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Research Note

Modelling Transient Unsaturated Flow Problems Using Control Volume Numerical Methods

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In this paper, an unsteady two-equation model that describes the flow of water in unsaturated zones is developed. The governing equations are based on general conservation laws of mass and momentum. Although the obtained governing equations are not amenable to being solved by analytical means, a control volume numerical solution using upwind scheme can be sought. In order to test the model equations, two test examples of one- and two-dimensional water-infiltration are simulated. The effects of discretization on spatial and temporal coordinates are demonstrated. Also, the results are compared with the results of previous studies.

INTRODUCTION

Prediction of water infiltration in unsaturated soil is important from the viewpoints of many branches of science and engineering, including water engineering, agricultural engineering and environmental engineering. In order to solve the relative problems accurately, Richards equation [1] in one of its three different forms, i.e., h-based, θ-based or mixed-based, is used. With few exceptions [2-5], almost all authors have used the h-based form or θ-based form of Richards equations and finite difference or finite element numerical models for their solutions. For instance, the models of Neuman [6], Haverkamp et al. [7], Huyakorn et al. [8], Celia et al. [9], Rathfelder and Abriola [10] and Pan Wierenga [11] can be enumerated. However, Celia et al. [9] showed that numerical analyses based on different forms of Richards equation [1] give rise to different solutions. They also showed that the time step marching has a profound effect on the final solution. It is also worth noting that Richards equation is a quasi-steady equation derived using Darcy law together with unsteady material balance equation. Furthermore, it has a severe non-linearity with the hydraulic conductivity such that different inter node averaging technique on the hydraulic conductivity may result in different solutions. Barker et al. [12] indicated that for very large fixed vertical space steps, the arithmetic mean could lose up to 1/2 of the cumulative mass infiltrated in problem of infiltration. Li [5] demonstrated that different inter node averaging scheme on the hydraulic conductivity may result in different solutions. Yu [13] showed that grid-spacing has an effect on hydraulic simulation.

Arastoopour and Semrau [14] have developed a new model to simulate low permeability porous media (less than 1 milidarcy). For their modelling, they used unsteady material balance equation [15,16] in conjunction with unsteady momentum equation [17]. Although the convective term of momentum transfer is negligible, Arastoopour and Semrau [14] also kept it in the unsteady equation. Then, they compared their model with experimental data as well as with the case in which conventional Darcy law is used as the momentum equation. They concluded that for low permeability porous media, the model momentum equation provides better results than the conventional Darcy law. Gray and O'Neill [17] had also shown that for slow, steady, incompressible and Newtonian fluid flow, the unsteady momentum equation reduces to the conventional Darcy law. Therefore, the momentum equation used by Arastoopour and Semrau [14] was a generalized equation or an extended form of the Darcy law. In addition, Mowla and Nasrifar [18] indicated that in high permeability porous media (more than 1 darcy), the inertial terms become important and Darcy law in its conventional form will not be accurate enough. Consequently, they extended the modified Darcy law in the form of unsteady momentum equation of Gray and O'Neill [17] and obtained good results in simulation of a two-phase flow unsaturated zone. Both.

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Arastoopour and Semrau [14] and Mowla and Nasrifar [18], used the method of lines for their numerical solution. The method of lines requires discritizing the material balance and momentum balance equations at each node and then integrating all of the discritized equations at all nodes simultaneously. Obviously, as the number of nodes and problem dimensions increase, the method becomes cumbersome and useless.

Therefore, in this study, attempts are made to use a general form of Darcy law in order to extend the model and overcome the problem with very low and very high permeability porous media. Many modified forms of Darcy law have been reported, e.g. [19]; however, here, the work of Mowla and Nasrifar [18] has been used. Obviously, the model is different from that of [1] and requires different numerical solution. Thus, a control volume numerical solution that seems to be suitable for model equations is considered. The model is, then, compared at limiting conditions with other models in order to show its applicability.

GOVERNING EQUATIONS

In this study for modelling an unsaturated problem, the conventional approach based on Richards equation [1] is not followed. Instead, the conservation equations of mass and momentum are used as the starting point.

