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Discretization of Problems Whose Weak Solutions are Uniquely Determined by Inequalities

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Many phenomena in physics or engineering lead to problems where global solutions can be expected to exist only in a weak sense. Examples are certain elliptic boundary value problems, different types of conservation laws, etc. A Functional Analysis of Discretizations used for the numerical construction of weak solutions was developed where the weak solutions are solutions of equations, eventually completed by a so-called entropy condition inequality (see [1]). In this paper, this theory is extended to problems where the weak solutions are uniquely determined by inequalities only, as is often the case. Examples show the application of the theoretical results.

INTRODUCTION

Problems arising from applications often lead to boundary or initial value problems; in this context, physicists, engineers or scientists of other fields are interested in global solutions which normally do not exist in a classic manner. Hence, problems of this type are often transformed into a weak form whose solutions exist globally and coincide with a classic solution as far as such a solution is sufficiently smooth. These global solutions are then called weak solutions of the original problems.

The weak form of an original problem is often constructed by formal multiplication of the given equation by test functions Φ and integration (by parts). However, this procedure sometimes leads to a loss of uniqueness of the (weak) solutions of the equation; therefore, additional conditions (e.g., entropy conditions) must be taken into account in order to select from the set of weak solutions a unique physically relevant one (called the entropy solution).

In this situation, it is the task of numerical analysis to construct procedures which are able to approximate this entropy solution.

Hints for minimal conditions discretizations of equations must be fulfilled in order to achieve the convergence which were mentioned by Stummel in [2]. This theory was later extended to more general problems (nonlinear problems, optmization problems etc.) resulting in establishment of a Functional Analysis of Discretization. Furthermore, the discretization of weakly formulated problems was included where the weak form is represented by an equality (eventually completed by an entropy condition inequality) (cf. [1,3]).

The loss of uniqueness can often be avoided if the weak representation of the original problem is given by a suitable inequality instead of an equality from the very beginning. In this case, the general theory needs some correction in order to fit this variant also.

SOME BASIC DEFINITIONS AND PROPERTIES

In order to formulate certain convergence results, some relations between originally given operators and operator sequences, which are expected to approximate the given operators, must be presented. (Some of these definitions and properties can already be found in [4].)

Let X and Y be topological spaces. Assume $\{X_n \mid n \in \mathbb{N}\}$ to be a sequence of subspaces of X . Let C be a mapping from X into Y and $\{C_n \mid n \in \mathbb{N}\}$ a sequence of mappings with $C_n : X_n \to Y$ $(n \in \mathbb{N})$.

Definition 1

The pair $[\{C_n\}\ ,\ C]$ is called asymptotically closed, if the following implication holds:

$$v_n \to v \text{ and } C_n v_n \to w \implies Cv = w\ . \quad (1)$$

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Definition 2

The sequence $\{C_n\}$ is called continuously convergent to C, if:

$$v_n \rightarrow v \implies C_n v_n \rightarrow Cv \,. \tag{2}$$

Remark 1

Obviously, if Statement 2 holds, Statement 1 is also fulfilled.

Definition 3

The sequence $\{C_n\}$ is called asymptotically regular, if

$$\{C_n\, v_n\} \text{ compact in } Y \implies \{v_n\} \text{ compact in } X. \tag{3}$$

Compact in Z means that every infinite subsequence contains a subsequence convergent in Z.

Definition 4

A sequence of sets $\{S_n \mid S_n \subset X\}$ is called asymptotically compact, if every sequence $\{u_n | u_n \in S_n, (n = 1, 2, \cdots)\}$ is compact in X.

The set of limit points of the convergent subsequences $\{u_n \mid n \in \mathrm{N}' \subset \mathrm{N}\}$ is denoted by:

$$\{S_n\}^* \,.$$

It is not necessarily expected that the approximate solutions u_n (for fixed n) are unique. However, conditions under which the sequence of sets $\{S_n\}$ of such solutions will converge to the (unique) entropy solution u_E are described in the following sense.

