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Research Note

Electromagnetic Wave Propagation in Square Law Wave Guides

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In this paper, propagation of an electromagnetic beam wrapped around the axis of a thin axisymmetric wave guide embedded in a square law medium, depending on a β^2 parameter small enough to make the β^{2n} terms negligible for $n \geq 2$ has been analyzed. To solve Maxwell's equations in this wave guide, a paraxial approximation of the wave equations, satisfied by the electric and magnetic components of the electromagnetic field, has been used. The solutions of the corresponding paraxial wave equations describe beam propagation by a series expansion of Gaussian modes.

INTRODUCTION

Wave guides, either natural, such as sound channels in underwater acoustics and the earth-ionosphere space in radiocommunications, or man made, such as optical fibers, are used to transmit energy and information over long distances with the requirement to keep undistorted, as far as possible, the conveyed signals along their way inside the guide. It is supposed, in this work, that electromagnetic wave propagation takes place along the z-axis of a thin axisymmetric wave guide with radius b and with the refractive index $n^2(r)$, $r = (x^2+y^2)^{1/2}$ of the following square law medium [1]:

$$n^2(r) \equiv \mu\varepsilon(r) = 1 \quad \beta^2(r/b)^2,$$
$$0 \leq r \leq b, \qquad \beta^2 < 1, \tag{1}$$

with constant permeability μ and permittivity $\varepsilon(r)$.

To solve Maxwell's equations in square law media requires some approximations. First, neglecting the β^{2n} terms for $n > 2$, the solutions are obtained as an $0(\beta^4)$ series expansion, 0 denoting the Landau symbol. Second, the main interest is in beams focused around the direction of propagation, which makes it possible to work with the paraxial approximation of wave equations. In fact, the paraxial wave equation plays a fundamental role in dealing with electromagnetic and acoustic wave propagation in different media, such as; laser beams in the atmosphere [1-3], optical beams in lenslike wave guides [1] and in dielectric fibers [4], radiowave propagation in the troposphere and ionosphere [5], as well as in turbulent media [6] and acoustic waves underwater [7] and in porous media [8].

The importance of the paraxial approximate solution of the wave equation, $(\Delta \quad c^{\ 2}\partial_t^2)\psi = 0$ and

$$\psi(r,z,t) = \exp[ik(ct \quad z)]\phi(r,z), \tag{2}$$

is that the scalar paraxial wave equation:

$$(\partial_r^2 + 1/r\partial_r \quad 2ik\partial_z)\phi(r,z) = 0, \tag{3}$$

has nondispersive solutions. The circular fundamental Gaussian modes [1]:

$$\phi_0(r,z,t) = u^{\ 1}\exp(\quad ikr^2/2u), \qquad u = z + i\zeta, \tag{4}$$

are well suited to describe the propagation of beams concentrated around oz and, also, the Laguerre-Gauss modes, $\phi_{m,n}(r,z) = \partial_x^m\partial_y^n\phi_0(r,z)$ and Hermite-Gauss modes, $\phi_j(r,z) = \partial_z^j\phi_0(r,z)$, $j,m,n = 1,2,3\cdots$ The positive constant, ζ, in the variable, u, makes the field bounded in the direction transverse to propagation.

This paper is organized as follows: The analysis of electromagnetic wave propagation inside an axisymmetric wave guide, with constant permeability μ and permittivity $\varepsilon(r)$, is performed in the Appendix. In the following section, first, the inhomogeneous wave equations are given, satisfied by the components E_z and H_z of the electromagnetic field, from which the other components are obtained through Maxwell's equations and, then, the $0(\beta^4)$ approximations of E_z and H_z are discussed, when $\varepsilon(r)$ has the form of Equation 1. After that, the paraxial approximations of these equations are obtained and, then, the Gaussian solutions of the paraxial equations are presented. Finally, the conclusive comments are given in the last section.

