order elliptic and second order parabolic and Navier-Stokes equations have been derived.

PRELIMINARIES

Let:

$$\Omega = \{x \in \mathbf{R}^n : x_1 \in \mathbf{R}^+,$$
$$x' = (x_2, x_3, ..., x_n) \in \tilde{\sigma}_{x_1} \subset \mathbf{R}^{n-1}\},$$

be the interior of a semi-infinite cylindrical domain where,

$$\sigma_\tau = \{(x_1, x') \in \Omega : \quad x_1 = \tau\},$$

and let $\tau \to \tilde{\sigma}_\tau$ be a mapping from $\mathbf{R}^+$ into the family of bounded domain subsets of $\mathbf{R}^{n-1}$; furthermore, suppose:

$$0 < \alpha \leq \inf_\tau mes\ \tilde{\sigma}_\tau \leq \sup_\tau mes\ \tilde{\sigma}_\tau \leq \beta,$$

and $\partial\sigma_\tau$ is smooth. Let:

$$\Omega_\tau = \{(x_1, x') \in \Omega : \quad 0 < x_1 < \tau\}.$$

The following initial-boundary value problem is considered:

$$u_t + \Delta^2 u = \Delta f(u),$$
$$(x_1, x', t) \in Q_T := \ \Omega \times [0, T), \tag{1}$$

$$u = \frac{\partial u}{\partial \nu} = 0, \quad (x_1, x', t) \in \partial\Omega \times [0, T), \tag{2}$$

$$u(x_1, x', 0) = 0, \quad (x_1, x') \in \Omega. \tag{3}$$

It is assumed that $f \in C^2(\mathbf{R})$, $f(0) = 0$,

$$f'(u) \geq 0 \quad \forall\ u \geq 0, \tag{4}$$

$$|f'(u)| \leq A_0 |u|^{\frac{m}{2}-1} \quad \forall\ u \in \mathbf{R}, \tag{5}$$

where $2 < m \leq \frac{2n}{n-2}$ for $n > 2$, $2 < m < \infty$ for $n = 2$ and A_0 is a positive constant.

Throughout the article, the following notations will be employed:

$$\partial_i = \frac{\partial}{\partial x_i}, \quad \partial_i^k = \frac{\partial^k}{\partial {x_i}^k}, \quad |\nabla u|^2 = \sum_{i=1}^n (\partial_i u)^2,$$

$$|\nabla^2 u|^2 = \sum_{i,j=1}^n (\partial_i \partial_j u)^2, \quad \Delta = \sum_{i=1}^n \partial_i^2,$$

$$\Delta^2 = \sum_{i,j=1}^n \partial_i^2 \partial_j^2, \quad ||u||_\Omega^2 = \int_\Omega u^2 dx,$$

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and $\frac{\partial u}{\partial \nu}$ will represent the exterior normal derivative of u.

For further considerations, the following technical lemmas are important.

Lemma 1

Let $D \subset \mathbf{R}^n$ be a bounded region. Then, $W_0^{2,2}(D) \subset L^p(D)$, for $2 < p \leq \frac{2n}{n-4}$ if $n > 4$ and $1 \leq p < \infty$ if $n \leq 4$. This means that a constant, C, exists which depends upon D, n and p such that:

$$\int_D |u|^p dx \leq C\Big\{\int_D |\nabla^2 u|^2 dx\Big\}^{\frac{p}{2}},$$

for every $u \in W_0^{2,2}(D)$ [14].

Lemma 2

Let Ψ be a monotone increasing function with $\Psi(0) = 0$, $\lim_{\tau\to\infty}\Psi(\tau) = +\infty$. Then $z(\tau) > 0$ satisfying $z(\tau) \leq \Psi(z'(\tau))$, $\tau \geq 0$, tends to $+\infty$ when $\tau \to \infty$. If $\Psi(\tau) \leq c_0\tau^p$ for some c_0 and $p > 1$ for $\tau \geq \tau_1$, then $\underline{lim}_{\tau\to\infty}\, \tau^{\frac{-p}{p-1}} z(\tau) > 0$ [8].

