Scientia Iranica, Vol. 8, No. 2, pp 81–91

SCIENTIA
IRANICA

Refined Non-Conforming Triangular Plate Element for Geometrically Non-Linear Analysis

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Considering total Lagrangian formulation, refined non-conforming triangular thin plate-bending element RT9 is used as the basis for refined geometrically non-linear non-conforming plate-bending element NRT15. Through possible introduction of various weak continuity conditions into the formulation of geometrically non-linear plate elements, the constraint condition of inter-element continuity can be satisfied to an average degree. A combined interpolation function is introduced into part of the formulation to improve the accuracy of computation. Numerical examples presented in this paper show that the present element possesses high accuracy.

INTRODUCTION

In recent years, considerable attention has been placed on the non-linear analysis of plates and, consequently, a number of non-linear plate elements have been developed.

Development of geometrically non-linear analysis of plate structures has been closely related to the achievement of FEM in the linear field. There are many methods that can be employed to construct the plate elements such as the displacement-based (conforming and non-conforming) method, the hybrid stress element method, etc., among which the non-conforming element method is regarded as an important one. This is due to the fact that for a C^1 type problems, in which the C^1 continuity conditions are required, the use of non-conforming element can prevent the difficulty of constructing the conforming displacement interpolation function. Moreover, it can improve the accuracy when used for the analyses of C^0 and C^0/C^1 problems. Recently, many different methods are available for constructing non-conforming elements in linear analyses such as the discrete Kirchhoff method [1], the free formulation element method [2], the natural element method [3] and the refined non-conforming element method [4-8].

Nine-parameter triangular bending plate element is a simple element which is widely used in engineering. A series of nine-parameter elements exists, which includes the non-conforming element BCIZ [9], the discrete Kirchhoff element DKT [10], the hybrid stress element HSM [11], the refined non-conforming element RT9 [5], etc. Batoz compared and evaluated [12] several popular nine-parameter bending elements and concluded that the DKT and hybrid stress model (HSM) elements seem to be the most effective nine DOF triangular elements available in thin plate bending analysis, with DKT element being to some extent better than the HSM element. In addition, he mentioned that the non-conforming BCIZ element also possesses high accuracy. However, since it cannot pass the patch test for some special meshes, it has not been widely used.

Recently, using the refined non-conforming element method, Cheung and Chen proposed a new element RT9 [5] on the basis of BCIZ. Numerical results show that the element RT9 can pass the patch test and yields higher accuracy compared with the DKT and HSM elements. Satisfactory results can also be obtained in dynamic and stability analyses [13].

With the success of RT9 element in the linear field, it is natural to apply it to the geometrically non-linear field. In geometrically non-linear problems, higher order terms of displacement gradients exist, which cause difficulty in establishing the proper convergence criteria. As a result, it is not an easy matter to use non-conforming elements in the geometrically non-linear field and it has been found that there are very few successful non-conforming elements in geometrically non-linear analysis.

Recently, Chen proposed a geometrically non-linear generalized variational principle [14], which

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relaxes the inter-element continuity requirement. Through introducing the first Piola stress tensor, the displacement gradient tensors and the common interpolation displacement function in the inter-element boundary, the continuity condition is satisfied to an average degree. The principle has been used successfully in non-linear analysis [15]. In this paper, a method is presented for establishing weak continuity conditions on the inter-element boundaries, which is equivalent to the geometrically non-linear generalized variational principle [14]. Considering such weak continuity conditions, the element RT9 can be used and extended to the non-linear analysis of plate structures. In the non-linear formulation of the proposed plate element NRT15, various weak continuity conditions can be given for different displacement gradients. It should be noted that parts of the continuity conditions needed in the non-linear formulation already exist in the element RT9, which makes the work here much simpler.

WEAK CONTINUITY CONDITIONS FOR THE GEOMETRICALLY NON-LINEAR BENDING PLATE ELEMENT

To ensure the convergence of non-linear non-conforming elements, various weak continuity conditions in the formulation should be satisfied according to the refined non-conforming method. To obtain the weak continuity condition on the inter-element boundaries, common displacements should be introduced. In the following, several weak continuity conditions are proposed according to the displacement gradients that will be found in the formulation of the geometrically non-linear plate element.

C^0 Weak Continuity Condition for the Displacement u and ν

The first-order gradients of the displacements u and ν for the membrane part of non-linear bending plate element exist, which should satisfy the C^0 continuity condition. According to the refined non-conforming method, the C^0 weak continuity condition for the alternative refined displacements u^* and ν^* can be expressed as follows,

$$\begin{aligned}
&\int_{\nu_e} \frac{\partial u^*}{\partial x} dxdy = \oint_{\partial\nu_e} \tilde{u} l ds,\\
&\int_{\nu_e} \frac{\partial u^*}{\partial y} dxdy = \oint_{\partial\nu_e} \tilde{u} m ds,\\
&\int_{\nu_e} \frac{\partial \nu^*}{\partial x} dxdy = \oint_{\partial\nu_e} \tilde{\nu} l ds,\\
&\int_{\nu_e} \frac{\partial \nu^*}{\partial y} dxdy = \oint_{\partial\nu_e} \tilde{\nu} l ds,
\end{aligned} \tag{1}$$

where $\tilde{u}$ and $\tilde{\nu}$ are the interpolation functions expressed in terms of nodal parameters on the inter-element boundary and l, m are the direction cosines of the element boundary.

To explain the rationality of the above equation, the left side of Equation 1 is integrated:

$$\begin{aligned}
&\int_{\nu_e} \frac{\partial u^*}{\partial x} dxdy = \oint_{\partial\nu_e} \tilde{u}^* l ds,\\
&\int_{\nu_e} \frac{\partial u^*}{\partial y} dxdy = \oint_{\partial\nu_e} \tilde{u}^* m ds,\\
&\int_{\nu_e} \frac{\partial \nu^*}{\partial x} dxdy = \oint_{\partial\nu_e} \tilde{\nu}^* l ds,\\
&\int_{\nu_e} \frac{\partial \nu^*}{\partial y} dxdy = \oint_{\partial\nu_e} \tilde{\nu}^* m ds.
\end{aligned} \tag{2}$$

Substituting Equation 2 into Equation 1, it is obtained that:

$$\begin{aligned}
&\oint_{\partial\nu_e} (\tilde{u}^* - \tilde{u}) l ds = 0,\\
&\oint_{\partial\nu_e} (\tilde{u}^* - \tilde{u}) m ds = 0,\\
&\oint_{\partial\nu_e} (\tilde{\nu}^* - \tilde{\nu}) l ds = 0,\\
&\oint_{\partial\nu_e} (\tilde{\nu}^* - \tilde{\nu}) m ds = 0.
\end{aligned} \tag{3}$$

The boundary displacements $\tilde{u}^*$ and $\tilde{\nu}^*$ are usually different from the common inter-element displacements $\tilde{u}$ and $\tilde{\nu}$, which are interpolated in terms of the nodal parameters. From Equation 3, it can be seen that the inter-element continuity condition is satisfied to some extent.