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WAVE AND PARAXIAL WAVE EQUATIONS IN SQUARE LAW MEDIA

Let the electromagnetic wave propagation be in the z-direction of an axisymmetric medium with constant permeability, μ, considered unity and the cylindrical coordinates r, ϕ and z be used. Also, it is proved in the Appendix that the components $E_z(r,z,t)$ and $H_z(r,z,t)$ (they do not depend on ϕ), from which the other components can be obtained, are solutions of the inhomogeneous wave equations, in which $\varepsilon' = \partial_r \varepsilon$ and $\Box_n = \partial_r^2 + 1/r\partial_r + \partial_z^2 - \varepsilon(r)c^{-2}\partial_t^2$.

$$\Box_n E_z = -\varepsilon'/\varepsilon \partial_z E, \qquad \Box_n H_z = -\varepsilon'/\varepsilon(\partial_z H_r - \partial_r H_z). \tag{5}$$

Then, it is assumed that the electromagnetic field has the form:

$$\{\mathbf{E},\mathbf{H}\}(r,z,t) = \exp(ikct)[\{\mathbf{E}^0,\mathbf{H}^0\} + \beta^2\{\mathbf{E}^1,\mathbf{H}^1\}](r,z) + 0(\beta^4), \tag{6}$$

and it is proved in the Appendix that, for square law permittivity (Equation 1), the components $\{E_z^0, H_z^0\}$ and $\{E_z^1, H_z^1\}$ satisfy the partial differential equations:

$$(\Delta + k^2)\{E_z^0, H_z^0\} = 0, \qquad \Delta = \partial_r^2 + 1/r\partial_r + \partial_z^2, \tag{7}$$

which is the Helmholtz equation in free space and

$$(\Delta + k^2)E_z^1 = (kr/b^2)E_z^0 + (2r/b^2)\partial_z E_r^0, \tag{8a}$$

$$(\Delta + k^2)H_z^1 = (kr/b^2)H_z^0 - (2r/b^2)(\partial_z H_r^0 - \partial_r H_z^0), \tag{8b}$$

while the components $\{E_r^0, H_r^0\}$ on the right hand side of Equations 8a and 8b, obtained from Maxwell's equations, are solutions of the inhomogeneous differential equation:

$$(\partial_z^2 + k^2)\{E_r^0, H_r^0\} - \partial_z\partial_r\{E_z^0, H_z^0\} = 0. \tag{9}$$

So, once the solutions of Equations 7 and 9 are known, one still needs to solve the inhomogeneous wave Equations 8a and 8b and, as discussed in the introduction, this difficult task can be made easier in the frame of the paraxial approximation. Then, one looks for the solutions of Equations 7 and 8 in the form:

$$\{\mathbf{E}^j, \mathbf{H}^j\}(r,z) = \exp(-ikz)\{\mathbf{e}^j, \mathbf{h}^j\}(r,z), \qquad j = 0, 1. \tag{10}$$

Also, it is supposed that the z-dependence of $\mathbf{e}^j$ and $\mathbf{h}^j$ can be assumed slow enough, compared to the rapid variations of the exponential factor, to make negligible $\partial_z^2 \mathbf{e}^j$, $\partial_z^2 \mathbf{h}^j$, so that Equation 7 becomes the paraxial wave equation (Equation 3).

$$\Delta^\dagger\{e_z^0, h_z^0\} = 0, \qquad \Delta^\dagger = \partial_r^2 + 1/r\partial_r - 2ik\partial_z, \tag{11}$$

while one obtains for Equation 9:

$$2ik\partial_z\{e_r^0, h_r^0\} = -(\partial_z - ik)\{\partial_r e_z^0, \partial_r h_z^0\}. \tag{12}$$

Then, taking into account Equation 12 to eliminate $\partial_z\{e_r^0, h_r^0\}$, the paraxial approximation of Equations 8a and 8b will be:

$$\Delta^\dagger e_z^1 = (kr/b)^2 e_z^0 + (r/kb^2)(k + i\partial_z)\partial_r e_z^0 - 2ikr/b^2 e_r^0, \tag{13a}$$

$$\Delta^\dagger h_z^1 = (kr/b)^2 h_z^0 + (r/kb^2)(k - i\partial_z)\partial_r h_z^0 + 2ikr/b^2 h_r^0, \tag{13b}$$

the right hand side of the above equations still depends on e_r^0, h_r^0.

