Dynamic Oligopolies Without Full Information

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Dynamic Cournot oligopolies are examined when the firms do not have accurate information about price function. After the mathematical model is formulated, the local asymptotical stability of the steady-state is proved under realistic conditions. Instability occurs when information lag is assumed. The possibility of limit cycles is examined, based on the Hopf bifurcation theorem.

INTRODUCTION

Cournot oligopolies are the most intensively discussed economic models since the pioneering work of Cournot [1]. The existence and uniqueness of the equilibrium was the main research area in the case of static models and the stability of the equilibrium in the case of dynamic extensions. Many researchers examined different variants of the original model of Cournot, including single-product oligopolies with and without product differentiation, multi-product models, labor-managed oligopolies, rent-seeking games and oligopsonies. A comprehensive summary of the results on single-product models and literature review are presented in Okuguchi [2] and multi-product models with several applications are the subject of the monograph by Okuguchi and Szidarovszky [3].

In most studies on static and dynamic oligopolies, full information is assumed, that is, it is assumed that all firms have exact knowledge of the market demand function (and, therefore, they know the exact price function as well) and each firm knows its own cost function accurately. In addition, the availability of instantaneous information on the output of the rivals is also assumed. In economic reality, none of these assumptions is realistic, since firms do not report production information to each other. Data on the output of the rivals is obtained via received prices, which are usually reported to the manufacturers with some time delay by the retailers and other sellers.

In this paper, both issues will be addressed: The inaccurate knowledge of the price functions, as well as time lags in obtaining and implementing information on the rivals outputs.

This paper is developed in the following manner: First, the mathematical models will be introduced. Then, discussion regarding the asymptotical behavior of the equilibrium is presented which is followed by the conclusion.

MATHEMATICAL MODELS

Assume that n firms produce the same product, or offer the same service, to the same market. The decision variable of each firm is the volume of its output, xi. Assume that the cost function of firm i is ci(xi), which is assumed to be known exactly by it. Let f be the unit price function, which depends on the output of the industry, Q = ∑j=1n xj. Therefore, the profit of firm i can be given as:

$$\pi_i(x_1, \ldots, x_n) = x_i f\left(\sum_{j=1}^{n} x_j\right) - c_i(x_i).$$  (1)

Let $Q_i = \sum_{j \neq i} x_j$ denote the output of the rest of the industry, then this profit function can be rewritten as:

$$\pi_i(x_i, Q_i) = x_i f(x_i + Q_i) - c_i(x_i).$$  (2)

In the theory of oligopoly, it is usually assumed that functions f and ci are twice continuously differentiable. Furthermore, for all $x_i \geq 0$ and $Q \geq x_i$,

$$x_i f''(Q) + f'(Q) < 0.$$  (A)

$$f'(Q) - c''_i(x_i) < 0.$$  (B)

These two assumptions imply that $\pi_i$ is concave in $x_i$ with fixed values of $Q_i$ and, if each firm has bounded capacity, then the Nikaido-Isoda theorem implies that there is at least one equilibrium $\bar{x} = (x_1^*, \ldots, x_n^*)$.
Dynamic Cournot Oligopolies

It satisfies the following condition for all \( i \) and feasible \( x_i \),

\[
\varphi_i(x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_n^*) \leq \varphi_i(x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_n^*). \tag{3}
\]

This condition shows that none of the firms can improve its profit by changing its output from equilibrium level \( x_i^* \). For any given value of \( Q_i \), the best response, or reaction, of firm \( i \) is the output \( x_i \), that maximizes its profit (Equation 2). The first order conditions are:

\[
f(x_i + Q_i) + x_i f'(x_i + Q_i) - c'_i(x_i) = 0, \tag{4}
\]

where corner optimum is excluded. The second order condition is satisfied by Conditions A and B. Let \( R_i(Q_i) \) denote the best response function. Its derivative can be obtained by implicitly differentiating Equation 4 to get:

\[
R'_i(Q_i) = -\frac{f'(Q) + x_i f''(Q)}{2f'(Q) + x_i f''(Q) - c'_i(x_i)}. \tag{5}
\]

Conditions A and B imply that:

\[-1 < R'_i(Q_i) < 0. \tag{6}\]