Definition 5

A sequence of sets $\{S_n \mid S_n \subset X\}$ is called set-convergent to a set $S \subset X$, if it is asymptotically compact with:

$$\{S_n\}^* \subset S,$$

and it is written that:

$$\{S_n\} \longrightarrow S.$$

Definition 6

A numerical method generating sets S_n of numerical solutions for every fixed $n \in \mathrm{N}$ is called convergent, if $\{S_n\}$ is set-convergent to a set S of (weak) solutions of the original problem.

A CONVERGENCE RESULT CONCERNING THE NUMERICAL TREATMENT OF PROBLEMS WEAKLY FORMULATED BY INEQUALITIES

Let the topological space Y be linear and semi-ordered and let J be an index set.

Assume that there is a unique solution $u_E \in X$ of the problem:

$$B(\Phi)u \geq 0 \ , \ \forall \ \Phi \in J \,, \tag{4}$$

where $B(\Phi)$ maps X into Y for every fixed $\Phi \in J$. The elements of J are called test elements.

u_E is approximated by solutions u_n $(n = 1, 2, \cdots)$ of equalities:

$$\hat{B}_n u_n = \hat{b}_n \quad (n = 1, 2, \cdots) \,, \tag{5}$$

where the mappings $\hat{B}_n : X \longrightarrow Y$ do not depend on the elements $\Phi \in J$ (because computers do not understand what test elements are) and with a sequence $\left\{\hat{b}_n\right\}$ compact in Y.

Let,

$$S_n = \{u_n \in X \mid u_n \text{ solves Equation 5}\}$$

be the set of approximate solutions for every fixed $n \in \mathrm{N}$ and assume that:

$$S_n \neq \emptyset \,. \tag{6}$$

It is also assumed that a weak inequality formulation

$$B_n(\Phi)u_n := b_n(\Phi) \geq 0 \ , \ \forall \Phi \in J \tag{7}$$

of Equation 5 exists for every $n \in \mathrm{N}$ such that every $u_n \in S_n$ fulfills not only Equation 5, but also Inequality 7 (not necessarily vice versa).

Herewith, the $B_n(\Phi)$ are also mappings from X into Y, and the sequences $\{b_n(\Phi)\}$ are assumed to converge to certain elements $b(\Phi)$ for every fixed $\Phi \in J$ and for every sequence $\{u_n \mid u_n \in S_n\}$. Based on Inequality 7:

$$b(\Phi) \geq 0. \tag{8}$$

Convergence Theorem

If:

(i) $\left\{\hat{B}_n\right\}$ is asymptotically regular,

(ii) $[\{B_n(\Phi)\} \, , \, B(\Phi)]$ is asymptotically closed for every fixed $\Phi \in J$,

then:

$$\{S_n\} \longrightarrow \{u_E\} \,. \tag{9}$$

Proof

$\left\{\hat{b}_n\right\}$ compact in $Y \Rightarrow \left\{\hat{B}_n u_n\right\}$ compact in Y. Because of assumption (i), $\{u_n\}$ is compact in X (independent of the particular choices of $u_n \in S_n$).

Hence, after such a particular choice of $\{u_n\}$, there is $\mathrm{N}' \subset \mathrm{N}$ and $\hat{u} \in X$ such that:

$$\{u_n \mid n \in \mathrm{N}'\} \longrightarrow \hat{u} \,.$$

u_n $(n \in \mathbb{N}')$ also fulfills

$$B_n(\Phi)u_n = b_n(\Phi)\,, \forall\, \Phi \in J\,,$$

which leads to:

$$B_n(\Phi)u_n \longrightarrow b(\Phi)\,, \forall \Phi \in J\,, \forall n \in \mathbb{N}'\,.$$

Because of assumption (ii),

$$B(\Phi)\hat{u} = b(\Phi) \geq 0\,, \forall\, \Phi \in J\,,$$

i.e.,

$$\hat{u} = u_E$$

(independent of the particular choice of $\{u_n\}$). ■

Remark 2

Assumption (i) was only needed in order to show the asymptotic compactness of $\{S_n\}$. Thus, in a concrete situation, the theorem is also valid if this asymptotic compactness can be shown by other arguments.