But, from now on, the fundamental circular Gaussian mode in Equation 4 is used as a solution to the paraxial wave equation (Equation 11), so that, with two arbitrary amplitudes $\{A_e, A_h\}$:

$$\{e_z^0, h_z^0\} = \{A_e, A_h\}[u^{-1}\exp(-ikr^2/2u)], \qquad u = z + i\zeta. \tag{14}$$

Then, using the relation:

$$\partial_r[u^{-1}\exp(-ikr^2/2u)] = -2r^{-1}\partial_z \exp(-ikr^2/2u), \tag{15}$$

Equations 13 transform to:

$$2ik\{e_r^0, h_r^0\} = 2r^{-1}(\partial_z - ik)\{ue_z^0, uh_z^0\}, \tag{16}$$

making it possible to eliminate $\{e_r^0, h_r^0\}$ from Equations 13a and 13b. This task is performed in the next section to get the solutions of these equations.

GAUSSIAN BEAMS

First, let Equation 13a be considered, with $A_e = 1$ and $\theta = kr^2/2u$ and Equation 14 be written as:

$$e^0 = u^{-1}\exp(-i\theta), \tag{17}$$

then, using ∂_u for ∂_z and taking into account Equation 16, Equation 13a becomes:

$$(\partial_r^2 + 1/r\partial_r - 2ik\partial_u)e_z^1(r,u) = \Pi(r,u)[u^{-1}\exp(-i\theta)], \tag{18a}$$

$$\Pi(r,u) = k^2r^2/b^2 + (r/kb^2)(k + i\partial_u)\partial_r + 2i/b^2(k + i\partial_u)u, \tag{18b}$$

and a simple calculation gives:

$$\Pi(r,u)[u^{\;1}\exp(\;i\theta)]=\left[\sum\nolimits_{j=0}^{4} a_j(r)u^{\;j}\right]\exp(\;i\theta), \tag{19a}$$

$$a_0 = 2ik/b^2, \qquad a_1 = k^2r^2/b^2, \qquad a_2 = \;\;2ikr^2/b^2,$$

$$a_3 = \;\;2r^2/b^2, \qquad a_4 = ikr^4/2b^2. \tag{19b}$$

The trial solution of Equation 18a is introduced as follows:

$$e_z^1(r,u) = f(r,u)\exp(\;\;i\theta), \tag{20}$$

transforming the left hand side of Equation 18a into:

$$(\partial_r^2 + 1/r\partial_r \;\;\; 2ik\partial_u)e_z^1(r,u) = [A(r)f + u^{\;1}B(r)f$$

$$2ik\partial_u f]\exp(\;\;i\theta), \tag{21a}$$

in which the differential operators, $A(r)$, $B(r)$ are:

$$A(r) = \partial_r^2 + 1/r\partial_r,$$

$$B(r) = \;\;2ikr(\partial_r + 1/r), \tag{21b}$$

and substituting Equations 19a and 20 into Equations 21a gives the partial differential equation:

$$[A(r)+u^{\;1}B(r)\;\;\;2ik\partial_u]f(r,u)=\sum\nolimits_{j=0}^{4} a_j(r)u^{\;j}. \tag{22}$$

Then, one looks for $f(r,u)$ as a series expansion:

$$f(r,u) = \sum\nolimits_{j=0}^{\infty} g_j(r)u^{\;j}. \tag{23}$$

Substituting Equation 23 into Equation 22 and using Equation 19b, one obtains the following system of differential equations:

$$Ag_0 = a_0 = 2ik/b^2,$$

$$Ag_1 + Bg_0 = a_1 = k^2r^2/b^2,$$

$$Ag_2 + (B + 2ik)g_1 = a_2 = \;\;2ikr^2/b^2,$$

$$Ag_3 + (B + 4ik)g_2 = a_3 = \;\;2r^2/b^2,$$

$$Ag_4 + (B + 6ik)g_3 = a_4 = ikr^4/2b^2,$$

$$Ag_n + [B + (2(n \;\;\; 1)ik)g_{n\;\;1} = 0,$$

$$n \geq 5. \tag{24}$$

The solution of this system is easy to get since, according to Equation 21b, each equation has the form:

