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Dynamic Oligopolies Without Full Information

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Dynamic Cournot oligopolies are examined when the firms do not have accurate information about price function. After the mathematical model is formulated, the local asymptotical stability of the steady-state is proved under realistic conditions. Instability occurs when information lag is assumed. The possibility of limit cycles is examined, based on the Hopf bifurcation theorem.

INTRODUCTION

Cournot oligopolies are the most intensively discussed economic models since the pioneering work of Cournot [1]. The existence and uniqueness of the equilibrium was the main research area in the case of static models and the stability of the equilibrium in the case of dynamic extensions. Many researchers examined different variants of the original model of Cournot, including single-product oligopolies with and without product differentiation , multi-product models, labor-managed oligopolies, rent-seeking games and oligopsonies. A comprehensive summary of the results on single-product models and literature review are presented in Okuguchi [2] and multi-product models with several applications are the subject of the monograph by Okuguchi and Szidarovszky [3].

In most studies on static and dynamic oligopolies, full information is assumed, that is, it is assumed that all firms have exact knowledge of the market demand function (and, therefore, they know the exact price function as well) and each firm knows its own cost function accurately. In addition, the availability of instantaneous information on the output of the rivals is also assumed. In economic reality, none of these assumptions is realistic, since firms do not report production information to each other. Data on the output of the rivals is obtained via received prices, which are usually reported to the manufacturers with some time delay by the retailers and other sellers.

In this paper, both issues will be addressed: The inaccurate knowledge of the price functions, as well as time lags in obtaining and implementing information on the rivals outputs.

This paper is developed in the following manner: First, the mathematical models will be introduced. Then, discussion reqarding the asymptotical behavior of the equilibrium is presented which is followed by the conclusion.

MATHEMATICAL MODELS

Assume that n firms produce the same product, or offer the same service, to the same market. The decision variable of each firm is the volume of its output, x_i. Assume that the cost function of firm i is $c_i(x_i)$, which is assumed to be known exactly by it. Let f be the unit price function, which depends on the output of the industry, $Q = \sum_{j=1}^{n} x_j$. Therefore, the profit of firm i can be given as:

$$\pi_i(x_1, \ldots, x_n) = x_i f\left(\sum_{j=1}^{n} x_j\right) - c_i(x_i). \tag{1}$$

Let $Q_i = \sum_{j \neq i} x_j$ denote the output of the rest of the industry, then this profit function can be rewritten as:

$$\pi_i(x_i, Q_i) = x_i f(x_i + Q_i) - c_i(x_i). \tag{2}$$

In the theory of oligopoly, it is usually assumed that functions f and c_i are twice continuously differentiable. Furthermore, for all $x_i \geqq 0$ and $Q \geqq x_i$,

$$x_i f''(Q) + f'(Q) < 0, \tag{A}$$

$$f'(Q) - c_i''(x_i) < 0. \tag{B}$$

These two assumptions imply that π_i is concave in x_i with fixed values of Q_i and, if each firm has bounded capacity, then the Nikaido-Isoda theorem implies that there is at least one equilibrium $\underline{x}^* = (x_1^*, \ldots, x_n^*)$

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(e.g. [4]). It satisfies the following condition for all i and feasible x_i,

$$\varphi_i(x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_n^*) \leqq \varphi_i(x_1^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_n^*). \qquad (3)$$

This condition shows that none of the firms can improve its profit by changing its output from equilibrium level x_i^*. For any given value of Q_i, the best response, or reaction, of firm i is the output x_i, that maximizes its profit (Equation 2). The first order conditions are:

$$f(x_i + Q_i) + x_i f'(x_i + Q_i) - c_i'(x_i) = 0, \qquad (4)$$

where corner optimum is excluded. The second order condition is satisfied by Conditions A and B. Let $R_i(Q_i)$ denote the best response function. Its derivative can be obtained by implicitly differentiating Equation 4 to get:

$$R_i'(Q_i) = -\frac{f'(Q) + x_i f''(Q)}{2f'(Q) + x_i f''(Q) - c_i''(x_i)}. \qquad (5)$$