The interpolation functions for the displacements u and ν can, usually, satisfy the C^0 continuity condition and no modification needs to be done in that case. However, if the displacement interpolations of u and ν cannot satisfy the C^0 continuity condition, then they should be modified according to the weak continuity condition expressed as Equation 1. Displacement functions $u(x, y)$ and $\nu(x, y)$ can be modified into the refined displacement functions $u^*(x, y)$ and $\nu^*(x, y)$, which satisfies the inter-element continuity requirement. The refined displacement functions $u^*(x, y)$ and $\nu^*(x, y)$ can be assumed as:

$$\begin{Bmatrix} u^* \\ \nu^* \end{Bmatrix} = \begin{Bmatrix} u \\ \nu \end{Bmatrix} + \begin{bmatrix} x & y & 0 & 0 \\ 0 & 0 & x & y \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}, \tag{4}$$

where:

$$\begin{Bmatrix} u \\ \nu \end{Bmatrix} = \mathbf{N^p q_e^p} \quad \begin{Bmatrix} u^* \\ \nu^* \end{Bmatrix} = \mathbf{N^{p^*} q_e^p}, \tag{5}$$

in which $\mathbf{q_e^p}$ are the nodal displacement parameters for the membrane part of non-linear plate element.

Substituting Equation 4 into Equation 1,

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \mathbf{B}_{\alpha c}\mathbf{q_e^p} - \mathbf{B}_{\alpha 0}\mathbf{q_e^p}, \tag{6}$$

where:

$$\mathbf{B}_{\alpha c}\mathbf{q_e^p} = \frac{1}{\Delta}\oint_{\partial\nu_e} \begin{Bmatrix} \tilde{u}l \\ \tilde{u}m \\ \tilde{\nu}l \\ \tilde{\nu}m \end{Bmatrix} ds,$$

$$\mathbf{B}_{\alpha 0}\mathbf{q_e^p} = \frac{1}{\Delta}\int_{\nu_e} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial \nu}{\partial x} \\ \frac{\partial \nu}{\partial y} \end{Bmatrix} ds, \tag{7}$$

in which:

$$\Delta = \int_{\nu_e} dxdy. \tag{8}$$

The strain matrices are given by:

$$\mathbf{B}_\alpha = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}^T \mathbf{N^p},$$

$$\mathbf{B}_\alpha^* = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}^T \mathbf{N^{p*}}, \tag{9}$$

and finally:

$$\mathbf{B}_\alpha^* = \mathbf{B}_\alpha - \mathbf{B}_{\alpha 0} + \mathbf{B}_{\alpha c}. \tag{10}$$

Based on the displacement gradient $\mathbf{B}_\alpha$, the refined displacement gradient $\mathbf{B}_\alpha^*$ is obtained directly using Equation 10. Obviously, the formulation is very simple and can be carried out easily.

C^1 Weak Continuity Condition for the Bending Part of a Non-Linear Plate Element

The C^1 weak continuity condition for the refined deflection w^* on the bending part of non-linear plate element can be expressed as follows:

$$\int_{\nu_e} \frac{\partial^2 w^*}{\partial x^2} dxdy = \oint_{\partial\nu_e} \left(\frac{\partial\tilde{w}}{\partial n} l^2 - \frac{\partial\tilde{w}}{\partial s} lm \right) ds,$$

$$\int_{\nu_e} \frac{\partial^2 w^*}{\partial y^2} dxdy = \oint_{\partial\nu_e} \left(\frac{\partial\tilde{w}}{\partial n} m^2 + \frac{\partial\tilde{w}}{\partial s} lm \right) ds,$$

$$\int_{\nu_e} 2\frac{\partial^2 w^*}{\partial x\partial y} dxdy = \oint_{\partial\nu_e} \left(2\frac{\partial\tilde{w}}{\partial n} lm + \frac{\partial\tilde{w}}{\partial s}(l^2 - m^2) \right) ds, \tag{11}$$

where $\frac{\partial\tilde{w}}{\partial n}$ and $\frac{\partial\tilde{w}}{\partial s}$ are the interpolation functions expressed in terms of the nodal parameters $\mathbf{q_e^b}$ on the element boundary and are the direction cosines of the element boundary and l, m are the direction cosines of the element boundary.

The rationality of Equation 11 can be explained as follows. By integrating the left side of Equation 11,

$$\int_{\nu_e} \frac{\partial^2 w^*}{\partial x^2} dxdy = \oint_{\partial\nu_e} \left(\frac{\partial\tilde{w}^*}{\partial n} l^2 - \frac{\partial\tilde{w}^*}{\partial s} lm \right) ds,$$

$$\int_{\nu_e} \frac{\partial^2 w^*}{\partial y^2} dxdy = \oint_{\partial\nu_e} \left(\frac{\partial\tilde{w}^*}{\partial n} m^2 + \frac{\partial\tilde{w}^*}{\partial s} lm \right) ds,$$

$$\int_{\nu_e} 2\frac{\partial^2 w^*}{\partial x\partial y} dxdy = \oint_{\partial\nu_e} \left(2\frac{\partial\tilde{w}^*}{\partial n} lm + \frac{\partial\tilde{w}^*}{\partial s}(l^2 - m^2) \right) ds. \tag{12}$$

Substituting Equation 12 into Equation 11,

$$\oint_{\partial\nu_e} \left[\left(\frac{\partial\tilde{w}^*}{\partial n} - \frac{\partial\tilde{w}}{\partial n}\right) l^2 - \left(\frac{\partial\tilde{w}^*}{\partial s} - \frac{\partial\tilde{w}}{\partial s}\right) lm \right] ds = 0,$$

$$\oint_{\partial\nu_e} \left[\left(\frac{\partial\tilde{w}^*}{\partial n} - \frac{\partial\tilde{w}}{\partial n}\right) m^2 + \left(\frac{\partial\tilde{w}^*}{\partial s} - \frac{\partial\tilde{w}}{\partial s}\right) lm \right] ds = 0.$$

$$\oint_{\partial\nu_e} \left[2\left(\frac{\partial\tilde{w}^*}{\partial n} - \frac{\partial\tilde{w}}{\partial n}\right) lm + \left(\frac{\partial\tilde{w}^*}{\partial s} - \frac{\partial\tilde{w}}{\partial s}\right)(l^2 - m^2) \right] ds = 0. \tag{13}$$

Equation 13 means that the inter-element continuity condition is satisfied to an average degree.

