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Diagonalization of ARMA Stationary Autocovariance Matrices

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In this paper, the problem of diagonalization of the autocovariance matrices of stationary processes is discussed. Basically, there is no explicit form for the diagonal matrices. An analytic solution has been provided for stationary ARMA models based on Band matrices. The results have been compared with Dargahi-Noubary [1] and Fuller [2] in a numerical study. It has been demonstrated that the results are more accurate compared to Dargahi-Noubary [1] and Fuller [2] approaches.

INTRODUCTION

Let $\mathbf{y}_t = \{y_1, y_2, \ldots, y_T\}$ be a random vector which follows a stationary $ARMA(p,q)$ process; i.e.:

$$\Phi(B)Y_t = \Theta(B)\epsilon_t,$$

where ϵ_t is a zero mean white noise process with variance σ^2 and:

$$\Phi(B) = 1 + \alpha_1 B + \alpha_2 B^2 + \cdots + \alpha_p B^p,$$

$$\Theta(B) = 1 + \beta_1 B + \beta_2 B^2 + \cdots + \beta_q B^q,$$

where $\alpha's$ and $\beta's$ are real constants and B is backward shift operator such as $BY_t = Y_{t-1}$. Suppose that the stationary condition is satisfied. The covariance matrix of the process is given by:

$$[\Sigma]_{rs} = \sigma_{rs} = \gamma(|r-s|), \tag{1}$$

which is a particular case of a Toeplitz matrix:

$$[\Sigma]_{rs} = \sigma_{rs} = \sigma_{r-s}.$$

These matrices arise frequently in statistical works as covariance matrices of wide-sense stationary processes in time series analysis, stochastic processes, nonparametric theory and some other areas. The diagonal form of these matrices is often used to estimate parameters, likelihood function, discriminant analysis and many other features especialy in time series.

In literature, some attempts have been made to obtain an exact or approximate diagonal form for producing expression for the elements of the inverse of the covariance matrix and its determinent, e.g. [3-9]. These approaches seem to require extremely heavy calculations and involve cumbersome matrices operations.

Using the diagonal form, can also be considered as involving eigenvalues and eigenvectors. It leads to the eigenproblems which are available only for special forms of matrices such as tridiagonal matrices, symmetric band, circular and companion matrices [10]. Dargahi-Noubary and Laycock [11] and Dargahi-Noubary [1] considered the problem as a eigenproblem and gave an approximately diagonal form using fast Fourier transform. A similar approach has been considered in [12-14]. Fuller gave a complementary approach to the full utilization of the covariance matrix in time series. His approach is based on companion and circular matrices. These two approaches made a further restriction, circular covariance matrix, to the stationary covariance matrix. It leads to a simple calculation, but fairly standard, especially for short processes (see [15]).

The purpose of this paper is to give an appropriate expression for the elements of the diagonal form of the covariance matrix and no more restriction has been made to the stationary matrix. The approach is based on a tridiagonal band matrix and eigenvalues and eigenvectors. The paper is organized as follows:

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In the next section, the band matrix of width $2q+1$ is discussed and a close approximation is obtained to the diagonal form of covariance matrix $MA(q)$ processes. The results are analytical and lead to exact diagonal form for $MA(1)$ processes. Then, the attention is devoted to obtaining a diagonal form to covariance matrix $ARMA(1,1)$. Furthermore, two important approaches to diagonalization (i.e. [1,2]) have been described.

Finally, a numerical comparison has been made between this approach and those presented in [1,2] to investigate the performance of the diagonalization of the autocovariance matrix based on this approach.

DIAGONALIZATION OF THE COVARIANCE MATRICES MA(q)

Consider the symmetric $T \times T$ band matrix, B_3, of width band three given by:

$$[B_3]_{rs} = \begin{cases} x_0 & r = s \\ x_1 & |r-s| = 1 \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

where $x_i, i = 0, 1$ is a real number.