A vector \( x^* = (x_1^*, \ldots, x_n^*) \) is a Nash-equilibrium if and only if, each component \( x_i^* \) is a feasible output of firm \( i \) and:

\[
x_i^* = R_i \left( \sum_{j \neq i} x_j^* \right), \tag{7}
\]

for all \( i \). The computation of any positive equilibrium is based on solving Equation 4 for \( i = 1, 2, \ldots, n \), which is a special system of nonlinear algebraic equations. By rewriting Equation 4 as:

\[
f(Q) + x_i f'(Q) - c'_i(x_i) = 0, \tag{8}
\]

and noticing that the left hand side is strictly decreasing in \( x_i \) with fixed values of \( Q \), it can be seen that in the neighborhood of the equilibrium, \( x_i \) is a single-valued function of \( Q, x_i = g_i(Q) \), which can be determined by the repeated solution of Equation 8 with a set of \( Q \) values. The derivative of function \( g \) can be also obtained by differentiating Equation 8 with respect to \( Q \) implicitly to have:

\[
g_i'(Q) = -\frac{f'(Q) + x_i f''(Q)}{f'(Q) - c'_i(x_i)} < 0,
\]

as a consequence of Conditions A and B. Next consider the single-variable nonlinear equation:

\[
Q = \sum_{i=1}^{n} g_i(Q) = 0. \tag{9}
\]

The left hand side is strictly increasing in \( Q \), so this equation can be solved by simple methods, such as the bisection or secant method. Let \( Q^* \) denote the solution, then \( x_i^* = g_i(Q^*) \) gives the \( i \)th component of the equilibrium point for \( i = 1, 2, \ldots, n \).

Assume next that the firms cannot assess the price functions accurately. Firm \( i \) (\( i = 1, 2, \ldots, n \)) believes that the unit price function is some \( f_i \) instead of the true price function \( f \). However, each firm observes the market price and estimates the output of the rest of the industry by computing it based on the unit price received and its own output. If \( Q_i^* \) denotes the estimated total output of the rivals, then the unit price equals:

\[
f(x_i + Q_i) = f_i(x_i + Q_i^*),
\]

which implies that:

\[
Q_i^* = (f_i^{-1} f)(x_i + Q_i) - x_i. \tag{10}
\]

The believed reaction function of firm \( i \) is the output that maximizes its believed profit:

\[
\pi_i^B(x_i, Q_i^*) = x_i f_i(x_i + Q_i^*) - c_i(x_i). \tag{11}
\]

By assuming that Conditions A and B are satisfied with \( f_i \) replacing \( f \) and by excluding corner optimum, it can be seen that the believed reaction function \( x_i = R_i^B(Q_i^*) \) satisfies equation:

\[
f_i(x_i + Q_i^*) + x_i f'_i(x_i + Q_i^*) - c'_i(x_i) = 0. \tag{12}
\]

Differentiate this equation implicitly to see that \( R_i^B(Q_i^*) \) has the same form as Equation 5 with \( f_i \) replacing \( f \), so:

\[-1 < R_i^B(Q_i^*) < 0. \tag{13}
\]

Assume now that each firm adjusts its output into the direction of its best response, then the following dynamic equations are obtained:

\[
\dot{x}_i(t) = k_i (R_i^B(Q_i^*(t)) - x_i(t)),
\]

where \( Q_i^* \) is given in Equation 10 and \( k_i > 0 \) is a given constant. Hence, the resulting dynamic system can be written as:

\[
\dot{x}_i(t) = k_i \left( R_i^B(f_i^{-1} f) \left( \sum_{j=1}^{n} x_j(t) - x_i(t) \right) - x_i(t) \right), \tag{14}
\]

for \( i = 1, 2, \ldots, n \).
Example 1

Assume that each firm has a linear cost function, \(c_i(x_i) = a_ix_i + b_i, \ i = 1, 2, \ldots, n\). Let \(Q = d(p)\) denote the market demand function and assume that each firm mistakenly believes that the demand function is \(\varepsilon_i d(p)\), where \(\varepsilon_i, i = 1, 2, \ldots, n\), is a positive parameter smaller, or larger, than 1. The believed price function by firm \(i\) is the solution of equation \(Q = \varepsilon_i d(p)\) which is:

\[
p = d^{-1}\left(\frac{Q}{\varepsilon_i}\right) = f\left(\frac{Q}{\varepsilon_i}\right),
\]

therefore, the following is obtained:

\[
f_i(Q) = f\left(\frac{Q}{\varepsilon_i}\right).
\]