For an arbitrary non-conforming displacement function $w(x,y)$, the refined displacement function $w^*(x,y)$ can be defined as:

$$w^*(x,y) = w(x,y) + P\beta = \mathbf{N}^{b^*}\mathbf{q_e^b}, \tag{14}$$

where:

$$w(x,y) = \mathbf{N^b q_e^b}, \tag{15}$$

$$p = \left[\frac{x^2}{2} \; \frac{y^2}{2} \; \frac{xy}{2} \right], \tag{16}$$

$$\beta = \{\beta_1 \; \beta_2 \; \beta_3\}^T. \tag{17}$$

Substituting Equation 14 into Equation 11 and using the strain matrices:

$$\mathbf{B}_\beta = \left[\frac{\partial^2}{\partial x^2} \; \frac{\partial^2}{\partial y^2} \; \frac{2\partial^2}{\partial x\partial y} \right]^T \mathbf{N^b},$$

$$\mathbf{B}_\beta^* = \left[\frac{\partial^2}{\partial x^2} \; \frac{\partial^2}{\partial y^2} \; \frac{2\partial^2}{\partial x\partial y} \right]^T \mathbf{N^{b^*}}, \tag{18}$$

and:

$$\mathbf{B}_\beta^* = \mathbf{B}_\beta - \mathbf{B}_{\beta 0} + \mathbf{B}_{\beta c}, \tag{19}$$

where:

$$\mathbf{B}_{\beta 0} = \frac{1}{\Delta}\int_{\nu_e} \mathbf{B}_\beta \, dxdy, \tag{20}$$

$$\mathbf{B}_{\beta c}\mathbf{q}_e^b = \frac{1}{\Delta}\oint_{\partial\nu_e}\left(\frac{\partial\tilde{w}}{\partial n}\begin{bmatrix} l^2 \\ m^2 \\ lm \end{bmatrix} + \frac{\partial\tilde{w}}{\partial s}\begin{bmatrix} -lm \\ lm \\ l^2 - m^2 \end{bmatrix}\right) ds. \tag{21}$$

Based on the strain matrix $\mathbf{B}_\beta$, which is often used in the standard displacement based FEM, the refined strain matrix $\mathbf{B}_\beta^*$ is obtained directly using Equation 19.

C^0 Weak Continuity Condition for the First-Order Gradients of Displacement W

In order to form the non-linear part of the incremental strain vector, it is necessary to use the first-order gradient of displacement w, which should satisfy the C^0 continuity condition. The C^0 weak continuity condition, here, can be expressed as follows:

$$\int_{\nu_e} \frac{\partial w^*}{\partial x} dxdy = \oint_{\partial\nu_e} \tilde{w} l ds,$$
$$\int_{\nu_e} \frac{\partial w^*}{\partial x} dxdy = \oint_{\partial\nu_e} \tilde{w} m ds, \tag{22}$$

where $\tilde{w}$ are the interpolation functions expressed in terms of nodal parameters $\mathbf{q}_e^b$ on the element boundary and l, m are the direction cosines of the element boundary.

The rationality can be explained in a similar way to that discussed in parts 1 and 2. If the first-order gradients of displacement are not conforming, then the displacement should be modified according to Equation 22. Assume that the refined displacement w^* is:

$$w^*(x,y) = w(x,y) + [x \;\; y]\begin{Bmatrix} \gamma_1 \\ \gamma_2 \end{Bmatrix}, \tag{23}$$

substituting Equation 23 into Equation 22 results in:

$$\mathbf{B}_\gamma^* = \mathbf{B}_\gamma - \mathbf{B}_{\gamma 0} + \mathbf{B}_{\gamma c}, \tag{24}$$

where:

$$\mathbf{B}_\gamma = \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \mathbf{N}^b, \quad \mathbf{B}_{\gamma c}\mathbf{q}_e^b = \frac{1}{\Delta}\oint_{\partial\nu_e}\begin{Bmatrix} \tilde{w} l \\ \tilde{w} m \end{Bmatrix} ds,$$

$$\mathbf{B}_{\gamma 0}\mathbf{q}_e^b = \frac{1}{\Delta}\int_{\nu_e}\begin{Bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{Bmatrix} dxdy. \tag{25}$$

Then, the refined displacement gradient given by Equation 24 is obtained according to the weak continuity condition expressed in Equation 22.

In the above discussion, weak continuity conditions are proposed which are used in the refined non-linear non-conforming plate element. Also the refined displacement gradients are derived in accordance with the weak continuity conditions. Note that the displacement gradients $\mathbf{B}_\alpha$, $\mathbf{B}_\beta$ and $\mathbf{B}_\gamma$ are the same as the gradients commonly used in the standard displacement-based non-linear methods. It should also be noted that the proposed weak continuity conditions in the paper are based on the geometrically non-linear generalized variational principle [14], from which the same formulations can be obtained.

In the following part of this paper, the formulation of the total Lagrangian method of standard displacement-based plate element and on that basis, the refined formulation of the proposed NRT15 non-linear plate element are presented.

TOTAL LAGRANGIAN STANDARD FORMULATION OF THE DISPLACEMENT-BASED NON-LINEAR PLATE ELEMENT

There are two kinds of description for geometrically non-linear analysis, total Lagrangian(T.L.) description and updated Lagrangian(U.L.) description. When the non-linear description is based on the initial configuration, it is called total Lagrangian description. In this paper, T.L. description is used to carry out the geometrically non-linear analysis.

Definition of Strain for Geometrically Non-Linear Plate

In this paper, using von-Karman hypothesis, the relationship between the strain and displacement of a non-linear plate can be given as:

$$\mathbf{e} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial \nu}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial \nu}{\partial x} \\ -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} + \begin{Bmatrix} \frac{1}{2}(\frac{\partial w}{\partial x})^2 \\ \frac{1}{2}(\frac{\partial w}{\partial y})^2 \\ (\frac{\partial w}{\partial x})(\frac{\partial w}{\partial y}) \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \mathbf{e}_0^p \\ \mathbf{e}_0^b \end{Bmatrix} + \begin{Bmatrix} \mathbf{e}_L^b \\ 0 \end{Bmatrix}, \tag{26}$$

where:

$$\mathbf{e}_0^p = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial \nu}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial \nu}{\partial x} \end{Bmatrix} \quad \mathbf{e}_0^b = \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

$$\mathbf{e}_L^b = \begin{Bmatrix} \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \end{Bmatrix} \tag{27}$$

Since non-linear strain items exist in the formulation, the incremental iterative analysis has to be employed.