The rth eigenvalue of B_3 given in Equation 2 is:

$$\xi_r = x_0 + 2x_1 \cos \frac{r\pi}{T+1}, \tag{3}$$

and the normalized associated eigenvector of B_3 given in Equation 2 is:

$$\zeta_r' = \sqrt{\frac{2}{T+1}} \left\{ \sin \frac{r\pi}{T+1}, \sin \frac{2r\pi}{T+1}, \ldots, \sin \frac{Tr\pi}{T+1} \right\}$$
$$(r = 1, 2, \ldots, T). \tag{4}$$

The $T \times T$ matrix of eigenvectors can be given by:

$$L' = \{\zeta_1, \zeta_2, \ldots, \zeta_T\}. \tag{5}$$

Therefore, the exact diagonal form can be obtained by:

$$L' B_3 L = \Lambda = \text{diag}\{\xi_1, \xi_2, \ldots, \xi_T\}.$$

For a $MA(1)$ process $y_t = \epsilon_t + \beta\epsilon_{t-1}$, the diagonal form of the covariance matrix is:

$$[\Lambda]_{rs} = \begin{cases} 1 + \beta^2 + 2\beta \cos \frac{r\pi}{T+1} & r = s \\ 0 & \text{otherwise}. \end{cases} \tag{6}$$

The symmetric $T \times T$ band matrix, B_{2q+1}, of width $2q+1$ is given by:

$$[B_{2q+1}]_{rs} = \begin{cases} x_{|r-s|} & |r-s| \le q \\ 0 & \text{otherwise}. \end{cases}$$

It is clear that a $MA(q)$ process has a B_{2q+1} form with:

$$x_{r-s} = \sum_{\tau=0}^{q-|r-s|} \beta_\tau \beta_{\tau+|r-s|},$$

where $\beta_0 = 1$. There is no diagonal form matrix for B_{2q+1}. However, B_3^i is approximately a symmetric band matrix of band width $2i+1$ band with some of the elements in the upper corner and the lower corner slightly different. This leads to the approximation of B_{2q+1} by a polynomial in B_3 of degree q. Chan [16] in a numerical study has shown that the best matrix with band 3 in the approximation of $(2q+1)$ symmetric band matrix is given by:

$$[B_3]_{rs} = \begin{cases} -1 & |r-s| = 1 \\ 0 & \text{otherwise}, \end{cases} \tag{7}$$

(see also [17]). The approximation is then:

$$B_{2q+1} \approx c_0 I + c_1 B_3 + c_2 B_3^2 + \cdots + c_q B_3^q, \tag{8}$$

where I is an T identity matrix.

The rth eigenvalue of $\xi_r = -2\cos\frac{r\pi}{T+1}$ and the associated eigenvector are as before. $c_0, c_1, \ldots, c_q$ are constants which can be obtained by equating Σ in Equation 8.

For example in $MA(2)$ processes:

$$x_0 = 1 + \beta_1^2 + \beta_2^2,$$

$$x_1 = \beta_1 + \beta_1\beta_2,$$

$$x_2 = \beta_2,$$

$$x_3 = x_4 = \cdots = x_T = 0.$$

The covariance matrix can be approximated by a power series of B_3 given in Equation 8 so that:

$$c_0 = 1 + \beta_1^2 + \beta_2^2 - 2\beta_2, \; c_1 = -\beta_1(1+\beta_2), \; c_2 = \beta_2.$$

It is easy to show that only two elements $[B_5]_{11}$ and $[B_5]_{TT}$ must be corrected to obtain an exact value. Because:

$$LB_3^m L' = L(B_3 L' L)(B_3 L' L) \ldots (B_3 L' L)$$
$$= (LB_3 L')(LB_3 L') \ldots (LB_3 L') = \Lambda^m,$$

the rth eigenvalue of $\Sigma = B_{2q+1}$ is:

$$LB_{2q+1} L' \approx c_0 I + c_1 \Lambda_3 + \cdots + c_q \Lambda^q,$$

where Λ^m is a diagonal matrix with the elements:

$$[\Lambda^m]_{rs} = \begin{cases} (-2\cos\frac{r\pi}{T+1})^m & \text{r=s} \\ 0 & \text{otherwise}. \end{cases}$$