In the literature, there are many forms of material balance equation which are essentially the same. Consider the following equation that was obtained using averaging technique by Whitaker [15] and later in a slightly different form by Gray [16]:

$$\frac{\partial}{\partial t}(\rho\theta) + \nabla.(\rho\theta\vec{\omega_s}) = -\nabla .J + \sigma + m, \tag{1}$$

where $\vec{\omega_s}$ is velocity vector. Equation 1 states that from the view point of a stationary observer, the moisture content increase per unit volume in a porous medium (the first term on the left hand side) is due to several reasons. They are: the convective movement (the second term on the left hand side), diffusion of water, water production due to chemical reaction and water gain due to phase change (the right hand terms in the equation, respectively).

In infiltration problems, there are no phase change and chemical reaction (m and σ are zero), and diffusive transport of water is much less than convective transport (J is negligible). Therefore, Equation 1 becomes:

$$\frac{\partial}{\partial t}(\rho\theta) + \nabla.(\rho\theta\vec{\omega_s}) = 0. \tag{2}$$

It is also assumed that the water and soil are incompressible and so ρ can be omitted from both sides. Consequently, Equation 2 for a two dimensional system becomes:

$$\frac{\partial\theta}{\partial t} + \frac{\partial}{\partial x}(\theta u) + \frac{\partial}{\partial y}(\theta \nu) = 0, \tag{3}$$

where u and v are water velocity components of $\vec{\omega_s}$ in x and y directions, respectively.

The momentum balance equation that is used in this study is a generalized form of Darcy law. It has been derived using an averaging technique [20]:

$$\frac{\partial}{\partial t}(\rho\theta\vec{\omega_s}) + \nabla.(\rho\theta\vec{\omega}.\vec{\omega_s}) = -\theta(\nabla.P - \rho\vec{g}) - \frac{\rho\theta^2 g}{K}\vec{\omega_s} + \mu\theta\nabla^2\vec{\omega_s}. \tag{4}$$

Here, the first two terms on the left-hand side of Equation 4 represent transient behavior and convective movement transfer, respectively. The three terms on the right-hand side account for momentum changes by pressure head and gravity (the first term), drag force due to porous media (the second term) and diffusion of momentum near porous wall (the last term), respectively. As expected, for limiting conditions stated by Gray and O'Neill [17], this reduces to Darcy law. Furthermore, where flow near solid boundary is important, such as in heat transfer in porous media, the diffusive term (the last term on the right hand side) becomes important and Equation 4 reduces to Brinkman equation [21].

For incompressible flow (ρ is constant), appropriate for infiltration problems, by substituting the material balance Equation 2 into momentum balance Equation 4, the following equation, after some mathematical manipulation results:

$$\frac{\partial\vec{\omega_s}}{\partial t} + \vec{\omega}.\nabla.\vec{\omega_s} = -\frac{1}{\rho}\nabla P + \vec{g} - \frac{\theta g}{K}\vec{\omega_s} + \nu\nabla^2\vec{\omega_s}. \tag{5}$$

In water infiltration problems, $\vec{\omega_s}$ is in order of 10^{-6} m/s or less and thus $\vec{\omega_s}.\nabla\vec{\omega_s}$ (the convective term of momentum) is negligible. Also, the diffusion of momentum near solid boundary is negligible with respect to drag forces. Therefore, the last term on the right hand side of the equation vanishes. In addition, taking $gx = 0$, $gy = -g$ and $h = P/\rho g$, the momentum equations in x and y directions reduces to:

$$\frac{1}{g}\frac{\partial u}{\partial t} = -\frac{\partial h}{\partial x} - \frac{\theta}{K}u, \tag{6}$$

$$\frac{1}{g}\frac{\partial \nu}{\partial t} = -\frac{\partial h}{\partial y} - \frac{\theta}{K}\nu - 1, \tag{7}$$

where u and ν are velocity components of velocity vector, $\vec{\omega_s}$ in x and y directions, respectively.

NUMERICAL MODELLING

In order to solve the obtained governing equations, Equations 3, 6 and 7, the control volume method described by Patankar [22] is used. Figure 1 shows the staggered grids that have been used in this study. The idea behind it is to use a different grid for each velocity component. The black points are the location of θ and h. Velocity components are calculated for locations that are on the faces of control volume. In the figure, the locations of velocity components are indicated with arrows. Vertical arrows denote locations for ν and horizontal arrows indicate the locations for u. Letters E, W, N and S (or e, w, n and s) on the figure stand for east, west, north and south, respectively.