Description of T.L. Iterating States

The whole loading process will be divided into a series of equilibrium states:

$$\Omega^{(0)} \to \Omega^{(1)} \to \cdots \Omega^{(k)} \to \Omega^{(k+1)} \to \cdots \Omega^{(f)}, \tag{28}$$

where $\Omega^{(0)}$ is the initial state, $\Omega^{(f)}$ is the final state and $\Omega^{(k)}$ is an arbitrary state between $\Omega^{(0)}$ and $\Omega^{(f)}$. The reference configuration is the initial configuration $\Omega^{(0)}$ for the total Langragian method and all subsequent formulations for the $K+1$th step are based on the assumption that all the states from $\Omega^{(0)}$ to $\Omega^{(k)}$ together with their components, such as stress, strain, displacement and loading history, can be determined.

Expression of Incremental Strain Tensor for Geometrically Non-Linear Plate

Using the incremental analysis method, the following are defined:

$$\mathbf{U_0} = \{u_0\ \nu_0\ w_0\}^T, \quad \mathbf{U} = \{u\ \nu\ w\}^T, \tag{29}$$

$$\mathbf{\Delta U} = \mathbf{U} - \mathbf{U_0}, \tag{30}$$

where $\mathbf{U_0}$ is the displacement vector at the end of the Kth iterating step and $\mathbf{U}$ is the displacement vector at the end of the $K+1$th iterating step.

Substituting Equations 29 and 30 into Equation 26, the incremental strain tensor is obtained, which can be expressed as:

$$\Delta\mathbf{e} = \begin{Bmatrix} \frac{\partial \Delta u}{\partial x} \\ \frac{\partial \Delta \nu}{\partial y} \\ \frac{\partial \Delta u}{\partial y} + \frac{\partial \Delta \nu}{\partial x} \\ -\frac{\partial^2 \Delta w}{\partial x^2} \\ -\frac{\partial^2 \Delta w}{\partial y^2} \\ 2\frac{\partial^2 \Delta w}{\partial x \partial y} \end{Bmatrix} + \begin{Bmatrix} \left(\frac{\partial w}{\partial x}\right)_0 & 0 \\ \left(\frac{\partial w}{\partial y}\right)_0 & 0 \\ \left(\frac{\partial w}{\partial x}\right)_0 & \left(\frac{\partial w}{\partial y}\right)_0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix} \begin{Bmatrix} \frac{\partial \Delta w}{\partial x} \\ \frac{\partial \Delta w}{\partial y} \end{Bmatrix}$$

$$+\frac{1}{2} \begin{Bmatrix} \left(\frac{\partial \Delta w}{\partial x}\right) & 0 \\ \left(\frac{\partial \Delta w}{\partial y}\right) & 0 \\ \left(\frac{\partial \Delta w}{\partial x}\right) & \left(\frac{\partial \Delta w}{\partial y}\right) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix} \begin{Bmatrix} \frac{\partial \Delta w}{\partial x} \\ \frac{\partial \Delta w}{\partial y} \end{Bmatrix}$$

$$= \begin{Bmatrix} \mathbf{\Delta e_0^P} \\ \mathbf{\Delta e_0^b} \end{Bmatrix} + \begin{Bmatrix} \mathbf{\Delta e_{L1}^P} \\ 0 \end{Bmatrix} + \begin{Bmatrix} \mathbf{\Delta e_{L2}^P} \\ 0 \end{Bmatrix}. \tag{31}$$

Incremental Constitutive Relation

The stress of the mid-side of the plate can be defined as follows:

$$\sigma = \{T_x\ T_y\ T_{xy}\ M_x\ M_y\ M_{xy}\}^T = \begin{Bmatrix} \sigma^p \\ \sigma^b \end{Bmatrix}, \tag{32}$$

and the increment as:

$$\Delta\sigma = \mathbf{A}\Delta\mathbf{e} = \begin{bmatrix} \mathbf{A^P} & 0 \\ 0 & \mathbf{A^b} \end{bmatrix} \Delta\mathbf{e}, \tag{33}$$

where:

$$\mathbf{A^P} = \frac{Et}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix},$$

$$\mathbf{A^b} = \frac{Et^3}{12(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}, \tag{34}$$

in which E is Young's modulus, μ is the Poisson's ratio and t is the thickness of the plate.

Incremental T.L. Virtual Work Equation

The incremental virtual work equation of the $K+1$th step can be given by:

$$\int_{\nu_e} \left[(\sigma^0 + \Delta\sigma)^T \delta\Delta\mathbf{e} - (\mathbf{\bar{F}^0} + \Delta\mathbf{\bar{F}})^T \delta\Delta\mathbf{u}\right] d\nu^0 - \int_{s\sigma} (\mathbf{\bar{T}^0} + \Delta\mathbf{\bar{T}})^T \delta\Delta\mathbf{u} ds^0 = 0, \tag{35}$$

where $d\nu^0$ and ds^0 are the body and surface elements of the initial configuration $\mathbf{\Omega}^{(0)}$, respectively, σ^0 is Kirchhoff stress tensor at the state $\mathbf{\Omega}^{(k)}$, $\mathbf{e^0}$ is Green strain tensor at the state $\mathbf{\Omega}^{(k)}$, $\mathbf{\bar{F}^0}$ and $\mathbf{\bar{T}^0}$ are body and surface force vectors at the end of state $\mathbf{\Omega}^{(k)}$, $\Delta\sigma$, $\Delta\mathbf{e}$, $\Delta\mathbf{\bar{F}}$ and $\Delta\mathbf{\bar{T}}$ are the incremental vectors of the above vectors at the $K+1$th step.

T.L. FORMULATION OF REFINED NON-CONFORMING PLATE ELEMENT NRT15 FOR GEOMETRICALLY NON-LINEAR ANALYSIS

Displacement Interpolation Function for the Proposed Element

The proposed element is based on the refined nine-parameter element RT9. The assumed displacement function can be given as:

$$\mathbf{U} = \begin{Bmatrix} u \\ \nu \\ w \end{Bmatrix}$$

$$= \begin{bmatrix} L_1 \; L_2 \; L_3 & & \\ & L_1 \; L_2 \; L_3 & \\ & & F_1 \; F_2 \; F_3 \; F_{x1} \; F_{x2} \; F_{x3} \; F_{y1} \; F_{y2} \; F_{y3} \end{bmatrix}$$

$$\times \begin{Bmatrix} \mathbf{q_e^p} \\ \mathbf{q_e^b} \end{Bmatrix}, \tag{36}$$

where:

$$\mathbf{q_e^p} = \{u_1 \; u_2 \; u_3 \; \nu_1 \; \nu_2 \; \nu_3\}, \tag{37}$$

$$\mathbf{q_e^b} = \{w_1 \; w_2 \; w_3 \; w_{x1} \; w_{x2} \; w_{x3} \; w_{y1} \; w_{y2} \; w_{y3}\}, \tag{38}$$

$$L_1 = (a_1 + b_1 x + c_1 y)/2\Delta, \tag{39}$$

$$a_1 = x_2 y_3 - x_3 y_2 \quad b_1 = y_2 - y_3 \quad c_1 = x_3 - x_2, \tag{40}$$

$$F_1 = L_1 + L_1^2 L_2 + L_1^2 L_3 - L_1 L_2^2 - L_1 L_3^2, \tag{41}$$

$$F_{x1} = c_3 L_1^2 L_2 - c_2 L_1^2 L_3 + (c_3 - c_2) L_1 L_2 L_3 / 2 \tag{42}$$

$$F_{y1} = b_2 L_1^2 L_3 - b_3 L_1^2 L_2 + (b_2 - b_3) L_1 L_2 L_3 / 2. \tag{43}$$