THEOREM 1

If Statements 4 and 5 hold, then for every positive t each nontrivial solution u of the initial-boundary value Problem 1 to 3 satisfies:

$$\underline{lim}_{\tau\to\infty}\tau^{-\frac{2}{m-2}} \int_0^T \| \,|\nabla^2 u(.,t)|\, \|^2_{\Omega_\tau} dt > 0.$$

It should be noted that Theorem 1 obtained here is different from Phragmen-Lindelöf type theorems in [2,13]. In those articles, Phragmen-Lindelöf type theorems are obtained under the condition that the relevent solutions tend to zero as $|x| \to \infty$ and the energy integral with respect to the whole domain is finite. The solutions considered here do not obey such restrictions. In particular, in the paper by Vafeades and Horgan [13], Phragmen-Lindelöf type theorem for Karman system under the condition of finiteness of the energy is established.

PROOF OF THEOREM 1

Let u be any nontrivial solution of the initial-boundary value Problem 1 to 3. Multiplying Equation 1 by u and integrating with respect to x over Ω_τ yields:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_\tau} u^2\, dx + \int_{\Omega_\tau} u\Delta^2 u\, dx = \int_{\Omega_\tau} u\Delta f(u) dx. \quad (6)$$

It is not difficult to see that:

$$u\Delta^2 u = \sum_{i=1}^n u\partial_i^4 u + 2\sum_{\substack{i,j=1\\ j>i}}^n u\partial_i^2\partial_j^2 u$$

$$= |\nabla^2 u|^2 + \sum_{i=1}^n \partial_i[u\partial_i^3 u - \partial_i u\partial_i^2 u$$

$$-2\sum_{j=i+1}^n \partial_j u\partial_i\partial_j u] + 2\sum_{\substack{i,j=1\\ j>i}}^n \partial_j(u\partial_i^2\partial_j u).$$

By Stokes formula and boundary Condition 2,

$$I_1 := \int_{\Omega_\tau} u\Delta^2 u\, dx$$

$$= \int_{\Omega_\tau} |\nabla^2 u|^2 dx + \int_{\sigma_\tau} [u\partial_1^3 u - \partial_1 u\partial_1^2 u$$

$$- 2\sum_{j=2}^n \partial_j u\partial_1\partial_j u] dx' \quad (7)$$

and:

$$I_2 := \int_{\Omega_\tau} u\Delta f(u) dx$$

$$= -\int_{\Omega_\tau} f'(u)\sum_{i=1}^n(\partial_i u)^2 dx + \int_{\sigma_\tau} uf'(u)\partial_1 u dx'. \quad (8)$$

Taking Conditions 4 and 5 into account, from Statement 8, the following inequality is obtained:

$$I_2 \leq A_0\int_{\sigma_\tau} |u|^{\frac{m}{2}}|\partial_1 u| dx'. \quad (9)$$

On the other hand,

$$u\partial_1^3 u - \partial_1 u\partial_1^2 u - 2\sum_{j=2}^n \partial_j u\partial_1\partial_j u$$

$$= \partial_1(u\partial_1^2 u) - \sum_{j=1}^n \partial_1(\partial_j u)^2$$

$$= \partial_1[u\partial_1^2 u - \sum_{j=1}^n (\partial_j u)^2].$$

Coupling these identities with Statement 7, it is obtained that:

$$I_1 = \int_{\Omega_\tau} |\nabla^2 u|^2 dx + \int_{\sigma_\tau} \partial_1[u\partial_1^2 u - \sum_{j=1}^n(\partial_j u)^2] dx'. \quad (10)$$

Equation 6 along with Statements 9 and 10 yields:

$$\frac{1}{2}\frac{d}{dt}||u(.,t)||^2_{\Omega_\tau} + || \, |\nabla^2 u(.,t)| \, ||^2_{\Omega_\tau}$$
$$\leq -\int_{\sigma_\tau} \partial_1[u\partial_1^2 u - \sum_{j=1}^{n}(\partial_j u)^2]dx'$$
$$+ A_0 \int_{\sigma_\tau} |u|^{\frac{m}{2}}|\partial_1 u|dx'.$$