Therefore, the rth eigenvalue of B_{2q+1} is:

$$\lambda_r \approx \sum_{k=0}^{T} c_k \left(-2\cos\frac{r\pi}{T+1}\right)^k \qquad r = 1,2,\ldots,T.$$

Again the associated eigenvector is as before. The approximately diagonal form for the covariance matrix of $MA(q)$ process is:

$$[\Lambda]_{rs} = \begin{cases} \sum_{k=0}^{T} c_k(-2\cos\frac{r\pi}{T+1})^k + O(\frac{1}{T}) & r = s \\ O(\frac{1}{T}) & r \neq s. \end{cases}$$

DIAGONALIZATION OF THE ARMA(1,1) COVARIANCE MATRIX

An $ARMA(1,1)$ process is given by:

$$x_t = \alpha x_{t-1} + \beta\epsilon_{t-1} + \epsilon_t \qquad (t = 1,2,\ldots,T).$$

The elements of the covariance matrix of this model are:

$$(\Sigma_{\beta,\alpha})_{rs} = \gamma(\tau) = \begin{cases} \frac{1+\beta^2+2\alpha\beta}{1-\alpha^2}\sigma^2 & \tau = 0 \\ \frac{(1+\alpha\beta)(\alpha+\beta)}{1-\alpha^2}\sigma^2 & \tau = 1 \\ \alpha\gamma(\tau-1) & \tau = 2,3,\ldots,T \end{cases} \tag{9}$$

where $\tau = |r-s|$. After some algebra it can be shown that:

$$\gamma(\tau) = \frac{\sigma^2}{1-\alpha^2}\left\{(1+\beta^2)\alpha^{|\tau|} + \beta(\alpha^{|\tau-1|} + \alpha^{|\tau+1|})\right\}$$
$$(\tau = 0,1,\ldots,T-1). \tag{10}$$

Diagonalization of the covariance matrix in $ARMA$ (1,1) process is usually considered in literature to obtain the inverse of the covariance matrix. Many authors have investigated the diagonalization of the covariance matrix and have suggested very complicated approximate matrices such as [4,7].

Without loss of generality consider $\sigma^2 = 1$. The matrix in Equation 9 can be expressed approximately as the product of the covariance matrix of the autoregressive process of order one and of the moving average process of order one, i.e., $\Sigma \approx AB$, where:

$$[A]_{rs} = \frac{\alpha^{|r-s|}}{1-\alpha^2}, \text{ and } [B]_{rs} = \begin{cases} 1+\beta^2 & r = s \\ \beta & |r-s| = 1 \\ 0 & \text{otherwise}, \end{cases} \tag{11}$$

which can be easily obtained from Equation 10 for $r,s = 1,2,\ldots,T$. Only some elements of the first row and the first column have to be corrected to give a precise matrix. The eigenvalues of a triangular band matrix, B_3, with band width 3, have been derived. However, the determinant of Σ is more complicated since it is the product of a band matrix, B, and a non band matrix, A. Calculating the roots of the characteristic equation:

$$|\Sigma - \lambda I| \approx |AB - \lambda I|,$$

requires some algebraic matrix operation manipulation. The approximate inverse matrix of A is given in [4,17,18] as:

$$[A_*^{-1}]_{rs} = \begin{cases} 1+\alpha^2 & r = s \\ -\alpha & |r-s| = 1 \\ 0 & \text{otherwise}. \end{cases}$$

Calculating the eigenvalues of $|AB - \lambda I|$ is equivalent to calculating the eigenvalues of $|B - \lambda A_*^{-1}|$.