Remark 3

Since u_E is unique, not only a subsequence of $\{u_n\}$, but also the full sequence converges to u_E.

EXAMPLES

Example 1

Consider the following problem:

$$-\Delta u + q(x)u - f(x) = 0 \quad \text{on} \quad G$$

$$u = 0 \quad \text{on} \quad \partial G\,,$$

with $f,\, q \in C(\bar{G})$, $q > 0$.

With the abbreviation:

$$Lu := -\Delta u + q(\cdot)u\,,$$

L turns out to be a self-adjoint and positive definite operator on $C_0^2(\bar{G})$ with respect to the scalar product,

$$\langle v, w\rangle := \int_G v(x)w(x)\, dG\,. \qquad (10)$$

By means of the generalized energy functional:

$$J(v) := \frac{1}{2}\langle Lv, v\rangle - \langle f, v\rangle\,, \qquad (11)$$

where $\langle Lv\,,\, v\rangle$ has (after partial integration and taking the boundary condition into account) to be read as:

$$\langle Lv, v\rangle = \int_G \left\{(\nabla v, \nabla v) + qv^2\right\} dG\,, \qquad (12)$$

a weak inequality formulation of the given problem is represented by:

$$B(\Phi)u := J(\Phi) - J(u) \geq 0 \quad, \quad \forall \Phi \in J\,, \qquad (13)$$

where $X = J = H_0^1(G)$ is a Hilbert space with respect to the scalar product

$$(v, w)_1 := \int_G \{q \cdot v \cdot w \,+\, (\nabla v, \nabla w)\}\, dG\,, \qquad (14)$$

and with $Y = \mathbb{R}$.

However, Statement 13 is equivalent to the following problem:

$$(u, \Phi)_1 = F(\Phi)\,, \qquad \forall \Phi \in X \qquad (15)$$

with:

$$F(\Phi) := \langle f\,,\, \Phi\rangle\,.$$

This problem (hence Statement 13 as well) has a unique solution u_E due to Lax-Milgram theorem.

For numerical purposes, a projection method is used. Let,

$$X_n = \text{span}\,\langle \Phi_1, \Phi_2, \cdots \Phi_n\rangle \subset X$$

be an n-dimensional subspace where the basis $\langle \Phi_1, \Phi_2, \cdots \Phi_n\rangle$ is an orthonormal one with respect to $(\cdot\,,\cdot)_1$, and $p_n : X \rightarrow X_n$ the projection operator.

Assume the numerical approximations $u_n \in X_n$ to be the solutions of:

$$(u_n\,,\, \Phi_i)_1 - F(\Phi_i) = 0 \quad (i = 1, 2, \cdots, n)\,. \qquad (16)$$

This equation also fulfills the conditions of Lax-Milgram theorem such that there is an (unique) approximate solution u_n, i.e., $S_n \neq \emptyset$, for every fixed $n \in \mathbb{N}$. (It is already known from classic results that the method converges, however, it is endeavored to show this fact by the arguments in this paper.)

Because of the fixed choice of the elements Φ_i $(i = 1, \cdots, n)$ for every particular n, Statement 16 can be written as:

$$\hat{B}_n u_n = 0 \qquad (n = 1, 2, \cdots), \qquad (17)$$

with operators $\hat{B}_n$ independent of a general test element $\Phi \in J$.