After an integration with respect to τ,

$$\int_0^\tau \Big\{\frac{1}{2}\frac{d}{dt}||u(.,t)||^2_{\Omega_s} + || \, |\nabla^2 u(.,t)| \, ||^2_{\Omega_s}\Big\}ds$$
$$\leq \int_{\sigma_\tau}[|\nabla u|^2 - u\partial_1^2 u]dx' + A_0\int_{\Omega_\tau}|u|^{\frac{m}{2}}|\partial_1 u|dx. \tag{11}$$

Integrating the above inequality with respect to t in the interval $[0,T)$ and utilizing initial Condition 3 gives:

$$\int_0^\tau \Big\{\frac{1}{2}||u(.,T)||^2_{\Omega_s} + \int_0^T || \, |\nabla^2 u(.,t)| \, ||^2_{\Omega_s}dt\Big\}ds$$
$$\leq \int_0^T \Big(\int_{\sigma_\tau}[|\nabla u|^2 - u\partial_1^2 u]dx'\Big)dt$$
$$+ A_0\int_0^T \Big(\int_{\Omega_\tau}|u|^{\frac{m}{2}}|\partial_1 u|dx\Big)dt. \tag{12}$$

Now, each term on the right hand side of the above statement will be considered separately. Schwarz inequality is used to get:

$$J_1 := \int_0^T \Big\{\int_{\sigma_\tau}[|\nabla u|^2 - u\partial_1^2 u]dx'\Big\}dt$$
$$\leq \int_0^T \Big\{\int_{\sigma_\tau}|\nabla u|^2 dx'$$
$$+ \Big(\int_{\sigma_\tau}u^2 dx'\Big)^{\frac{1}{2}}\Big(\int_{\sigma_\tau}|\partial_1^2 u|^2 dx'\Big)^{\frac{1}{2}}\Big\}dt, \tag{13}$$

and:

$$J_2 := A_0\int_0^T \Big\{\int_{\Omega_\tau}|u|^{\frac{m}{2}}|\partial_1 u|dx\Big\}dt$$
$$\leq A_0\int_0^T \Big\{\Big(\int_{\Omega_\tau}|u|^m dx\Big)^{\frac{1}{2}}\Big(\int_{\Omega_\tau}|\partial_1 u|^2 dx\Big)^{\frac{1}{2}}\Big\}dt. \tag{14}$$

Recall inequality $2|A|\,|B| \leq \varepsilon(A)^2 + \frac{1}{\varepsilon}(B)^2$, which holds for positive A, B and ε, thus:

$$J_1 \leq \int_0^T \Big\{\int_{\sigma_\tau}|\nabla u|^2 dx' + \frac{\varepsilon}{2}\int_{\sigma_\tau}u^2 dx'$$
$$+ \frac{1}{2\varepsilon}\int_{\sigma_\tau}|\partial_1^2 u|^2 dx'\Big\}dt \tag{15}$$

and:

$$J_2 \leq A_0\int_0^T \Big\{\frac{\varepsilon}{2}\int_{\Omega_\tau}|u|^m dx + \frac{1}{2\varepsilon}\int_{\Omega_\tau}|\partial_1 u|^2 dx\Big\}dt. \tag{16}$$

Recall Poincaré-Friedrichs inequality:

$$\lambda\int_D u^2 dx \leq \int_D |\nabla u|^2 dx, \tag{17}$$

where λ is the first eigenvalue of the Laplacian operator in D with homogeneous Dirichlet boundary conditions. Because of the above inequality,

$$J_1 \leq \Big(\lambda_1^{-1}(\tau) + \frac{\varepsilon}{2}\lambda_1^{-2}(\tau) + \frac{1}{2\varepsilon}\Big)$$
$$\int_0^T \Big(\int_{\sigma_\tau}|\nabla^2 u|^2 dx'\Big)dt, \tag{18}$$

and:

$$J_2 \leq A_0\int_0^T \Big\{\frac{\varepsilon}{2}\int_{\Omega_\tau}|u|^m dx$$
$$+ \frac{1}{2\varepsilon}\lambda_2^{-1}(\tau)\int_{\Omega_\tau}|\nabla^2 u|^2 dx\Big\}dt, \tag{19}$$

where $\lambda_2(\tau)$ depends on Ω_τ.