Through cyclic permutation, the expressions for Equations 41 to 43, F_2, F_3, F_{x2}, F_{x3}, F_{y2} and F_{y3} can be obtained.

From Equation 36, it can be seen that linear interpolation functions are used for the displacements u and ν and the interpolation function for the displacement w is the same as that of element RT9.

Expression of Incremental Strain

Substituting Equation 36 into Equation 31, the formulation of incremental strain tensor is obtained, which can be expressed as follows:

$$\Delta \mathbf{e} = \mathbf{B_L} \Delta \mathbf{q} = (\mathbf{B_{L1}} + \mathbf{B_{L2}} + \mathbf{B_N}) \Delta \mathbf{q}, \tag{44}$$

where:

$$\mathbf{B_{L1}} = \begin{bmatrix} \mathbf{B_0^p} & 0 \\ 0 & \mathbf{B_0^p} \end{bmatrix} = \begin{bmatrix} \mathbf{C_0^p B_\alpha} & 0 \\ 0 & \mathbf{C_0^b B_\beta} \end{bmatrix}, \tag{45}$$

$$\mathbf{B_{L2}} = \begin{bmatrix} 0 & \mathbf{B_L^b} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{G B_\gamma} \\ 0 & 0 \end{bmatrix}, \tag{46}$$

$$\mathbf{B_N} = \begin{bmatrix} 0 & \mathbf{B_N^b} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{G_1 B_\gamma} \\ 0 & 0 \end{bmatrix}, \tag{47}$$

$$\mathbf{C_0^p} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \mathbf{C_0^b} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{48}$$

$$\mathbf{G} = \begin{bmatrix} a_1^T & 0 \\ 0 & a_2^T \\ a_2^T & a_1^T \end{bmatrix} \quad \mathbf{G_1} = \frac{1}{2} \begin{bmatrix} b_1^T & 0 \\ 0 & b_2^T \\ b_2^T & b_1^T \end{bmatrix}, \tag{49}$$

$$a_i^T = \mathbf{q_e^b}^T \mathbf{B_N}^T \quad b_i^T = \Delta \mathbf{q_e^b}^T \mathbf{B_N}^T. \tag{50}$$

Expression of Refined Strain Tensors in the Proposed Element NRT15

In this part, the refined displacement gradients are derived in detail for the proposed non-linear nonconforming plate element.

First, due to the fact that linear interpolation functions, which satisfy the C^0 continuity condition strictly, are used for the displacements u and ν in the proposed element, no modification is necessary for the first-order gradient $\mathbf{B}_\alpha$.

Secondly, the displacement function of w cannot satisfy the C^1 continuity condition itself, therefore it must be modified here to ensure convergence. In order to obtain the refined second-order gradient tensor of displacement w, the following forms of $\frac{\partial \bar{w}}{\partial s}, \frac{\partial \bar{w}}{\partial n}$ are assumed:

$$\begin{Bmatrix} \frac{\partial \bar{w}}{\partial n} \\ \frac{\partial \bar{w}}{\partial s} \end{Bmatrix} =$$

$$- \begin{bmatrix} 0 & L_1 & 0 & 0 & L_2 & 0 \\ 6L_1L_2/L & 0 & L_1(L_1 - 2L_2) & -6L_1L_2/L & 0 & -L_2(2L_2 - L_1) \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & -l & -m & & & \\ 0 & m & -l & & & \\ & & & 1 & 0 & 0 \\ & & & 0 & -l & -m \\ & & & 0 & m & -l \end{bmatrix} \begin{Bmatrix} w_1 \\ w_{x,1} \\ w_{y,1} \\ w_2 \\ w_{x,2} \\ w_{y,2} \end{Bmatrix} \tag{51}$$

Substituting Equation 51 into Equation 19, the refined second-order gradient of displacement w is obtained,

$$\mathbf{B}_\beta^* = \mathbf{B}_\beta - \mathbf{B}_{\beta 0} + \mathbf{B}_{\beta c}, \tag{52}$$

where:

$$\mathbf{B}_{\beta c} = [\mathbf{B}_{\beta c}^1 \; \mathbf{B}_{\beta c}^2 \; \mathbf{B}_{\beta c}^3], \tag{53}$$

$$\mathbf{B}_{\beta c}^1 = \frac{1}{\Delta} \times$$

$$\begin{bmatrix} l_1 m_1 - l_3 m_3 & (l_1^2 y_{21} + l_3^2 y_{13})/2 & (l_1^2 x_{12} + l_3^2 x_{31})/2 \\ l_3 m_3 - l_1 m_1 & (m_1^2 y_{21} + m_3^2 y_{13})/2 & (m_1^2 x_{12} + m_3^2 x_{31})/2 \\ 2(m_1^2 - m_2^2) & l_1^2 x_{12} + l_3^2 x_{31} & m_1^2 y_{21} + m_3^2 y_{13} \end{bmatrix} \tag{54}$$

$$l_1 = y_{21} / \sqrt{x_{12}^2 + y_{21}^2} \quad m_1 = x_{12} / \sqrt{x_{12}^2 + y_{21}^2} \tag{55}$$

$$y_{21} = y_2 - y_1 \quad x_{12} = x_1 - x_2. \tag{56}$$

Through cyclic permutation, the expression of the matrices $\mathbf{B}^2_{\beta c}$, $\mathbf{B}^3_{\beta c}$ and, consequently, the matrix $\mathbf{B}_{\beta c}$ can be obtained.