(15)

Notice that \(f_i^{-1}(p)\) is the solution of equation:

\[
f\left(\frac{Q}{\varepsilon_i}\right) = p,
\]

which is:

\[
Q = \varepsilon_i f_i^{-1}(p) = \varepsilon_id(p).
\]

(16)

Therefore, from Equation 10 it is observed that:

\[
Q_i^\varepsilon = \varepsilon_i(x_i + Q_i) - x_i = (\varepsilon_i - 1)x_i + \varepsilon_iQ_i.
\]

(17)

The believed reaction function is the solution of Equation 12, which has now the special form:

\[
f\left(\frac{x_i + Q_i^\varepsilon}{\varepsilon_i}\right) + \frac{x_i}{\varepsilon_i}f'\left(\frac{x_i + Q_i^\varepsilon}{\varepsilon_i}\right) + a_i = 0,
\]

and if this equation is compared with Equation 4 it is concluded that:

\[
\frac{x_i}{\varepsilon_i} = R_i\left(\frac{Q_i^\varepsilon}{\varepsilon_i}\right),
\]

implying that:

\[
R_i^B(Q_i^\varepsilon) = \varepsilon_iR_i\left(\frac{Q_i^\varepsilon}{\varepsilon_i}\right).
\]

(18)

Based on Equations 17 and 18, System 14 can be rewritten as follows:

\[
\dot{x}_i(t) = k_i\left(\varepsilon_iR_i\left(\frac{\varepsilon_i - 1}{\varepsilon_i} x_i(t) + \sum_{j \neq i} x_j(t)\right) - x_i(t)\right),
\]

(19)

for \(i = 1, 2, \ldots, n\).

As a particular case, assume that \(f(Q) = \frac{A}{Q}\), so, by assuming positive reaction the true reaction function of firm \(i\) is the solution of Equation 4, which has now the form:

\[
\frac{x_i}{Q_i} - x_i\left(\frac{A}{x_i + Q_i}\right)^2 - a_i = 0,
\]

implying that:

\[
x_i = R_i(Q_i)\sqrt{\frac{AQ_i}{a_i} - Q_i}.
\]

(20)

Any positive equilibrium is, therefore, the solution of the following system of nonlinear algebraic equations:

\[
x_i = \sqrt{\frac{A}{a_i} \sum_{j \neq i} x_j - \sum_{j \neq i} x_j}, \ i = 1, 2, \ldots, n.
\]

As a further special case, assume that the marginal costs, \(a_i\), of the firms are identical, \(a_1 = \cdots = a_n = a\). Then, any symmetric equilibrium \((x^*, \ldots, x^*)\) is the solution of equation:

\[
x = \sqrt{\frac{A}{a} (n-1)x - (n-1)x},
\]

that is:

\[
x^* = \frac{A(n-1)}{an^2}.
\]

(21)

Next, the symmetric steady-state of dynamic System 19 will be found. If \(\varepsilon_1 = \cdots = \varepsilon_n = \varepsilon\), then any symmetric steady-state \((x^*(\varepsilon), \ldots, x^*(\varepsilon))\) is the solution of equation:

\[
x = \varepsilon \left(\sqrt{\frac{A}{a} (\frac{\varepsilon - 1}{\varepsilon} x + (n-1)x)} - \left(\frac{\varepsilon - 1}{\varepsilon} x + (n-1)x\right)\right),
\]

which is:

\[
x^*(\varepsilon) = \frac{A(n\varepsilon - 1)}{a\varepsilon n^2}.
\]

(22)

Notice that if \(\varepsilon \neq 1, x^*(\varepsilon) \neq x^*\) showing that the inaccurate knowledge of the price function results in a different system equilibrium.

In the following sections, the asymptotic properties of the solutions of the dynamic System 14 will be examined.