Lemma 1 and Inequality 19 yield:

$$J_2 \leq A_0\Big\{\frac{\varepsilon}{2}C(\tau)\Big(\int_0^T || \, |\nabla^2 u(.,t)| \, ||^2_{\Omega_\tau}dt\Big)^{\frac{m}{2}}$$
$$+ \frac{\lambda_2^{-1}(\tau)}{2\varepsilon}\int_0^T || \, |\nabla^2 u(.,t)| \, ||^2_{\Omega_\tau}dt\Big\}. \tag{20}$$

Neglecting the first term on the left hand side of Statement 12 and using Inequalities 18 and 20, it is deduced that:

$$\int_0^\tau \Big(\int_0^T || \, |\nabla^2 u(.,t)| \, ||^2_{\Omega_s}dt\Big)ds$$
$$\leq d_1(\tau)\int_0^T \Big(\int_{\sigma_\tau}|\nabla^2 u|^2 dx'\Big)dt$$
$$+ d_2(\tau)\int_0^T || \, |\nabla^2 u(.,t)|\,||^2_{\Omega_\tau}dt$$
$$+ \frac{\varepsilon A_0 C(\tau)}{2}\Big\{\int_0^T || \, |\nabla^2 u(.,t)| \, ||^2_{\Omega_\tau}dt\Big\}^{\frac{m}{2}}, \tag{21}$$

where,

$$d_1(\tau) := \lambda_1^{-1}(\tau) + \frac{\varepsilon}{2}\lambda_1^{-2}(\tau) + \frac{1}{2\varepsilon}, \tag{22}$$

$$d_2(\tau) := \frac{A_0\lambda_2^{-1}(\tau)}{2\varepsilon}. \tag{23}$$

Substituting Statements 22 and 23 into Statement 21 gives an inequality involving,

$$E(\tau,T) := \int_0^T \| \, |\nabla^2 u(.,t)| \, \|^2_{\Omega_\tau} dt, \tag{24}$$

the strain energy contained in Ω_τ. Precisely, the following nonlinear integro-differential inequality is obtained:

$$\int_0^\tau E(s,T)ds \le d_1(\tau)E'(\tau,T) + d_2(\tau)E(\tau,T) + \frac{\varepsilon A_0 C(\tau)}{2}[E(\tau,T)]^{\frac{m}{2}}, \tag{25}$$

where,

$$E'(\tau,T) = \frac{d}{d\tau}\int_0^T \| \, |\nabla^2 u(.,t)| \, \|^2_{\Omega_\tau} dt = \int_0^T \Big(\int_{\sigma_\tau} |\nabla^2 u|^2 dx'\Big)dt. \tag{26}$$

The next objective is to solve the nonlinear integro-differential Inequality 25. Introducing,

$$F(\tau,T) = E(\tau,T) + \int_0^\tau E(s,T)ds,$$

in Inequality 25 results in:

$$F(\tau,T) \le (1 + d_1(\tau) + d_2(\tau))F'(\tau,T) + \frac{\varepsilon A_0 C(\tau)}{2}[F'(\tau,T)]^{\frac{m}{2}}. \tag{27}$$

Hence, Lemma 2 provides:

$$\underline{\lim}_{\tau\to\infty} \tau^{-\frac{m}{m-2}} F(\tau,T) > 0,$$

and:

$$\underline{\lim}_{\tau\to\infty} \tau^{-\frac{2}{m-2}} E(\tau,T) > 0, \tag{28}$$

which completes the proof of Theorem 1.

Remark 1

Estimate 28 will also be valid if Ω is considered as a real cylinder of the form:

$$\Omega = \{(x_1, x') \in \mathbf{R}^n : \ -\infty < x_1 < \infty, \ x' \in \tilde{\sigma}_{x_1}\},$$

with:

$$\Omega_\tau = \{(x_1, x') \in \Omega : \ |x_1| < \tau\}.$$

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