Finally, if the same third-order displacement interpolation function is used for displacement w as the one used in the linear analysis, the C^0 continuity condition can be satisfied naturally. However, numerical results show that the accuracy is too low in solving the typical cantilever problem. In order to improve accuracy, when forming $\mathbf{B}_\gamma$ of the proposed element NRT15, a combination of the third-order and linear interpolation functions are used to define displacement function w, which is given by:

$$w^* = w_3 + \alpha(w_3 - w_1), \tag{57}$$

where:

$$w_1 = \sum_{i=1}^{3} L_i w_i \quad (i = 1, 3). \tag{58}$$

It is known that both interpolation functions w_1 and w_3 satisfy the C^0 continuity condition strictly. Consequently, the new combination form w^* satisfies the converging condition also.

It should also be noted that the different choice of the variable α might also effect the accuracy of the results. In this paper, the results have been compared according to the different values of α. Numerical results show that NRT15 has the highest accuracy when the value of α is considered as -0.89.

With the new interpolation function w^*, the new first-order gradient vector $\mathbf{B}^*_\gamma$ is obtained, which can be expressed as:

$$\mathbf{B}^*_\gamma = \begin{Bmatrix} \frac{\partial w^*}{\partial x} \\ \frac{\partial w^*}{\partial y} \end{Bmatrix}. \tag{59}$$

Replacing $\mathbf{B}_\beta$ and $\mathbf{B}_\gamma$ with the refined gradient vectors $\mathbf{B}^*_\beta$ and $\mathbf{B}^*_\gamma$, the refined strain matrices $\mathbf{B}^*_{L1}$, $\mathbf{B}^*_{L2}$ and $\mathbf{B}^*_{\mathrm{N}}$ are derived, which can be expressed as:

$$\mathbf{B}^*_{\mathrm{L1}} = \begin{bmatrix} \mathbf{B}^{\mathrm{p}}_0 & 0 \\ 0 & \mathbf{B}^{\mathrm{b}^*}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{C}^{\mathrm{p}}_0\mathbf{B}_\alpha & 0 \\ 0 & \mathbf{C}^{\mathrm{b}}_0\mathbf{B}^*_\beta \end{bmatrix}, \tag{60}$$

$$\mathbf{B}^*_{\mathrm{L2}} = \begin{bmatrix} 0 & \mathbf{B}^{\mathrm{b}^*}_{\mathrm{L}} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{G}\mathbf{B}^*_\gamma \\ 0 & 0 \end{bmatrix}, \tag{61}$$

$$\mathbf{B}^*_{\mathrm{N}} = \begin{bmatrix} 0 & \mathbf{B}^{\mathrm{b}^*}_{\mathrm{N}} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{G}_1\mathbf{B}^*_\gamma \\ 0 & 0 \end{bmatrix}, \tag{62}$$

then:

$$\Delta\mathbf{e}^* = (\mathbf{B}^*_{\mathrm{L1}} + \mathbf{B}^*_{\mathrm{L2}} + \mathbf{B}^*_{\mathrm{N}})\Delta\mathbf{q}. \tag{63}$$

T.L. Incremental Equilibrium Equation for the Proposed Element NRT15

Substituting Equations 64 and 33 into Equation 35, the discrete equilibrium equation of the plate element is obtained, which can be expressed as:

$$(\mathbf{K}_{\mathrm{L}} + \mathbf{K}_\sigma)\Delta\mathbf{q} = \Delta\mathbf{p} - \Delta\mathbf{R} - \mathbf{R}^0, \tag{64}$$

where:

$$\begin{aligned} \mathbf{K}_{\mathrm{L}} &= \int_{\nu_e} \mathbf{B}^T_{\mathrm{L}}\mathbf{A}\mathbf{B}_{\mathrm{L}} d\nu^0 \\ &= \int_{\nu_e} \mathbf{B}^{*}_{\mathrm{L1}}{}^T\mathbf{A}\mathbf{B}^*_{\mathrm{L1}} d\nu^0 \\ &+ \int_{\nu_e} (\mathbf{B}^{*}_{\mathrm{L1}}{}^T\mathbf{A}\mathbf{B}^*_{\mathrm{L2}} + \mathbf{B}^{*}_{\mathrm{L2}}{}^T\mathbf{A}\mathbf{B}^*_{\mathrm{L1}} + \mathbf{B}^{*}_{\mathrm{L2}}{}^T\mathbf{A}\mathbf{B}^*_{\mathrm{L2}}) d\nu^0 \\ &= \mathbf{K}_0 + \mathbf{K}_{\mathrm{LG}}, \end{aligned} \tag{65}$$

in which $\mathbf{K}_0$ is the same as the linear stiffness matrix and $\mathbf{K}_{\mathrm{LG}}$ is called initial displacement matrix. $\mathbf{K}_0$ and $\mathbf{K}_{\mathrm{L}}$ can be presented in detail as follows:

$$\mathbf{K}_0 = \int_{\nu_e} \begin{bmatrix} \mathbf{K}^{\mathrm{p}}_0 & 0 \\ 0 & \mathbf{K}^{\mathrm{b}}_0 \end{bmatrix} d\nu^0$$

$$\mathbf{K}_{\mathrm{LG}} = \int_{\nu_e} \begin{bmatrix} 0 & \mathbf{K}^{\mathrm{pG}}_{0\mathrm{L}} \\ \mathbf{K}^{\mathrm{Gp}}_{0\mathrm{L}} & \mathbf{K}^{\mathrm{G}}_{\mathrm{L}} \end{bmatrix} d\nu^0 \tag{66}$$

in which:

$$\mathbf{K}^{\mathrm{p}}_0 = \mathbf{B}^{\mathrm{p}}_0{}^T\mathbf{A}^{\mathrm{p}}\mathbf{B}^{\mathrm{b}}_0, \tag{67}$$

$$\mathbf{K}^{\mathrm{b}}_0 = \mathbf{B}^{\mathrm{b}*}_0{}^T\mathbf{A}^{\mathrm{b}}\mathbf{B}^{\mathrm{b}*}_0, \tag{68}$$

$$\mathbf{K}^{\mathrm{pG}}_{0\mathrm{L}} = \mathbf{B}^{\mathrm{p}}_0{}^T\mathbf{A}^{\mathrm{p}}\mathbf{B}^{\mathrm{b}*}_L, \tag{69}$$

$$\mathbf{K}^{\mathrm{Gp}}_{0\mathrm{L}} = \mathbf{B}^{\mathrm{b}*}_{\mathrm{L}}{}^T\mathbf{A}^{\mathrm{p}}\mathbf{B}^{\mathrm{p}}_0, \tag{70}$$

$$\mathbf{K}^{\mathrm{G}}_{\mathrm{L}} = \mathbf{B}^{\mathrm{b}*}_{\mathrm{L}}{}^T\mathbf{A}^{\mathrm{p}}\mathbf{B}^{\mathrm{b}*}_{\mathrm{L}}, \tag{71}$$

where the matrix $\mathbf{K}_\sigma$ is called the initial stress matrix, expressed as:

$$\mathbf{K}_\sigma = \int_{\nu_e} \mathbf{B}^{*T}\mathbf{Q}\mathbf{B}^* d\nu^0, \tag{72}$$

while the matrix $\mathbf{Q}$ can be expressed as:

$$\mathbf{Q} = \begin{bmatrix} T_x & T_{xy} \\ T_{xy} & T_y \end{bmatrix}. \tag{73}$$