In this example, $\left\{\hat{b}_n\right\} = (0, 0, \cdots)$, hence compact. Obviously,

$$u_n = \sum_{j=1}^{n} F(\Phi_j)\Phi_j\,,$$

such that:

$$\|u_{\nu+s} - u_\nu\|_1{}^2 = \sum_{j=\nu+1}^{\nu+s} F^2(\Phi_j)$$

Statement 15 shows that $F(\Phi_j)$ $(j = 1, 2, \cdots)$ are Fourier coefficients of u_E in X with respect to the system $\{\Phi_1, \Phi_2, \cdots\}$. Due to the Bessel inequality, $\{u_n\}$ is a Cauchy sequence in X, hence compact (even convergent) because X is complete. Thus, $\{S_n\}$ is asymptotically compact in X (cf. Remark 2).

A weak formulation of Statement 17 reads as:

$$
\begin{aligned}
&(p_n \hat{B}_n u_n\,,\ \Phi)_1 = (\hat{B}_n u_n\,,\ p_n \Phi)_1 \\
&\quad = \langle L u_n - f\,,\ p_n \Phi \rangle = 0\ , \\
&\quad \forall \Phi \in J\ ,\ (n = 1, 2, \cdots)\ ,
\end{aligned}
\tag{18}
$$

where $\langle L u_n, p_n \Phi \rangle$ must be understood in the sense of the right hand side of Statement 12.

Obviously, every approximate solution u_n is also a solution of Equation 18 for its particular n. But the solutions of Equation 18 also fulfill:

$$
\begin{aligned}
&B_n(\Phi) u_n := J(p_n \Phi) - J(u_n) =: b_n(\Phi) \geq 0\ , \\
&\quad \forall \Phi \in J\ ,\ (n = 1, 2, \cdots)\ .
\end{aligned}
\tag{19}
$$

Because of:

$$
J(w) = \frac{1}{2}(w, w)_1 - \langle f, w \rangle = \frac{1}{2}\|w\|_1^2 - \langle f, w \rangle\ ,
$$

it is obtained that:

$$
\begin{aligned}
&|b_{\nu+s}(\Phi) - b_\nu(\Phi)| \\
&\quad = \Big| \frac{1}{2}\{\|p_{\nu+s}\Phi\|_1 + \|p_\nu \Phi\|_1\}\{\|p_{\nu+s}\Phi\|_1 - \|p_\nu \Phi\|_1\} \\
&\quad - \langle f, p_{\nu+s}\Phi - p_\nu \Phi \rangle \\
&\quad - \frac{1}{2}\{\|u_{\nu+s}\|_1 + \|u_\nu\|_1\}\{\|u_{\nu+s}\|_1 - \|u_\nu\|_1\} \\
&\quad + \langle f, u_{\nu+s} - u_\nu \rangle \Big|\ .
\end{aligned}
$$

This leads to:

$$
\begin{aligned}
&|b_{\nu+s}(\Phi) - b_\nu(\Phi)| \\
&\qquad \leq \|\Phi\|_1\ \|p_{\nu+s}\Phi - p_\nu \Phi\|_1 \\
&\qquad + \|f\|_1\ \|p_{\nu+s}\Phi - p_\nu \Phi\|_1 \\
&\qquad + \frac{1}{2}\{\|u_{\nu+s}\|_1 + \|u_\nu\|_1\}\ \|u_{\nu+s} - u_\nu\|_1 \\
&\qquad + \|f\|_1\ \|u_{\nu+s} - u_\nu\|_1\ .
\end{aligned}
$$

For all sufficiently great values of ν and s,

$$
\begin{aligned}
&|b_{\nu+s}(\Phi) - b_\nu(\Phi)| \\
&\qquad \leq (\|\Phi\|_1 + \|f\|_1)\ \|p_{\nu+s}\Phi - p_\nu \Phi\|_1 \\
&\qquad + (c + \|f\|_1)\ \epsilon\ ,
\end{aligned}
\tag{20}
$$

where c is an upper bound for the convergent sequence $\{\|u_n\|_1\}$.