According to the control volume method [22], each equation is integrated over a control volume with a weight function of 1 and then the resulting equation is discretized based on one of the different schemes. Starting with material balance Equation 3:

$$(\theta_P - \theta_{P0})\Delta x \Delta y + [(\theta u)_e - (\theta u)_w]\Delta y \Delta t + [(\theta \nu)_n - (\theta \nu)_s]\Delta x \Delta t = 0. \tag{8}$$

Each term at staggered grid location is to be defined regularly. Consider $(\theta u)_e$ for instance. Since the dispersion term has been neglected in the material balance Equation 3, taking upwind scheme formulation is a suitable assumption. Therefore, the value of θ at point e is considered equal to the value of θ at point $P(\theta_P)$ if the velocity direction is from left to right and equal to the value of θ at point $E(\theta_E)$ if the velocity direction is from right to left. In an abstract form without paying attention to the velocity direction, $(\theta u)_e$ may be given by:

$$(\theta u)_e = \theta_P||u_e, 0|| - \theta_E|| - u_e, 0||, \tag{9}$$

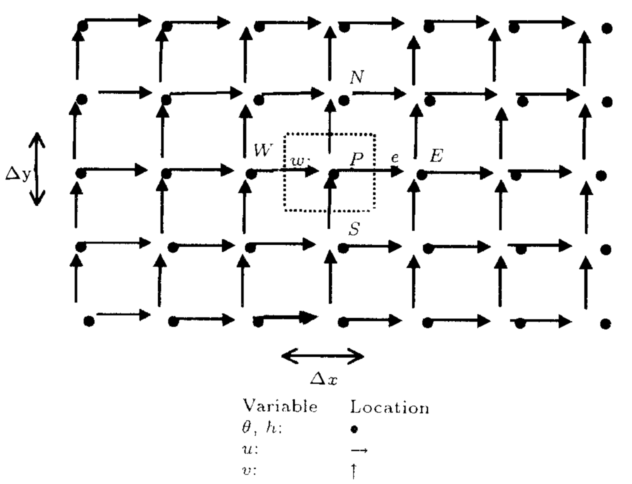

Figure 1. Staggered grid and spatial location of variables.

where $||a, b||$ is an operator that selects the larger value of a and b. Equation 9 automatically reduces to the proper value of $(\theta u)_e$ depending upon the direction of u. Using similar definitions for $(\theta u)_w$, $(\theta \nu)_n$ and $(\theta \nu)_s$ and putting them into Equation 8 yields:

$$c_P\theta_P = c_E\theta_E + c_W\theta_W + c_N\theta_N + c_S\theta_S + f, \tag{10}$$

where c_i's and f are some functions of velocity, spatial coordinates and initial moisture content.

Now consider Equation 6, the momentum balance equation in x-direction. Taking the source term, Darcy forces, equal to S_P, integrating over the control volume and then rearranging the terms gives:

$$\left(\frac{\Delta x.\Delta y}{\Delta t} - S_{Pe}.\Delta x.\Delta y\right)u_e = u_{e0}\frac{\Delta x.\Delta y}{\Delta t} + (h_P - h_E)\Delta y.g, \tag{11}$$

where S_{Pe} may be any averaging scheme on the values of S_P at grid locations of P and E. By defining:

$$a_e = \frac{\Delta x.\Delta y}{\Delta t} - S_{Pe}\Delta x.\Delta y, \tag{12}$$

and:

$$b_e = u_{e0}\frac{\Delta x.\Delta y}{\Delta t}, \tag{13}$$

Equation 6 becomes:

$$a_e u_e = b_e + (h_P - h_E).\Delta y.g. \tag{14}$$

By performing the same procedure for other grid points and inserting the resulted equations in Equation 8:

$$L_P h_P^o = L_E h_E^o + L_W h_W^o + L_N h_N^o + L_S h_S^o + O, \tag{15}$$

is obtained, where $h_i^o = h_i - h_i^*$ and h_i^* is the pressure head for another condition. The coefficients are some functions of moisture content, spatial coordinates, hydraulic conductivity and time. To solve the algebraic equations numerically, SIMPLE (Semi-Implicit Method for Pressure Linked Equations) algorithm is used. For more detailed numerical procedure, the interested reader can refer to [22].