Since A^{-1} is positive definite, there exists an orthogonal matrix such that $E'A^{-1}E = I$. Define $E = L$ where L is given in Equation 5. Since the matrix A_*^{-1} is a triangular band matrix, the following can be given:

$$LA_*^{-1}L' = [LA_*^{-1}L']_{rs}$$
$$= \begin{cases} 1+\alpha^2 - 2\alpha\cos\frac{\pi r}{T+1} & \text{r=s} \\ 0 & \text{otherwise} \end{cases}$$
$$= (LA_*^{-\frac{1}{2}})(LA_*^{-\frac{1}{2}})'.$$

Now define the matrix $F = LA_*^{-\frac{1}{2}}$, consequently, $F' = \Lambda^{-1/2}L$,

$$F'BF = \Lambda^{-\frac{1}{2}}LB(L'\Lambda^{-\frac{1}{2}}) = \Lambda^{-\frac{1}{2}}(LBL')\Lambda^{-\frac{1}{2}}$$

and:

$$[F'BF]_{rs} = \begin{cases} \frac{1+\beta^2+2\beta\cos\frac{\pi}{T+1}}{1+\alpha^2-2\alpha\cos\frac{\pi}{T+1}} & r = s \\ 0 & \text{otherwise}. \end{cases}$$

Due to the fact that B is a triangular matrix and LBL' is a diagonal matrix given in Equation 6,

$$L\Sigma L' \approx LABL' = \Lambda, \tag{12}$$

where Λ is a diagonal matrix of approximate eigenvalues of Σ, i.e.:

$$\lambda_r \approx \frac{1+\beta^2+2\beta\cos\frac{r\pi}{T+1}}{1+\alpha-2\alpha\cos\frac{r\pi}{T+1}}.$$

DIAGONALIZATION OF THE COVARIANCE MATRIX OF AN ARMA(1,1) PROCESS USING FULLER AND DARGAHI-NOUBARY METHODS

There are other solutions suggested by different authors like Fuller [2], Dargahi-Noubary and Laycock [11] and Dargahi-Noubary [1]. In this section, the performance of the diagonalization of the autocovariance matrix based on (i) The L matrix method, (ii) Fuller approach, namely the F matrix, and (iii) Dargahi-Noubary approach, namely the D matrix, will be studied.

Fuller Approach

The $T \times T$ covariance matrix of a stationary process, say Σ can be written as Equation 1. Fuller [2] made a further assumption that the covariance matrix is such that, $\gamma(j) = \gamma(T-j)$ called a "circular autocovariance matrix" which has been well used in simplifying many analytical problems involving the covariance matrix. Note that this is the covariance matrix of a circular process, $x_{T+t} = x_t$.

This restriction on the process becomes less as T increases (For properties of a circular process see [15,19]). Since Σ is positive semidefinite covariance matrix, there exists an F matrix such that $F'F = I$ and $F'\Sigma F = \Lambda$, where Λ is diagonal and λ_j ($j = 1,2,\ldots,T$) are the characteristic roots of Σ. Fuller's investigation demonstrated that for large T, λ_j are approximately equal to $2\pi f(\omega_j)$, where $f(\omega_j)$ is the spectral density of $\mathbf{x}$ with $\omega_j = \frac{2\pi j}{T}$, ($j = 1,2,\ldots,T-1$). The matrix F suggested by Fuller [2], for providing an approximate diagonal matrix, is:

$$[F]_{rs} = \begin{cases} \frac{1}{\sqrt{2}} & r=1 \\ \sqrt{\frac{2}{T}}\cos\frac{\pi(r-1)s}{T} & r=2,4,\ldots,T-1 \\ \sqrt{\frac{2}{T}}\sin\frac{\pi(r-1)(s-1)}{T} & r=3,5,\ldots,T. \end{cases} \quad (13)$$

for odd n. When n is even, an additional row:

$$\frac{1}{\sqrt{T}}[1,-1,1,\ldots,1,1], \quad (14)$$

is added to F. Fuller also proved that the elements of $F'\Sigma F$ converge to $2\pi\Lambda$ and every element of $F'\Sigma F - 2\pi\Lambda$ is less than $\frac{4d}{T}$ in magnitude value, where d is a finite and given by:

$$\sum_{\tau=-\infty}^{\infty} |\tau||\gamma(\tau)| = d.$$

Dargahi-Noubary Approach

Dargahi-Noubary and Laycock [11] and Dargahi-Noubary [1] gave the discrimination of two stationary processes in the spectral domain. Again they considered the processes to be circular. Dargahi-Noubary [1] used the finite Fourier transform, which is defined as:

$$W_{\mathbf{x}}^{(T)}(\omega_n) = \frac{1}{\sqrt{2\pi T}}\sum_{t=1}^{T} x_t \exp\{-it\omega_n\}$$

for the sequence $\{x_t\}$, $t = 1,2,\ldots,T$ where $\omega_n = \frac{2\pi n}{T}$. The separation of the real and imaginary part leads to:

$$C_{\mathbf{x}}^{(T)}(\omega_n) = \begin{cases} W_{\mathbf{x}}^{(T)}(\omega_n) & n=0 \\ \sqrt{2}\,\mathrm{Re}\,W_{\mathbf{x}}^{(T)}(\omega_n) & n=1,2,\ldots,k \\ \sqrt{2}\,\mathrm{Im}\,W_{\mathbf{x}}^{(T)}(\omega_n) & n=k+1,\ldots,T-1, \end{cases}$$

where Re and Im denote real and imaginary parts, respectively and $T = 2k+1$. In fact the suggested matrix for diagonalizing the autocovariance matrix is:

$$[D]_{r,s} = \begin{cases} \frac{1}{\sqrt{2}} & r=1 \\ \cos\frac{rs}{T} & r=2,3,\ldots,k \\ \sin\frac{(r-k)s}{T} & r=k+1,\ldots,T. \end{cases} \quad (15)$$

Then, he showed that in this case,

$$[D\Sigma D'] = diag[f(\omega_0), f(\omega_1), \ldots, f(\omega_k), f(\omega_1), \ldots, f(\omega_k)]$$

where $f(\omega_t)$ denotes the spectrum of x_t.

NUMERICAL STUDY DIAGONALIZATION OF THE COVARIANCE MATRIX OF ARMA(1,1) PROCESSES USING FULLER'S; DARGAHI-NOUBARY'S; AND THE L METHOD

A numerical study was carried out to compare the performance of the L method with those of Fuller and Dargahi-Noubary. The first example is an $ARMA(1,1)$ process with $T = 5$ for which then $T = 20$, 50 and 100 are used. The three approaches to diagonalize the autocovariance matrix are:

1. $L\Sigma L'$ where "L" is given in Equation 12,
2. $F\Sigma F'$ where "F" is given by Fuller (Equation 13 or 14),
3. $D\Sigma D'$ where "D" is given by Dargahi-Noubary (Equation 15).

For each approach the sum of the principal diagonal elements (SPD) and sum of off-diagonal elements (SOD) have been calculated and then the ratio of the principal sum to the off-diagonal sum has been considered as a criterion for comparing of the three approaches.

Then the example was repeated for $T = 5$, 20, 50 and 100 with different values of parameters. Results are shown in Tables 1 and 2. As can be seen from the tables, L diagonalize Σ better than F or D for all values of α and β. In the case of $\alpha = 0, ARMA(0,1)$(i.e., $MA(1)$) it can be seen that the elements of the off-diagonal $L\Sigma L'$ are zero. This confirms the results found previously where it has been shown that for the $MA(1)$ processes the matrix $L\Sigma L'$ is completely diagonal. Comparison between Fuller approach and Dargahi-Noubary approach shows that Fuller approach gives slightly better results. As T increases both approaches provide better results. However, in all cases the L-method is better than the other two.