Again, because of the Bessel inequality, for every fixed $\Phi \in J$,

$$
\|p_{\nu+s}\Phi - p_\nu \Phi\|_1 < \epsilon,
$$

for all sufficiently great values of ν and s.

Together with Statement 20, $\{b_n(\Phi)\}$ is a Cauchy sequence in $\mathbb{R}$ with a limit $b(\Phi)$, and because of Statement 19:

$$
b(\Phi) \geq 0,
$$

i.e., Conditions 7 and 8 are fulfilled.

It is now necessary to show that assumption (ii) of the Convergence Theorem also holds.

For this purpose, assume $\{v_n\} \subset X$ and $\{B_n(\Phi)\ v_n\} \subset \mathbb{R}$ to be convergent sequences:

$$
\begin{aligned}
&\exists v \in X\ ,\quad \exists w \in \mathbb{R} : \\
&v = \lim_{n\to\infty} v_n\ ,\quad w = \lim_{n\to\infty} B_n(\Phi) v_n\ .
\end{aligned}
$$

It has to be shown that: $B(\Phi)v = w$, $\forall \Phi \in J$.

Indeed, in the example presented:

$$
\begin{aligned}
B(\Phi)v - w &= J(\Phi) - J(v) - \lim_{n\to\infty}(J(p_n\Phi) - J(v_n)) \\
&= \lim_{n\to\infty}(J(\Phi) - J(p_n\Phi)) - \lim_{n\to\infty}(J(v) - J(v_n))\ ,
\end{aligned}
$$

and both limits vanish because of the same arguments by which the convergence of $\{b_n(\Phi)\}$ was shown.

This ends the proof of the convergence of the projection method by means of the convergence result described in the previous section.

Example 2

Let $\Omega = \{(x,t) | x \in \mathbb{R}\ ,\ 0 \leq t \leq T\}$ be a strip of the upper x-t-halfplane and $X \subset L_1^{\mathrm{loc}}$. Moreover, let $Y = \mathbb{R}$ and:

$$
J = \left\{ \Phi := (c, \hat{\Phi}) | \hat{\Phi} \in C_0^1(\Omega)\ ,\ \hat{\Phi} \geq 0\ ,\ c \in \mathbb{R} \right\}\ ,
$$

where $C_0^1(\Omega)$ is the set of C^1-functions with compact support in Ω (test functions).

As a weak formulation of the initial value problem for the scalar conservation law which does not only occur as simplifications of systems of conservation laws in CFD but also as direct models in many other fields (cf. [5,6] etc.).

$$
\begin{aligned}
&u_t + \frac{\partial}{\partial x} f(u) = 0\ ,\ \forall (x,t) \in \Omega, \\
&u(x,0) = u_0(x)\ ,\ \forall x \in \mathbb{R} \text{ with } u_0 \in BV(\mathbb{R}) \cap L_\infty(\mathbb{R}),
\end{aligned}
\tag{21}
$$

now, Kruzkov approach [7] is used (see also [8]).

Find $u \in X$ such that:

$$B(\Phi) = \int_{\Omega} \Big\{ |u-c| \frac{\partial \hat{\Phi}}{\partial t} + \operatorname{sgn}(u-c)\,[f(u) - f(c)] \frac{\partial \hat{\Phi}}{\partial x} \Big\} d\Omega \geq 0, \quad \forall \Phi \in J, \tag{22}$$

where X is the particular subset of L_1^{loc}, (hence a topological space) the elements of which show the property:

$$\int_{-r}^{r} |u(x,t) - u_0(x)| dx \to 0 \text{ for } t \to 0+,$$

$$t \in (0,T) \setminus \mathcal{C} \,,\; \forall r > 0 \,. \tag{23}$$

Herewith, $\mathcal{C} \subset (0,T)$ is a set of measure 0.

In a paper with Panov, Kruzkov showed the existence of a unique solution u_E for Inequality 22 provided that $f \in C(\mathbb{R})$ (cf. [9]).