The right part of Equation 56 can be defined as the unbalanced force vector and its various items can be

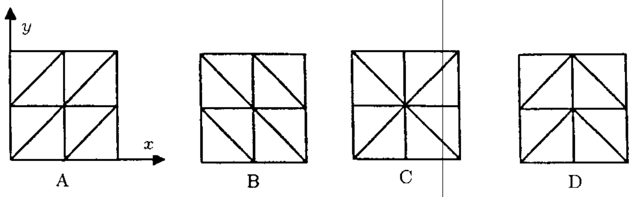

Figure 1. Four different kinds of regular mesh division.

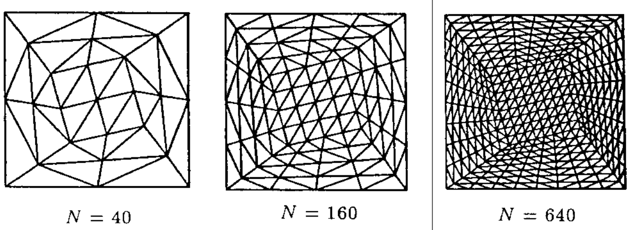

Figure 2. Three types of mesh for E orientation.

expressed by:

$$\Delta \mathbf{p} = \int_{\nu_e} \mathbf{N}^{*T} \Delta \bar{\mathbf{F}} d\nu^0 + \int_{s\sigma} \mathbf{n}^{*T} \Delta \bar{\mathbf{T}} ds^0, \tag{74}$$

$$\Delta \mathbf{R} = \int_{\nu_e} \mathbf{B}_{\mathbf{L}2}^{*}{}^T \sigma^0 d\nu^0,$$

$$\mathbf{R}^0 = \int_{\nu_e} (\mathbf{B}_{\mathbf{L}1}^{*}{}^T \sigma^0 - \mathbf{N}^{*T} \bar{\mathbf{F}}^0) d\nu^0 - \int_{s\sigma} \mathbf{n}^{*T} \bar{\mathbf{T}}^0 ds^0. \tag{75}$$

It should be noted that when using the linearization method in small incremental steps, the higher-order vector $\mathbf{B}_{\mathbf{N}}^{*}$ of the incremental strain vector $\mathbf{B}_{\mathbf{L}}$ could be omitted, i.e.,

$$\mathbf{B}_{\mathbf{L}} = \mathbf{B}_{\mathbf{L}1}^{*} + \mathbf{B}_{\mathbf{L}2}^{*}.$$

NUMERICAL EXAMPLES

Example 1

A simply supported square plate is subjected to a uniformly distributed load $p = 1.0$, with length $L = 100.0$, thickness $t = 1.0$, Young's modulus $E = 2.1 \times 10^6$ and Poisson's ratio $\mu = 0.25$. In this example, five different mesh divisions A-E are considered. For mesh A-D, only one-quarter of the plate is used in the analysis because of symmetry (Figure 1). However, for the irregular mesh E, the whole plate is divided into 40, 160 and 640 elements, respectively (Figure 2). The results are compared with DKT, BCIZ and the analytical solution in Tables 1 and 2. It should be noted that α is considered equal to -0.89 as stated previously.

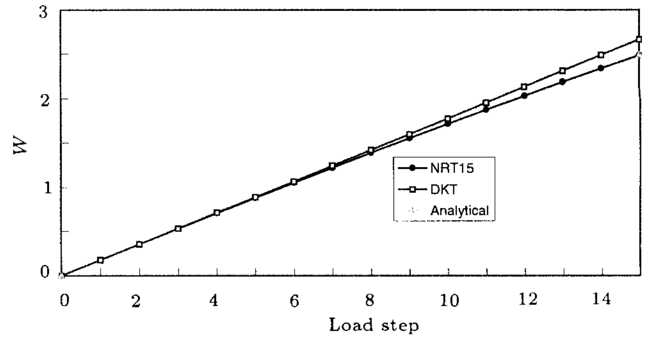

Figure 3. Deflection of cantilever plate.

Example 2

A clamped square plate is subjected to uniformly distributed load $p = 3.0$, with $L = 100.0$, $t = 1.0$, $E = 2.1 \times 10^6$ and $\mu = 0.316$. The results for mesh division A are presented in Table 3 and compared with those of the elements DKT and BCIZ.

Example 3

A cantilever square plate is subjected to a loading of 1) $P = 400$ N or 2) $P = 1400$ N at the free end (opposite of the clamped side). The Young's modulus is 210×10^6 N/m^2, $L = 10$ m, $t = 0.1$ m and $\mu = 0.0$. In this problem, the regular mesh division D and the irregular mesh division E have been used for the computation. The results are compared with DKT and are presented in Table 4 and Figure 3.

Example 4

A clamped circular plate is subjected to a uniform load q, assuming Poisson's ratio $\mu = 0.3$, Young's modulus $E = 10^7$, radius $R = 100$, and thickness $t = 2$. A quarter of the plate is modelled with 6, 24 and 96 elements, respectively. The different types of mesh are shown in Figure 4. The load factor (qR^4/Et^4) is from 1 to 15, while 15 equal load steps are used. The constant α is again considered as -0.89 for the proposed element NRT15. The results are presented in Table 5 and Figure 5 and compared with the analytical solution as well as the other results. The element QS is an 8-node ISO-parameter non-linear plate element using a 2×2 reduced integration.

Example 5

A square plate, with two opposite edges clamped and the others simply supported, is subjected to a uniform lateral load q, with $\mu = 0.3$, $E = 0.3 \times 10^7$ and $t/a = 0.01$. The load factor (qa^4/Et^4) is from 0.91575 to 9.1575 in 20 equal steps. Five types of mesh division, identical to that of Example 1, are considered here.

Table 1. The results of a simply supported square plate subject to uniformly distributed load (mesh division A).

	Analytical Solution [16]	Computational Results			
			2 × 2	4 × 4	8 × 8
Linear	2.176	NRT15	2.1550	2.1723	2.1755
		DKT	2.1718	2.1773	2.1769
		BCIZ	2.1550	2.1724	–
Non-Linear	0.940	NRT15	0.9737	0.9493	0.9433
		DKT	0.9741	0.9496	0.9434
		BCIZ	0.9947	1.0275	–

Table 2. The results of Example 1 (mesh division B-E).