**ASYMPTOTIC BEHAVIOR WITHOUT TIME LAGS**

In this section, it is assumed that, at each time period, instantaneous information is available to all firms about their own outputs, as well as about the unit price obtained by the industry. This is the situation that is described by System 14. The asymptotical behavior of its solutions can be examined by investigating the locations of the eigenvalues of the Jacobian at the
steady-state. For the sake of simple notation, let \( h_i = f_i^{-1} \). Notice that in the case of exact knowledge of the price function, \( h_i \) is the identity function. Simple differentiation shows that:

\[
J = \begin{bmatrix}
k_1 r_1 h'_1 & k_1 r_1 h'_1 & \cdots & k_1 r_1 h'_1 \\
k_2 r_2 h'_2 & k_2 r_2 h'_2 & \cdots & k_2 r_2 h'_2 \\
\vdots & \vdots & \ddots & \vdots \\
k_n r_n h'_n & k_n r_n h'_n & \cdots & k_n r_n h'_n \\
\end{bmatrix}
\]

where:

\[
h'_i = h'\left(\sum_{j=1}^{n} x_j^*(\varepsilon)\right),
\]

and:

\[
r_i = R_i^T \left(h_i \left(\sum_{j=1}^{n} x_j^*(\varepsilon)\right) - x_i^*(\varepsilon)\right),
\]

with \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \). Notice that \( J \) has a special structure:

\[
J = D + a1^T,
\]

with \( 1^T = (1,1, \ldots, 1) \), \( D = \text{diag}\left(-k_1(r_1 + 1), \ldots, -k_n(r_n + 1)\right) \) and:

\[
a = \begin{bmatrix}
k_1 r_1 h'_1 \\
k_2 r_2 h'_2 \\
\vdots \\
k_n r_n h'_n \\
\end{bmatrix}.
\]

It is realistic to assume that both the true and the assessed price functions are strictly decreasing, so \( h_i \) strictly increases, implying that \( h'_i > 0 \) for all \( i \). Similarly, from Relation 13 it is known that \( r_i < 0 \) and \( r_i + 1 > 0 \). The characteristic polynomial of \( J \) can be written as follows:

\[
\det \left((D + a1^T - \lambda I)\right) = \det(D - \lambda I)\det(1 + \left((D - \lambda I)^{-1} a1^T\right) = \det(D - \lambda I)[1 + \left((D - \lambda I)^{-1} a\right)]
\]

\[
= \pi_{i=1}^n (-k_i(r_i + 1) - \lambda) \left[1 + \sum_{i=1}^{n} \frac{k_i r_i h'_i}{-k_i(r_i + 1) - \lambda}\right],
\]

(24)

where the fact was utilized that with any \( u, v \in \mathbb{R}^n \),

\[
\det(I + uu^T) = 1 + u^T u,
\]

which can be proved by finite induction with respect to \( n \).

The main result of this section is as follows.

**Theorem**

Under the conditions given above, any positive steady-state of System 14 is always locally asymptotically stable.

**Proof**

It is sufficient to prove that all eigenvalues of \( J \) are real and negative. Notice first that for all \( i, -k_i(r_i + 1) < 0 \), so the roots of the first factor are all negative. The bracketed factor can be rewritten as:

\[
g(\lambda) = 1 + \sum_{j=1}^{s} \frac{w_j}{\delta_j - \lambda},
\]

where \( \delta_1 < \delta_2 < \cdots < \delta_s \) denote the different \(-k_i(r_i + 1)\) values and \( w_j = \sum_{i=1}^{\{i \mid -k_i(r_i + 1) = \delta_j\}} k_i r_i h'_i < 0 \). Clearly, \( \lim_{\lambda \to \pm\infty} g(\lambda) = 1, \lim_{\lambda \to \delta_j \pm 0} g(\lambda) = \pm\infty \) and:

\[
g'(\lambda) = \sum_{j=1}^{s} \frac{w_j}{(\delta_j - \lambda)^2} < 0.
\]

The graph of function \( g \) is shown in Figure 1. Equation \( g(\lambda) = 0 \) is equivalent to a polynomial equation of degree \( s \) and there is a root before \( \delta_1 \) and a root between each \( \delta_j \) and \( \delta_{j+1}, j = 1, 2, \ldots, s - 1 \). Hence, \( s \) negative roots were found. Since there are no more roots, the proof is complete.