TEST EXAMPLES

In the absence of experimental data and for evaluating the performance of governing equations and their solutions via control volume method, two test examples are considered. First, the problem of one-dimensional infiltration of water in a soil column is considered. The relevant parameters are [5]:

$$\theta(h) = \frac{\alpha(\theta_s - \theta_r)}{\alpha + |h|^\beta} + \theta_r, \tag{16}$$

$$K(h) = K_s . \frac{A}{A + |h|^\gamma}. \tag{17}$$

Here $\alpha = 1.611 \times 10^6$, $\theta_s = 0.287$, $\theta_r = 0.075$, $\beta = 3.96$, $K_s = 9.4 \times 10^{-5}$ m/s, $A = 1.175 \times 10^6$ and $\gamma = 4.74$. The initial condition is $h(y, 0) = -0.615$ m and the boundary conditions are $h(0, t) = h_{\text{top}} = -0.207$ cm, $h(0.4, t) = h_{\text{bottom}} = -0.615$ m. Infiltration is considered to occur over a period of 360 seconds.

The pressure head computed as a function of distance from the top of the soil column is compared with the finite element results of [5] in Figure 2. An up stream weighting on the source term is used in the present results, and the finite element results of [5] were obtained over a dense grid. There are good agreements between the two models except for the region of the flood front where the present results lag behind those of [5]. This is because of difference in modelling, discretization and method of solution. Nevertheless, the two models are comparable.

The influence of averaging scheme on the source term (S_P) is illustrated in Figure 3. It is recalled that the source term in the present formulation is to be evaluated at staggered grid locations (e, w, n, s). The source term, however, is a function of pressure head that is evaluated at main grid locations. Therefore, different definitions for S_P have been evaluated. Figure 3 compares numerical solutions obtained using arithmetic, harmonic, upwind and midstream averaging techniques for S_P. The midstream average (Mid) is the arithmetic mean of the upstream and downstream pressure head. It is observed that the results produced are independent of the averaging scheme, except for the region of the flood front where the upwind technique leads the others. These results suggest that since the control volume formulation presented herein is based on an upwind scheme, it is appropriate to use upwinding on the source term in all further results.

The effects of discretization in time and space are demonstrated in Figures 4 and 5. Figure 4 shows that pressure head profiles for a fixed nodal spacing steepen and converge as the time step is shown in Figure 5. It

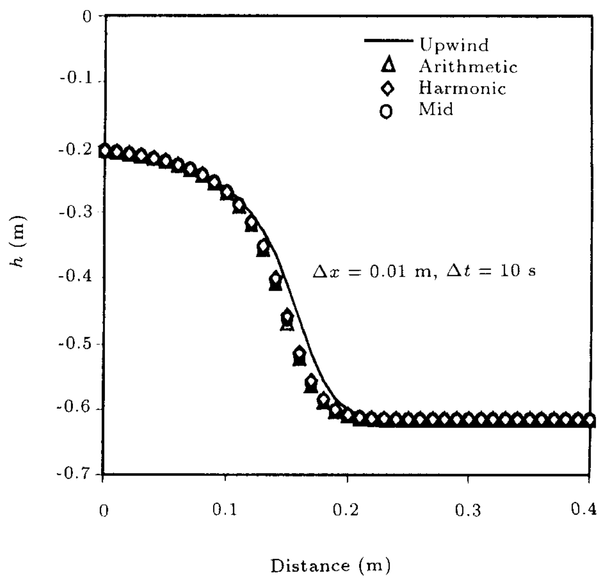

Figure 3. The effect of averaging scheme on the source term of momentum transfer on the water pressure head profile.

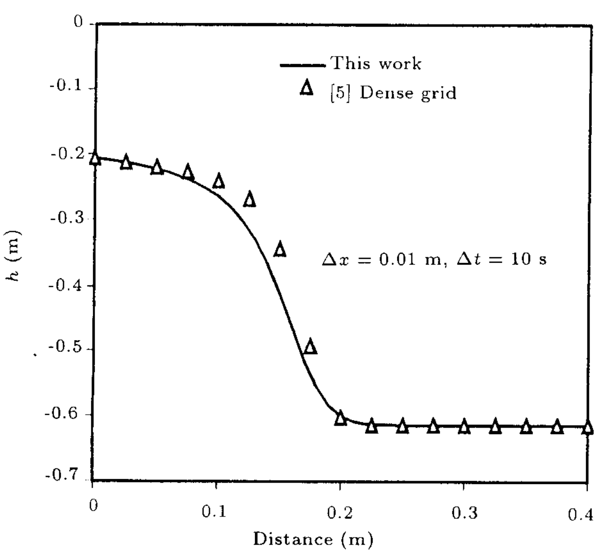

Figure 2. Comparison of the predictions of water pressure head with the graph results of Li [5]