$$(\partial_r^2 + 1/r\partial_r)g_n(r) = \alpha_n,$$

$$\alpha_n = a_n \;\;\; [B + (2(n \;\;\; 1)ik)g_{n\;\;1}, \tag{25}$$

with the solution:

$$g_n = \alpha_n r^{n+2}/(n+2)^2. \tag{26}$$

One obtains, for instance;

$$g_0 = ikr^2/2b^2, \qquad g_1 = \;\;2k^2r^4/4^2b^2,$$

$$g_2 = \;\;ikr^4/4^2b^2 \;\;\; ik^3r^6/6^2b^2. \tag{27}$$

The convergence of Series 23 is an open question and the parameter, ζ, is important to make the component in Equation 20 bounded in the transverse direction.

Assuming the form of Equation 17 for h_z^0, the component h_z^1 satisfies Equation 18a, in which, according to Equation 13b, the function, $\Pi(r,u)$, is now:

$$\Pi(r,u) = k^2r^2/b^2 + (r/kb^2)(k \;\;\; i\partial_u)\partial_r$$

$$2i/b^2(k + i\partial_u)u, \tag{28}$$

so that, in Series 19a, the functions $a_j(r)$ are:

$$a_0 = \;\;2ik/b^2, \qquad a_1 = k^2r^2/b^2, \qquad a_2 = 0,$$

$$a_3 = 2r^2/b^2, \qquad a_4 = \;\;ikr^4/2b^2, \tag{29}$$

and, from that, the calculations are the same as for e_z^1.

In summary, one may write, according to Equations 20 and 23 and taking into account Equation 17;

$$\{e_z^1(r,z), h_z^1(r,z)\}=\{uf_e(r,u)e_z^0(r,z), uf_h(r,u)h_z^0(r,z)\}, \tag{30}$$

with f labeled by the subscripts e, h, in order to avoid confusion, so that, according to Equation 10;

$$\{E_z^1(r,z), H_z^1(r,z)\}$$

$$= \{uf_e(r,u)E_z^0(r,z), uf_h(r,u)E_z^0(r,z)\}, \tag{31}$$

in which $\{E_z^0, H_z^0\} = \{A_e, A_h\}u^{\;1}\exp(\;\;ikz\;\;\;ikr^2/2u)$. Then, according to Equation 6, the $0(\beta^4)$ paraxial approximation of the electromagnetic field z-component is for a Gaussian beam in a square law medium:

$$E_z(r,z,t), = A_e\exp(ikct)[1 + \beta^2 uf_e(r,u)]E_z^0(r,z)$$

$$+ 0(\beta^4),$$

$$H_z(r,z,t), = A_h\exp(ikct)[1 + \beta^2 uf_h(r,u)]H_z^0(r,z)$$

$$+ 0(\beta^4). \tag{32}$$

One must use the Maxwell Equations A3 and A4 of the Appendix for the other components of the electromagnetic field and one needs to deal with first order differential equations such as for the ϕ-components:

$$(\partial_r + 1/r)E_\phi + ik\mu H_z = 0,$$

$$(\partial_r + 1/r)H_\phi \;\;\; ik\varepsilon E_z = 0, \tag{33a}$$

with the solutions;

$$E_\phi(r,z) = \quad (ik\mu/r)\int_0^r s\ ds\ H_z(s,z),$$

$$H_\phi(r,z) = (ik/r)\int_0^r s\ ds\ \varepsilon(s)E_z(s,z), \tag{33b}$$

requiring a numerical quadrature.

DISCUSSION

Optical and acoustic waves in a square law medium are inclined to propagate in a well collimated way, giving rise to paraxial beams; a result which has been the source of many works in the past [1]. Also, it is proved here that paraxial approximation is a powerful tool to solve inhomogeneous wave equations, deduced from Maxwell's equations for z-components of the electromagnetic field.