This assumption is much weaker than the properties of the flux f normally assumed to be valid. Because of this weak assumption, Kruzkov result is really important from the point of view of mathematical models of certain problems of nonlinear elasticity, oil prospection etc.

However, in this paper, such stronger conditions, namely $f \in C^1$ and f strongly convex with minimum $f(0) = 0$ are only considered.

Let,

$$|f'|_{\infty}^{*} := \max\left\{ |f'(u)| \,,\; |u| \leq \|u_0\|_{L_\infty} \right\} < \infty \,. \tag{24}$$

Concerning the numerical procedure, explicit finite $(2k+1)$-point difference equations of conservation form are used, i.e.,

$$\frac{u_j^{\nu+1} - u_j^{\nu}}{\Delta x} + \frac{g_{j+\frac{1}{2}}^{\nu} - g_{j-\frac{1}{2}}^{\nu}}{\Delta t} = 0 \,, \tag{25}$$

where u_j^{ν} is expected to become an approximation of $u(x_j\,, t_\nu)$ with

$$x_j = j\,\Delta x \,,\; t_\nu = \nu \Delta t \,;$$

$$j = 0,\; \pm 1,\; \pm 2, \cdots \,,\; \nu = 0,1,2,\cdots \,,$$

and where $g_{j+\frac{1}{2}}^{\nu}$ abbreviates

$$g_{j+\frac{1}{2}}^{\nu} := g(u_{j-k+1}^{\nu}, u_{j-k}^{\nu}, \cdots, u_j^{\nu}, u_{j+1}^{\nu}, \cdots, u_{j+k}^{\nu})$$

with a lipschitz continuous numerical flux $g : \mathbb{R}^{2k} \to \mathbb{R}$. $k \in \mathbb{N}$ is fixed.

In order to make Equation 25 an approximation of Statement 21, g is assumed to fit the consistency condition:

$$g(v, v, \cdots, v) = f(v) \,,\; \forall v \in \mathbb{R} \,. \tag{26}$$

The discrete initial values are:

$$u_j^0 = \frac{1}{\Delta x} \int_{(j-\frac{1}{2})\Delta x}^{(j+\frac{1}{2})\Delta x} u_0(x)\, dx \,,$$

and the step ratio is assumed to fulfill Courant-Friedrichs-Lewy condition:

$$0 < \lambda := \frac{\Delta t}{\Delta x} = \text{const} \leq \frac{1}{|f'|_{\infty}^{*}} \,. \tag{27}$$

Assume $\Delta x = O(\frac{1}{n})$, $n \in \mathbb{N}$ and put:

$u_n(x,t) = u_j^{\nu}$ for

$$\begin{cases} (j - \frac{1}{2})\Delta x \leq x < (j + \frac{1}{2})\Delta x, j = 0, \pm 1, \pm 2, \cdots \\ \nu \Delta t \leq t < (\nu+1)\Delta t, \nu = 0,1,2, \cdots \left[\frac{T}{\Delta t}\right] - 1. \end{cases} \tag{28}$$

If X_n denotes the set of functions $u : \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$, which are constant on the rectangles mentioned on the right hand side of Statement 28 and which fulfill Statement 23, obviously $u_n \in X_n \subset X$ and Statement 25 or, equivalently:

$$u_j^{\nu+1} - u_j^{\nu} + \lambda \left[g_{j+\frac{1}{2}}^{\nu} - g_{j-\frac{1}{2}}^{\nu} \right] = 0$$

$$(j = 0, \pm 1, \pm 2, \cdots \,,\quad \nu = 0,1,2, \cdots \left[\frac{T}{\Delta t}\right] - 1) \,,$$

can be written as:

$$\hat{B}_n u_n = 0 \quad (n = 1, 2, \cdots), \tag{29}$$

with operators $\hat{B}_n : X_n \to \mathbb{R}$.

Thus, Statement 25 is of the form of Statement 5 and there is an (unique) approximate solution u_n for every $n \in \mathbb{N}$ because the method is explicit (i.e., : $S_n \neq \emptyset$).