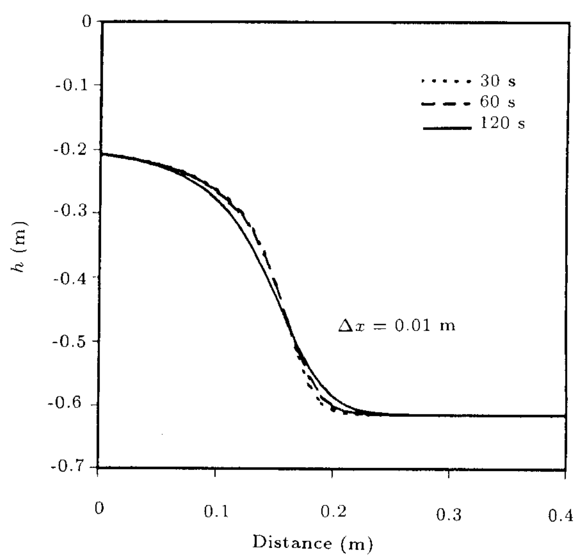

Figure 4. The effect of time step size on the water pressure head profile.

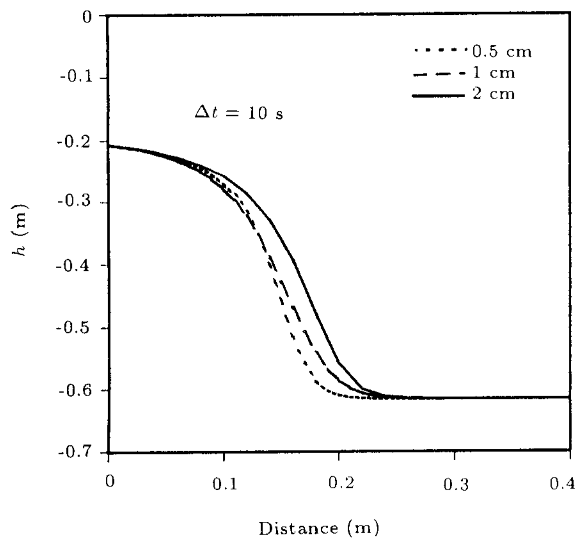

Figure 5. The effect of mesh refinement on the water pressure head profile.

is clear that as the nodal spacing decreases the profiles steepen and become similar at the flood front.

The second example is a two-dimensional infiltration problem depicted in Figure 6. The physical properties are:

$$K(\theta) = 1.734 \times 10^{-7}(\theta - 0.1498), \tag{18}$$

$$\frac{\partial h}{\partial \theta} = 3.33. \tag{19}$$

The initial condition is $h(x, y, 0) = -0.9$ m and the boundary conditions and the system characteristics are shown in the figure.

In the second example two-dimensional infiltration under gravity is considered (see Figure 6). Comparison is made of results obtained in the present study and the finite element and finite difference solutions of [6] and [23] in Figures 7 and 8.

Figure 7 shows the computed pressure head profile along the base of the soil at a distance of 0.01 m above the base and Figure 8 illustrates the same profile along the top of the soil and at a distance of 0.005 m below the top. Close to the inflow boundary, the pressure head predicted by the control volume formulation is slightly higher than the finite difference or finite element predictions. As the distance from the inflow boundary increases, the pressure head in the present simulation becomes higher than the others, the greatest differences are in the region of the flood front itself. In another words, the solution presented here is comparable with those of earlier investigators.

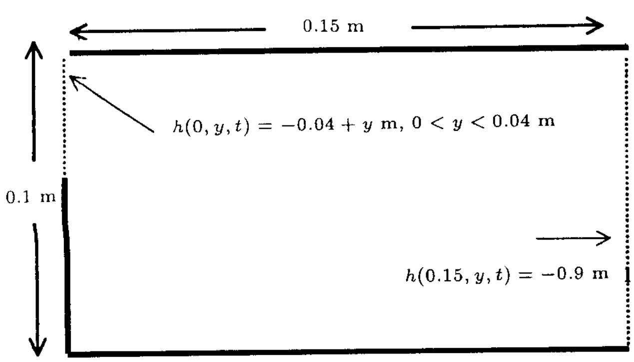

Figure 6. Schematic diagram for a two-dimensional water infiltration problem.