Table 1. Comparison between Fuller, Dargahi-Noubary and the L methods in diagonalizing the covariance matrix of $ARMA(1,1)$, $T = 5$, 20.

		Ratio SPD over SOD ($T = 5$)			Ratio SPD over SOD ($T = 20$)		
α	β	***L***	***F***	***D***	***L***	***F***	***D***
0.2	0.2	324.876	24.124	1.897	228.244	1.117	0.715
0.2	0.4	102.104	7.582	1.802	71.734	1.047	0.735
0.3	0.7	182.419	20.066	1.956	125.640	1.115	0.733
0.1	0.8	55.010	2.010	1.297	39.084	0.710	0.654
-0.2	0.3	18.680	1.291	1.006	13.124	0.547	0.532
-0.4	0.8	5.202	0.556	0.524	3.635	0.266	0.311
-0.5	0.8	3.877	0.488	0.463	2.608	0.230	0.281
-0.1	-0.7	57.521	2.049	1.088	40.869	0.765	0.581
0.5	0.5	12.353	2.239	1.166	7.899	0.817	0.530
-0.2	-0.4	102.104	7.054	1.638	71.734	1.065	0.728
0.2	-0.7	13.018	0.912	0.654	9.357	0.430	0.344
0.1	-0.5	35.260	1.288	0.827	25.052	0.565	0.440
-0.8	-0.9	2.885	0.627	0.559	1.397	0.290	0.290
0.8	0.6	2.858	0.751	0.731	1.384	0.432	0.264
0.7	0.1	3.640	0.887	0.771	2.001	0.474	0.300
0.0	0.5	∞	1.688	1.179	∞	0.644	0.613
0.0	0.3	∞	2.453	1.359	∞	0.787	0.667
0.5	0.3	7.059	1.279	0.894	4.514	0.599	0.402
0.8	0.0	2.526	0.663	1.166	1.223	0.388	0.243
-0.5	-0.5	12.353	1.936	0.681	7.899	0.751	0.568

Table 2. Comparison between Fuller, Dargahi-Noubary and the L methods in diagonalizing the covariance matrix of $ARMA(1,1)$, $T = 50$, 100.

		Ratio of SPD over SOD ($T = 50$)			Ratio of SPD over SOD($T = 100$)		
α	β	***L***	***F***	***D***	***L***	***F***	***D***
0.2	0.2	216.782	1.032	0.668	213.359	0.998	0.645
0.2	0.4	68.131	0.938	0.678	67.056	0.889	0.660
0.3	0.7	119.077	1.023	0.681	117.131	0.989	0.665
0.1	0.8	37.165	0.606	0.590	36.589	0.551	0.569
-0.2	0.3	12.465	0.462	0.473	12.268	0.417	0.454
-0.4	0.8	3.468	0.225	0.290	3.420	0.204	0.285
-0.5	0.8	2.482	0.201	0.262	2.447	0.184	0.257
-0.1	-0.7	38.862	0.646	0.552	38.260	0.585	0.543
0.5	0.5	7.421	0.695	0.497	7.283	0.632	0.488
-0.2	-0.4	68.131	0.951	0.684	-	0.901	0.671
0.2	-0.7	8.934	0.351	0.328	8.808	0.311	0.329
0.1	-0.5	23.822	0.467	0.414	23.453	0.417	0.406
-0.8	-0.9	1.240	0.253	0.233	1.201	0.233	0.218
0.8	0.6	1.229	0.352	0.231	1.190	0.312	0.221
0.7	0.1	1.835	0.388	0.271	1.791	0.344	0.261
0.0	0.5	∞	0.546	0.550	∞	0.495	0.529
0.0	0.3	∞	0.677	0.603	∞	0.620	0.582
0.5	0.3	4.241	0.494	0.372	1.051	0.441	0.363
0.8	0.0	1.086	0.314	0.211	12.353	0.279	0.200
-0.5	-0.5	7.421	0.648	0.505	7.283	0.595	0.485

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