	Analytical Solution [16]	Present Element Solution by Different Meshes				
			2 × 2	4 × 4	6 × 6	8 × 8
Non-Linear	0.940	Mesh B	0.8760	0.9263	0.9347	0.9390
		Mesh C	0.9445	0.9403	0.9407	0.9410
		Mesh D	0.9284	0.9371	0.9395	0.9407
		Mesh E	0.8902*	0.9341**	0.9397***	

*, ** and ***, the whole plate is divided into 40, 160 and 640 elements, respectively.

Table 3. The results of a clamped square plate subject to a uniformly distributed load.

	Analytical Solution [17]	Computational Results			
			2 × 2	4 × 4	8 × 8
Linear	1.9443	NRT15	2.2676	2.043	1.9757
		DKT	2.3887	2.0795	1.9853
		BCIZ	–	2.0431	1.9755
Non-Linear	1.151	NRT15	1.3365	1.1960	1.1550
		DKT	1.3450	1.2053	1.1580
		BCIZ	–	1.2962	1.3155

Table 4. A cantilever square plate subject to a concentrated load P.

Analytical Solution [18]		Mesh Type		DKT	Present Solution		
					$\alpha = -0.89$	$\alpha = -0.0$	$\alpha = -1.0$
Linear	$P = 400$ N	Regular	2 × 2	0.7670	0.753	0.753	0.753
			4× 4	0.7624	0.760	0.760	0.760
	$W = 0.7619$		8 × 8	0.7620	0.7616	0.7616	0.7616
Non-Linear	$P = 400$ N $W = 0.7547$	Regular	2 × 2	0.7652	0.6776	0.2956	0.7510
			4 × 4	0.7629	0.7316	0.4095	0.7599
			8 × 8	0.7620	0.7561	0.5819	0.7615
		Irregular	40		0.6733	0.3730	0.6820
			160		0.7478	0.5175	0.7583
			640		0.7587	0.6370	0.7615
	$P = 1400$ N $W = 2.495$	Regular	2 × 2	2.680	1.6506	0.5139	2.6284
			4 × 4	2.670	2.1041	0.7447	2.6599
			8 × 8	2.669	2.4847	1.2293	2.6633

Table 5. The results of clamped circular plate under a uniformly distributed load.

Load Step	Analytical Solution [19]	Present solution by Different Meshes			DKT	QS [20]
		6	24	96	24	12
1	0.169	0.1624	0.1685	0.1686	0.1715	0.1614
2	0.323	0.3155	0.3247	0.3241	0.3300	0.3116
3	0.457	0.4543	0.4630	0.4610	0.4697	0.4453
6	0.761	0.7852	0.7828	0.7752	0.7908	0.7566
10	1.035	1.0924	1.0747	1.0600	1.0824	1.0417
15	1.279	1.3654	1.3334	1.3143	1.3417	1.2950

Table 6. A square plate with two opposite edges clamped and the others simply supported under a uniform lateral load (mesh division A).

Load	Classical Theory [21]	Present Solution			DKT
		4×4	6×6	8×8	6×6
0.91575	0.019149	0.01959	0.01936	0.01928	0.0195
4.57880	0.095120	0.09733	0.09619	0.09577	0.0969
6.86810	0.141550	0.14486	0.14315	0.14252	0.1442
9.15750	0.186710	0.19114	0.18880	0.18800	0.1902

$*\alpha = -0.89$ for the proposed element

The results are presented in Tables 6 and 7. It should be noted that α is taken as -0.89 in Table 7.

CONCLUSION AND DISCUSSION

1. Various weak continuity conditions, according to displacement gradients, can be introduced into the formulation of geometrically non-linear non-conforming plate elements to ensure convergence.
2. Combined displacement function can be used to formulate the geometrical stiffness matrix, which can improve the accuracy of geometrically non-linear analysis.
3. A refined geometrically non-linear non-conforming plate element NRT15 is proposed on the basis of the weaker continuity conditions and the combined displacement function. From Examples 1 and 2, it can be seen that the non-refined element BCIZ is less accurate compared with the refined element NRT15 in non-linear analysis. From the computed examples, it can be also noticed that both NRT15 and DKT elements possess higher accuracy for most of the problems. For the cantilever problem in Example 3, both elements can converge to the analytical solution in the linear field. However, for the corresponding non-linear problem, although DKT element still converges to a linear solution, the proposed element RT15 demonstrates better accuracy. Furthermore, it is possible to conclude that the proposed element RT15 possesses higher

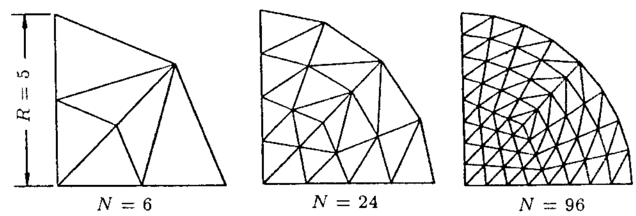

Figure 4. Three meshes for a circular plate quadrant.

Table 7. A square plate with two opposite edges clamped and the others simply supported under a uniformly distributed load (mesh divisions B-E).

Load	Classical Theory [21]	Present Solution				
			2×2	4×4	6×6	8×8
9.1575	0.18671	Mesh B	0.1682	0.1822	0.1848	0.1861
		Mesh C	0.1853	0.1867	0.1869	0.1869
		Mesh D	0.1825	0.1852	0.1863	0.1868
		Mesh E	0.1780*	0.1847**	0.1865***	

*, ** and ***, the whole plate is divided into 40, 160 and 640 elements, respectively.

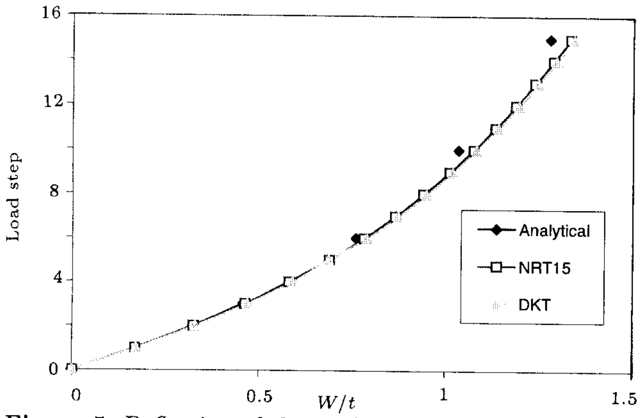

Figure 5. Deflection of clamped circular plate under uniform load.

accuracy compared with DKT and BCIZ and is, therefore, an excellent element for geometrically non-linear analysis.

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