In the strictly convex case (i.e., $f \in C^1(\mathbb{R})$ and f strictly convex), the Kruzkov solution coincides with the unique entropy solution in the sense of Lax/Oleinik (cf. p.e. [8] , Theorem 3.5.); moreover, it is well-known that Lax/Oleinik solution is of bounded variation with respect to x for every $t > 0$ provided that in the strictly convex case under consideration, u_0 fulfills the conditions mentioned in Statement 21. Furthermore, the bound is independent of $t \in [0,T]$.

If g is constructed in such a way that the total variation of a numerical solution is bounded as well, the method is called TV-stable. Monotone schemes (which are of first order only) like Engquist-Osher scheme [10] or so-called TVD-methods introduced by Harten [11] are TV-stable, and it was LeVeque ([12], p 164) who showed that TV-stable methods lead to d-compact sets S_n provided that u_0 fulfills the assumptions of Statement 21. Thus, by Remark 2, assumption (i) of the abstract Convergence Theorem is replaced suitably by a fulfilled property.

For convenience, let only the case $k = 1$ be considered. As a weak formulation of Statement 29, the inequality to be fulfilled is introduced:

$$B_n(\Phi) := \int_{\Delta t}^{T} \int_{-\infty}^{\infty} \frac{\hat{\Phi}(x,t) - \hat{\Phi}(x,t-\Delta t)}{\Delta t}$$

$$|u_n(x,t)-c|\,dx\,dt + \frac{1}{\Delta t}\int_0^{\Delta t} \sum_i \int_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}} \hat{\Phi}(x,t)$$

$$\left| \frac{1}{\Delta x} \int_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}} u_0(\xi)\,d\xi - c \right| dx\,dt$$

$$+ \int_{\Omega} \frac{\hat{\Phi}(x+\frac{\Delta x}{2},t) - \hat{\Phi}(x-\frac{\Delta x}{2},t)}{\Delta x}$$

$$G\left(u_n(x-\frac{\Delta x}{2},t), u_n(x+\frac{\Delta x}{2},t); c\right)\, d\Omega \geq 0\,, \tag{30}$$

where G is a numerical entropy flux which has to be defined in such a way that the following entropy consistency condition holds:

$$G(v,v;c) = \tilde{F}(v;c) \qquad \forall\,(v,c) \in \mathbb{R}^2. \tag{31}$$

Herewith,

$$\tilde{F}(v;c) := \operatorname{sgn}(v-c)\,[f(v) - f(c)] \tag{32}$$

denotes the particular Kruzkov entropy flux.

The flux splitting choice:

$$G(\alpha,\beta;c) = \tilde{F}_+(\alpha;c) + \tilde{F}_-(\beta;c)$$

with

$$\tilde{F}_+(\alpha;c) := \begin{cases} \tilde{F}(\alpha;c) & ,\alpha \geq 0 \\ 0 & ,\alpha < 0 \end{cases}$$

$$\tilde{F}_-(\beta;c) := \begin{cases} 0 & ,\beta \geq 0 \\ \tilde{F}(\beta;c) & ,\beta < 0 \end{cases}$$

shows an example and a solution of a TV-stable method (Statement 29), then fulfills Statement 30, as well.

Analogously to similar considerations in [3], it can be shown that $\{B_n(\Phi)\}$ converges continuously to $B(\Phi)$ for every fixed $\Phi \in J$.

Thus, also assumption (ii) of the Convergence Theorem is fulfilled (cf. Remark 1) such that the TV-stable method under consideration is convergent (as already known from less general convergence theorems).

Remark 4

It is a remaining task to construct operators $\hat{B}_n$ for problems with only continuous flux functions f in such a way that the assumption (i) of the general Convergence Theorem can be fulfilled in this interesting situation, too. (The proof given here to show the validity of assumption (ii) does already fit this case as well.)

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