**Remark**

The believed price function \( f_i \) can be a very inaccurate approximation of the true price function. If \( f_i \) and \( f_i \) \( (i = 1, 2, \ldots, n) \) satisfy Conditions A and B and are strictly decreasing, the assertion of the theorem remains true.
ASYMPTOTIC BEHAVIOR WITH TIME LAGS

In the case of the model of the previous section it was assumed that at each time period each firm knows the instantaneous price. This assumption is not realistic, since price information is given to the manufacturers by the retailers and other sellers with some delay. The exact value of the delay is usually not known, so continuously distributed lags were assumed. The firms do not observe \( Q_i \) directly, since they do not know the exact price function. Instead of \( Q_i \), the values of \( Q_i^d \), given in Equation 10, are only observed with delay. As in [5] a weighted average of past values is assumed:

\[
Q_i^D(t) = \int_0^t w(t-s, T_i, m_i)Q_i^d(s)ds,
\]

(25)

where the weighting function is assumed to have the form:

\[
w(t-s, T, m) = \begin{cases} 
\frac{1}{m!}e^{-\frac{t}{mT}} & \text{if } m = 0 \\
\frac{1}{m!} \frac{m}{T} (\frac{m}{T})^m e^{-\frac{t}{mT}} & \text{if } m \geq 1.
\end{cases}
\]

(26)

Here, \( T \) is a positive real parameter and \( m \geq 0 \) is an integer. Notice that this weighting function has the following properties:

i) The area under the weighting function is unity for all \( T \) and \( m \);

ii) For \( m = 0 \), weights are exponentially declining with the most weight given to the most current value. If \( m \geq 1 \) then zero weight is given to the most current value, rising to maximum at \( t-s = T \) and decreasing exponentially afterwards;

iii) As \( m \) increases, the weighting function becomes more peaked around \( t-s = T \). As \( m \to \infty \), the weighting function converges to the Dirac delta function centered at \( t-s = T \);

iv) As \( T \to \infty \), the weighting function tends to the Dirac delta function for all \( m \geq 0 \).

With delayed information, dynamic System 14 is modified in the following way:

\[
\dot{x}_i(t) = k_i(R^B(Q_i^D(t)) - x_i(t)),
\]

(27)

where \( Q_i^D(t) \) is given in Equation 25. This is a system of nonlinear integro-differential equations, the asymptotical behavior of which can be examined by linearization. The linearized equation can be written as:

\[
\dot{x}_{i\delta}(t) = k_i(r_i \int_0^t w(t-s, T_i, m_i)(h'_i \left( \sum_{j=1}^n x_{j\delta}(s) \right)
- x_{i\delta}(s))ds - x_{i\delta}(t),
\]

(28)

where \( r_i \) and \( h'_i \) are the same as in the previous section and \( x_{j\delta} \) is the deviation of \( x_j \) from its steady-state level. As in [6], the solution is sought in the form:

\[
x_{j\delta}(t) = v_j e^{\lambda t}, \quad j = 1, 2, \ldots, n.
\]

(29)

Substitute this solution into Equation 28 and let \( t \to \infty \) to have:

\[
\left[ \lambda + k_i \left( 1 - r_i(h'_i - 1) \left( \frac{\lambda T_i}{q_i} + 1 \right)^{-(m_i+1)} \right) \right] v_i
\]

\[
- k_i r_i h'_i \left( \frac{\lambda T_i}{q_i} + 1 \right)^{-(m_i+1)} \sum_{j \neq i} v_j = 0,
\]

(30)

where:

\[
q_i = \begin{cases} 
1 & \text{if } m_i = 0 \\
\frac{m_i}{m_i} & \text{if } m_i \geq 1.
\end{cases}
\]