Conditions A and B imply that:

$$-1 < R_i'(Q_i) < 0. \qquad (6)$$

A vector $\underline{x}^* = (x_1^*, \ldots, x_n^*)$ is a Nash-equilibrium if and only if, each component x_i^* is a feasible output of firm i and:

$$x_i^* = R_i\left(\sum_{j \neq i} x_j^*\right), \qquad (7)$$

for all i. The computation of any positive equilibrium is based on solving Equation 4 for $i = 1, 2, \ldots, n$, which is a special system of nonlinear algebraic equations. By rewriting Equation 4 as:

$$f(Q) + x_i f'(Q) - c_i'(x_i) = 0, \qquad (8)$$

and noticing that the left hand side is strictly decreasing in x_i with fixed values of Q, it can be seen that in the neighborhood of the equilibrium, x_i is a single-valued function of $Q, x_i = g_i(Q)$, which can be determined by the repeated solution of Equation 8 with a set of Q values. The derivative of function g can be also obtained by differentiating Equation 8 with respect to Q implicitly to have:

$$g_i'(Q) = -\frac{f'(Q) + x_i f''(Q)}{f'(Q) - c_i''(x_i)} < 0,$$

as a consequence of Conditions A and B. Next consider the single-variable nonlinear equation:

$$Q - \sum_{i=1}^{n} g_i(Q) = 0. \qquad (9)$$

The left hand side is strictly increasing in Q, so this equation can be solved by simple methods, such as the bisection or secant method. Let Q^* denote the solution, then $x_i^* = g_i(Q^*)$ gives the ith component of the equilibrium point for $i = 1, 2, \ldots, n$.

Assume next that the firms cannot assess the price functions accurately. Firm i ($i =, 1, 2, \ldots, n$) believes that the unit price function is some f_i instead of the true price function f. However, each firm observes the market price and estimates the output of the rest of the industry by computing it based on the unit price received and its own output. If Q_i^e denotes the estimated total output of the rivals, then the unit price equals:

$$f(x_i + Q_i) = f_i(x_i + Q_i^e),$$

which implies that:

$$Q_i^e = (f_i^{-1} o f)(x_i + Q_i) - x_i. \qquad (10)$$

The believed reaction function of firm i is the output that maximizes its believed profit:

$$\pi_i^B(x_i, Q_i^e) = x_i f_i(x_i + Q_i^e) - c_i(x_i). \qquad (11)$$

By assuming that Conditions A and B are satisfied with f_i replacing f and by excluding corner optimum, it can be seen that the believed reaction function $x_i = R_i^B(Q_i^e)$ satisfies equation:

$$f_i(x_i + Q_i^e) + x_i f_i'(x_i + Q_i^e) - c_i'(x_i) = 0. \qquad (12)$$

Differentiate this equation implicitly to see that $R_i^{B'}(Q_i^e)$ has the same form as Equation 5 with f_i replacing f, so:

$$-1 < R_i^{B'}(Q_i^e) < 0. \qquad (13)$$

Assume now that each firm adjusts its output into the direction of its best response, then the following dynamic equations are obtained:

$$\dot{x}_i(t) = k_i(R_i^B(Q_i^e(t)) - x_i(t)),$$

where Q_i^e is given in Equation 10 and $k_i > 0$ is a given constant. Hence, the resulting dynamic system can be written as:

$$\dot{x}_i(t) = k_i\left(R_i^B\left((f_i^{-1} o f)\left(\sum_{j=1}^{n} x_j(t)\right) - x_i(t)\right) - x_i(t)\right), \qquad (14)$$

for $i = 1, 2, \ldots, n$.

Example 1

Assume that each firm has a linear cost function, $c_i(x_i) = a_i x_i + b_i$, $i = 1, 2, \ldots, n$. Let $Q = d(p)$ denote the market demand function and assume that each firm mistakenly believes that the demand function is $\varepsilon_i d(p)$, where $\varepsilon_i, i = 1, 2, \ldots, n$, is a positive parameter smaller, or larger, than 1. The believed price function by firm i is the solution of equation $Q = \varepsilon_i d(p)$ which is:

$$p = d^{-1}\left(\frac{Q}{\varepsilon_i}\right) = f\left(\frac{Q}{\varepsilon_i}\right),$$

therefore, the following is obtained:

$$f_i(Q) = f\left(\frac{Q}{\varepsilon_i}\right). \tag{15}$$

Notice that $f_i^{-1}(p)$ is the solution of equation:

$$f\left(\frac{Q}{\varepsilon_i}\right) = p,$$

which is :

$$Q = \varepsilon_i f^{-1}(p) = \varepsilon_i d(p). \tag{16}$$

Therefore, from Equation 10 it is observed that:

$$Q_i^e = \varepsilon_i(x_i + Q_i) - x_i = (\varepsilon_i - 1)x_i + \varepsilon_i Q_i. \tag{17}$$