The situation is a bit different for acoustic waves propagating inside a square law medium, since they satisfy the scalar Helmholtz equation ($\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$):

$$\Delta\psi + n^2k_0^2\psi = 0, \qquad n^2 = 1 \quad \beta^2r^2/b^2. \tag{34}$$

The solutions of Equation 34 are obtained in the form:

$$\psi(x) = f(\mathbf{x})g(y)\exp(ikz). \tag{35}$$

Substituting Equation 35 into Equation 34 leads, after some mathematical manipulations, to differential equations for f and g of the type obtained in quantum mechanics for harmonic oscillators [9]. The paraxial approximation of these solutions gives the Hermite-Gauss modes mentioned in the introduction.

These results are sometimes extended to electromagnetic fields, whose scalar approximations are possible only as long as polarization effects can be neglected. It is often considered [1] that scalar solutions could be used as approximations to solutions of Maxwell's equations, provided that the refractive index varies very slowly over the distance of one optical wavelength: An approximation not satisfactory since, as proved in the Appendix, no component of the electromagnetic field satisfies the Helmholtz equation (Equation 34).

Now, when is the $0(\beta^4)$ approximation justified? Among different situations concerning, for instance, laser propagation in air, $\beta \cong 3.10 \quad ^4$ [3] over a wide range of values for temperature, power and electric field. On the other hand, $\beta = 0,136$ [5] for radiowave propagation in the atmosphere, while, for high electromagnetic waves in the ionosphere, the Appleton-Hartree formula for the refractive index becomes simple in some situations [10] with $\beta < 1$.

Recently, the existence of metamaterials with negative permittivity, permeability and refractive index, fancied by Veselago [11] many years ago, has been proved [12], giving rise since then to an explosion of published works [13]. So, it is natural to wonder what would happen to an electro-magnetic field propagating in a metamaterial endowed with the square law medium (Equation 1), in which $\varepsilon < 0$, $\mu < 0$. The transformation $\varepsilon, \mu \Rightarrow \quad \varepsilon, \quad \mu$ applied to Maxwell Equations A3 and A4, gives:

$$(E_{r,z}, H_{r,z}) \Rightarrow (E_{r,z}, H_{r,z}),$$

$$(E_\phi, H_\phi) \Rightarrow \quad (E_\phi, H_\phi), \tag{36}$$

and, for plane waves $\exp(ikz) \Rightarrow \exp(\quad ikz)$, so that the paraxial wave Equation 3 becomes:

$$(\partial_r^2 + 1/r\partial_r + 2ik\partial_z)\psi(r,z) = 0. \tag{37}$$

In metamaterials, the phase velocity is antiparallel to the Poynting vector and, in a recent work [14] on wave propagation inside a waveguide filled with a metamaterial, the authors concluded possible applications in the design of novel devices, such as ultra-thin waveguides, which are thinner than a diffraction limited size.

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APPENDIX

The propagation of electromagnetic waves in the z-direction of an axisymmetric medium, with constant permeability μ and permittivity $\varepsilon(r)$, $r = (x^2 + y^2)^{1/2}$, is considered. Then, using the cylindrical coordinates, r, φ, z, Maxwell equations, written with the nabla symbol ∇:

$$\nabla \wedge \mathbf{E} + \mu c \;\, {}^{1}\partial_t \mathbf{H} = 0, \qquad \nabla \wedge \mathbf{H} \quad \varepsilon c \;\, {}^{1}\partial_t \mathbf{E} = 0,$$

$$\varepsilon \equiv \varepsilon(r), \tag{A1}$$

$$\nabla.(\mu \mathbf{H}) = 0, \qquad \nabla.(\varepsilon \mathbf{E}) = 0, \tag{A2}$$

become (since the fields do not depend on φ):

$$\partial_z E_\varphi + \mu c \;\, {}^{1}\partial_t H_r = 0, \tag{A3a}$$

$$(\partial_r + 1/r)E_\varphi + \mu c \;\, {}^{1}\partial_t H_z = 0, \tag{A3b}$$

$$\partial_z H_r \quad \partial_r H_z \quad \varepsilon c \;\, {}^{1}\partial_t E_\varphi = 0, \tag{A3c}$$

$$(\partial_r + 1/r)H_r + \partial_z H_z = 0, \tag{A3d}$$

$$\partial_z H_\varphi \quad \varepsilon c \;\, {}^{1}\partial_t E_r = 0, \tag{A4a}$$