Notice first that this equation can be rewritten in the following way:

\[
\left[ \left( \lambda + k_i \left( \frac{\lambda T_i}{q_i} + 1 \right)^{m_i+1} \right) - k_i r_i (h'_i - 1) \right] v_i
\]

\[
- k_i r_i h'_i \sum_{j \neq i} v_j = 0,
\]

which is equivalent to the determinant equation:

\[
\begin{vmatrix}
A_1(\lambda) & B_1(\lambda) & \ldots & B_n(\lambda) \\
B_2(\lambda) & A_2(\lambda) & \ldots & B_n(\lambda) \\
\vdots & \vdots & \ddots & \vdots \\
B_n(\lambda) & B_n(\lambda) & \ldots & A_n(\lambda)
\end{vmatrix} = 0,
\]

(31)

with:

\[
A_i(\lambda) = (\lambda + k_i) \left( \frac{\lambda T_i}{q_i} + 1 \right)^{m_i+1} - k_i r_i (h'_i - 1),
\]

and:

\[
B_i(\lambda) = -k_i r_i h'_i.
\]

The structure of Determinant 31 is similar to the structure of the Jacobian given earlier in Equation 23. Therefore, by using the same idea as in deriving the
closed form Representation 24 of its characteristic polynomial, it can be shown that Equation 31 is
equivalent to a nonlinear algebraic equation:
\[
\tau_{i=1}^n (A_i(\lambda) - B_i(\lambda)) \left[ 1 + \sum_{j=1}^n \frac{B_j(\lambda)}{A_j(\lambda) - B_j(\lambda)} \right] = 0.
\] (32)
Notice, in addition, that in the special case of \(T_i = 0\) (without time delay) for all \(i\), Equation 31 formally
reduces to the characteristic equation of the Jacobian (Equation 23).

Equation 32 is satisfied if either:

\[
A_i(\lambda) - B_i(\lambda) = 0,
\] (33)

or:

\[
1 + \sum_{j=1}^n \frac{B_j(\lambda)}{A_j(\lambda) - B_j(\lambda)} = 0.
\] (34)

Since \(B_i(\lambda)\) is a constant and \(A_i(\lambda)\) is a polynomial, both equations are polynomial.

The roots of these
equations can be obtained by using standard numerical
techniques. For an introduction to the solution of nonlinear equations see, for example [7]. In general
cases, analysis of the locations of the roots needs numerical
methods, however, in certain special cases, it is possibly analytical.

Now the special case is considered where the firms
are identical,

\[
k_1 = k_2 = \cdots = k_n = k, \quad r_1 = \cdots = r_n = r, \quad
h'_1 = \cdots = h'_n = h', \quad T_1 = \cdots = T_n = T
\]
\[
m_1 = \cdots = m_n = m.
\]

Then \(q_1 = \cdots = q_n = q\) and Equations 33 and 34 are
simplified as:

\[
(\lambda + k) \left( \frac{\lambda T}{q} + 1 \right)^{m+1} + kr = 0,
\] (35)

and:

\[
(\lambda + k) \left( \frac{\lambda T}{q} + 1 \right)^{m+1} + kr(1 - nh') = 0.
\] (36)

Consider first the case of \(m = 0\). Then there are two
quadratic equations:

\[
\lambda^2 T + \lambda(1 + kT) + k(1 + r) = 0,
\] (37)

and:

\[
\lambda^2 T + \lambda(1 + kT) + k(1 + r - rh') = 0.
\] (38)

Notice that under the assumptions given earlier, all
coefficients are positive, so all eigenvalues have negative
real parts implying the local asymptotical stability of
the steady-state.

Assume next that \(m = 1\). Then two cubic
equations are obtained:

\[
\lambda^3 T^2 + \lambda^2 (2T + kT^2) + \lambda(1 + 2kT) + k(1 + r) = 0,
\] (39)

\[
\lambda^3 T^2 + \lambda^2 (2T + kT^2) + \lambda(1 + 2kT) + k(1 + r - rh') = 0.
\] (40)