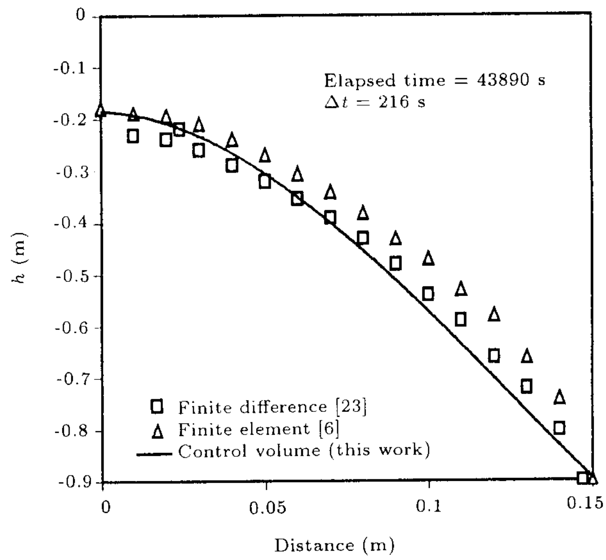

Figure 7. Water pressure head (m) along the length of the soil at a distance of 0.01 m above the base ($\Delta x =$ 0.01 m, $\Delta y = 0.005$ m).

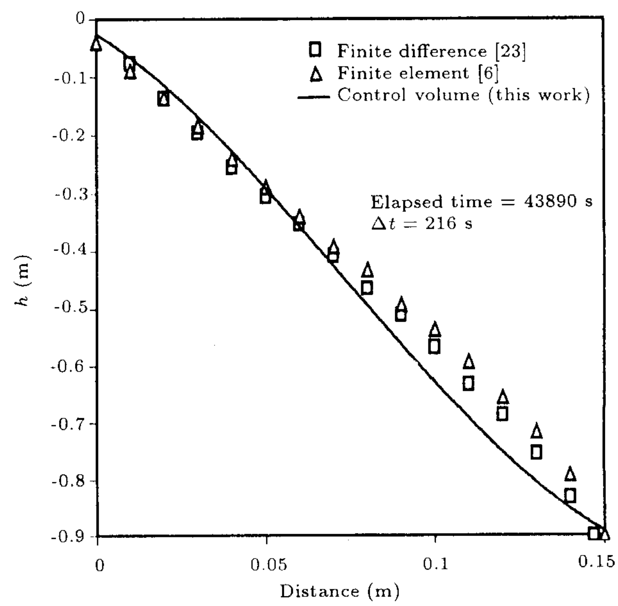

Figure 8. Water pressure head (m) along the length of the soil at a distance of 0.005 m below the top ($\Delta x =$ 0.01 m, $\Delta y = 0.005$ m).

CONCLUSION

In this paper, instead of using Richards equations for water infiltration in unsaturated zone, a different unsteady state model has been employed. Equations governing such infiltration were developed from the basic principles of mass and momentum transfer [18]. This two-equation model does not raise the ambiguity in selection of one of the different types of Richards equation [1]. Because in the presented formulation the source term of momentum (S_i, the most nonlinear term) has appeared as a single term, it has been considered as the source of nonlinearity. Comparison demonstrates that different types of inter node averaging (except for upwinding) do not lead to large differences. This is against the conclusion that has been reached by Li [5] for Richards equation. To solve the obtained equations, a control volume numerical model has been developed. To demonstrate the applicability of this approach, two test examples were simulated. The results were compared with those of other studies. The agreements were good. The convergence properties on the spatial coordinate and time were demonstrated. In summary, a different model other than Richards equation [1] with a control volume numerical solution was developed, which can describe unsaturated flow in an unsaturated zone accurately.

NOMENCLATURE

a, b, c, d, f, L, O	variable coefficients
g	gravitational constant, 9.806 m/s^2
h	water pressure head, m
J	Diffusional mass flux, kg/m^2.s
K	hydraulic conductivity, m/s
m	mass gain due to a source of water, kg/m^3.s
P	water pressure, Pa
S_p	source term in momentum equation s/m
t	time, sec.
u	horizontal velocity, m/s
ν	vertical velocity, m/s
x, y, z	spatial coordinate, m
$\rightarrow$	vector sign

Greek Letters

ρ	water density, kg/m^3
θ	moisture content
σ	mass gain due to chemical reaction, kg/m^3.s
ω_s	velocity in an arbitrary s-direction

Subscripts

e, w, n, s	east, west, north and south, respectively
E, W, N, S	east, west, north and south, respectively
P	center of control volume

Superscripts

$*$	previous conditions
o	difference terms

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