$$(\partial_r + 1/r)H_\varphi \quad \varepsilon c \;\, {}^{1}\partial_t E_z = 0, \tag{A4b}$$

$$\partial_z E_z \quad \partial_r E_z + \mu c \;\, {}^{1}\partial_t H_\varphi = 0, \tag{A4c}$$

$$(\partial_r + 1/r)(\varepsilon E_r) + \varepsilon \partial_z E_z = 0, \tag{A4d}$$

so that, once E_z, H_z is known, the components E_φ, H_φ and E_r, H_r are supplied, respectively, by Equations A3b, A4b, A4d and A3d. Now, one looks for the wave equations satisfied by E_z, H_z.

One gets, from Equations A4b and A4c:

$$\mu \varepsilon c \;\, {}^{2}\partial_t^2 E_z = (\partial_r + 1/r)\partial_r E_z \quad \partial_z(\partial_r + 1/r)E_r, \tag{A5}$$

and, according to Equation A4d:

$$(\partial_r + 1/r)E_r + \partial_z E_z + \varepsilon'/\varepsilon E_r = 0, \qquad \varepsilon' = \partial_r \varepsilon. \tag{A6}$$

Then, substituting Equation A6 into Equation A5 gives, with $\Box_n \equiv \partial_r^2 + r \;\, {}^{1}\partial_r + \partial_z^2 \quad \mu\varepsilon(r)c \;\, {}^{2}\partial_t^2$, the inhomogeneous wave equation:

$$\Box_n E_z = \quad \varepsilon'/\varepsilon \partial_z E_r. \tag{A7}$$

Similarly, using Equation A3b and Equation A3c, one obtains:

$$\mu \varepsilon c \;\, {}^{2}\partial_t^2 H_z = \quad (\partial_r + 1/r)(\partial_z H_r \quad \partial_r H_z)$$
$$\varepsilon'/\varepsilon(\partial_z H_r \quad \partial_r H_z), \tag{A8}$$

and substituting Equation A3d into Equation A8, one finally gets:

$$\Box_n H_z = \quad \varepsilon'/\varepsilon(\partial_z H_r \quad \partial_r H_z). \tag{A9}$$

From now on, it is supposed that $\mu = 1$ and one looks for the $0(\beta^4)$ solutions of the inhomogeneous Equations A7 and A9 in the form:

$$\{E, H\}(r, z, t) = \exp(ikct)[\{E^0, H^0\}$$
$$+ \beta^2\{E^1, H^1\}](r, z) + 0(\beta^4). \tag{A10}$$

Substituting Equation A10 into Equations A7 and A9 and taking into account Expression 1 of $\varepsilon(r)$, which implies $\varepsilon'/\varepsilon = \quad 2\beta^2 r/b^2$ and making null the β^{2n} terms for $n = 0, 1$, one gets the partial differential equations satisfied by $\{E_z^0, H_z^0\}$ and $\{E_z^1, H_z^1\}$:

$$(\Delta + k^2)\{E_z^0, H_z^0\} = 0, \qquad \Delta = \partial_r^2 + 1/r\partial_r + \partial_z^2, \tag{A11}$$

which is the Helmholtz equation in free space and:

$$(\Delta + k^2)E^1 = (kr/b)^2 E^0 + (2r/b^2)\partial_z E_r^0,$$

$$(\Delta + k^2)H^1 = (kr/b)^2 H^0 \quad (2r/b^2)(\partial_z H_r^0 \quad \partial_r H_z^0), \tag{A12}$$

in which E_r^0, H_r^0, which do not depend on β, are solutions of the free space Maxwell equations obtained by imposing $\varepsilon = \mu = 1$ in Equations A3 and A4, with, in addition, $E_z = E_z^0$ and $H_z = H_z^0$. Then, eliminating E_ϕ between Equations A3a and A3c and H_ϕ, between Equations A4a and A4c give:

$$(\partial_z^2 + k^2)\{E_r^0, H_r^0\} \quad \partial_z \partial_r \{E_z^0, H_z^0\} = 0. \tag{A13}$$

So, once the solutions of Equation A11 and Equation A13 are known, one just has to solve the inhomogeneous Helmholtz Equation A12.