All coefficients are positive, however, the local asymptotical
stability of the steady-state is not guaranteed, in
general, since the Routh-Hurwitz criterion is not necessarily satisfied (see, e.g. [8]). In case of instability
cyclic behavior of the solution is possible. This phenomenon
is known as limit cycles. The Hopf bifurcation theorem
gives sufficient conditions for the birth of limit cycles
around the steady-state (see, e.g. [9]). Limit cycles
eexist if there is a pure complex eigenvalue \(\lambda^* = i\alpha^*\) with
some \(\alpha^* \neq 0\) and the derivative of this eigenvalue, with
respect to a bifurcation parameter (what is selected as \(r\)), is nonzero at \(\lambda^*\). A number \(\lambda = i\alpha\) is an eigenvalue
if either:

\[
-i\alpha^3 T^2 - \alpha^2 (2T + kT^2) + i\alpha(1 + 2kT) + k(1 + r) = 0,
\]
or:

\[
-i\alpha^3 T^2 - \alpha^2 (2T + kT^2) + i\alpha(1 + 2kT)
+ k(1 + r - rh') = 0.
\]

Equating the real and imaginary parts to zero, the
following relations are obtained:

\[
\alpha^2 = \frac{k(1 + r)}{2T + kT^2} = \frac{1 + 2kT}{T^2},
\]
or:

\[
\alpha^2 = \frac{k(1 + r - rh')}{2T + kT^2} = \frac{1 + 2kT}{T^2}.
\] (41)

In both cases \(\alpha^2 > 0\), so real \(\alpha\) exists. The critical value
of the bifurcation parameter is obtained by solving
these equations for \(r\). The solutions are:

\[
r^* = \frac{(1 + 2kT)(2T + kT^2) - kT^2}{kT^2}
\]
\[
= 2 \left( \frac{1}{kT} + T + 2 \right),
\]

\[
r^* = \frac{(1 + 2kT)(2T + kT^2) - kT^2}{kT^2(1 - nh')}
\]
\[
= \frac{2}{nh' - 1} \left( \frac{1}{kT} + kT + 2 \right).
\] (42)

Since \(-1 < r < 0\), the first solution is not feasible.
Figure 2 shows the value of \(r\) as a function of \(kT\). In
order to have feasible solution condition
\[ -\frac{8}{nh' - 1} > -1, \]

must be satisfied, which is equivalent to \( nh' > 9 \). That is, if \( h' \) is very close to 1, which is its value in the case of the knowledge of the true price function, then \( n = 9 \) or 10 is the possible smallest value of \( n \), depending on the condition that \( h' \) is less or larger than 1. If \( h' \) increases, then the value of the smallest feasible \( n \) decreases and if \( h' \) decreases, then the smallest \( n \) increases. The derivative of \( \lambda \) with respect to \( r \), can be obtained by differentiating Equation 40 implicitly:
\[ 3\lambda^2 T^2 + 2\lambda(2T + kT^2) + \lambda(1 + 2kT) + k(1 - nh') = 0, \]

where the simplifying notation \( \dot{\lambda} = d\lambda/dr \) was used. Consequently,
\[ \dot{\lambda} = \frac{k(nh' - 1)}{3\lambda^2 T^2 + 2\lambda(2T + kT^2) + (1 + 2kT)}, \]
and the real part of its value at \( \lambda^* = i\alpha^* \), where \( \alpha^* \) is given in Equation 41, is as follows:
\[ \Re \dot{\lambda}|_{\lambda = \lambda^*} = \frac{-k(nh' - 1)}{2(2kT + 1) + 2(2 + kT)} \neq 0, \]

if \( h' \neq \frac{1}{n} \). This is a realistic condition, since, in full information case, \( h' = 1 \). Hence, the Hopf bifurcation theorem implies that there is a limit cycle for \( r \) in the neighborhood of the steady-state. This negative derivative indicates that if the value of \( r \) decreases from \( r^* \), then the real part of the corresponding eigenvalue becomes positive, showing instability, and if \( r \) increases then \( \Re \lambda \) decreases and becomes negative, implying the local asymptotical stability of the steady-state.

**CONCLUSIONS**

In this paper, dynamic oligopolies were examined without knowledge of the price function. Under realistic assumptions on the exact and approximating price and cost functions, it could be proven that if instantaneous information is available to each firm about the market price, then the steady-state is always locally asymptotically stable. If time lag is assumed in obtaining and implementing price information, then instability might occur. Assuming continuously distributed time lags, the possibility of the birth of limit cycles was examined, based on the Hopf bifurcation theorem.

**REFERENCES**