The believed reaction function is the solution of Equation 12, which has now the special form:

$$f\left(\frac{x_i + Q_i^e}{\varepsilon_i}\right) + \frac{x_i}{\varepsilon_i} f'\left(\frac{x_i + Q_i^e}{\varepsilon_i}\right) - a_i = 0,$$

and if this equation is compared with Equation 4 it is concluded that:

$$\frac{x_i}{\varepsilon_i} = R_i\left(\frac{Q_i^e}{\varepsilon_i}\right),$$

implying that:

$$R_i^B(Q_i^e) = \varepsilon_i R_i\left(\frac{Q_i^e}{\varepsilon_i}\right). \tag{18}$$

Based on Equations 17 and 18, System 14 can be rewritten as follows:

$$\dot{x}_i(t) = k_i\left(\varepsilon_i R_i\left(\frac{\varepsilon_i - 1}{\varepsilon_i} x_i(t) + \sum_{j \neq i} x_j(t)\right) - x_i(t)\right), \tag{19}$$

for $i = 1, 2, \ldots, n$.

As a particular case, assume that $f(Q) = \frac{A}{Q}$, so, by assuming positive reaction the true reaction function of firm i is the solution of Equation 4, which has now the form:

$$\frac{A}{x_i + Q_i} - x_i \frac{A}{(x_i + Q_i)^2} - a_i = 0,$$

implying that:

$$x_i = R_i(Q_i)\sqrt{\frac{AQ_i}{a_i}} - Q_i. \tag{20}$$

Any positive equilibrium is, therefore, the solution of the following system of nonlinear algebraic equations:

$$x_i = \sqrt{\frac{A}{a_i}\sum_{j \neq i} x_j} - \left(\sum_{j \neq i} x_j\right), \quad i = 1, 2, \ldots, n.$$

As a further special case, assume that the marginal costs, a_i, of the firms are identical, $a_1 = \cdots = a_n = a$. Then, any symmetric equilibrium $(x^*, \ldots, x^*)$ is the solution of equation:

$$x = \sqrt{\frac{A}{a}(n-1)x} - (n-1)x,$$

that is:

$$x^* = \frac{A(n-1)}{an^2}. \tag{21}$$

Next, the symmetric steady-state of dynamic System 19 will be found. If $\varepsilon_1 = \cdots = \varepsilon_n = \varepsilon$, then any symmetric steady-state $(x^*(\varepsilon), \ldots, x^*(\varepsilon))$ is the solution of equation:

$$x = \varepsilon\left(\sqrt{\frac{A}{a}\left(\frac{\varepsilon - 1}{\varepsilon}x + (n-1)x\right)} - \left(\frac{\varepsilon - 1}{\varepsilon}x + (n-1)x\right)\right),$$

which is:

$$x^*(\varepsilon) = \frac{A(n\varepsilon - 1)}{a\varepsilon n^2}. \tag{22}$$

Notice that if $\varepsilon \neq 1, x^*(\varepsilon) \neq x^*$ showing that the inaccurate knowledge of the price function results in a different system equilibrium.

In the following sections, the asymptotic properties of the solutions of the dynamic System 14 will be examined.

ASYMPTOTIC BEHAVIOR WITHOUT TIME LAGS

In this section, it is assumed that, at each time period, instantaneous information is available to all firms about their own outputs, as well as about the unit price obtained by the industry. This is the situation that is described by System 14. The asymptotical behavior of its solutions can be examined by investigating the locations of the eigenvalues of the Jacobian at the steady-state. For the sake of simple notation, let $h_i = f_i^{-1} o f$. Notice that in the case of exact knowledge of the price function, h_i is the identity function. Simple differentiation shows that:

$$\underline{J}=\begin{bmatrix} k_1(r_1(h_1'-1)-1) & k_1r_1h_1' & \dots & k_1r_1h_1' \\ k_2r_2h_2' & k_2(r_2(h_2'-1)-1) & \dots & k_2r_2h_2' \\ \vdots & \vdots & & \vdots \\ k_nr_nh_n' & k_nr_nh_n' & \dots & k_n(r_n(h_n'-1)-1) \end{bmatrix}, \tag{23}$$

where:

$$h_i' = h_i'\left(\sum_{j=1}^{n} x_j^*(\underline{\varepsilon})\right),$$

and:

$$r_i = R_i^{B'}\left(h_i\left(\sum_{j=1}^{n} x_j^*(\underline{\varepsilon})\right) - x_i^*(\underline{\varepsilon})\right),$$

with $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$. Notice that $\underline{J}$ has a special structure:

$$\underline{J} = \underline{D} + \underline{a}\underline{1}^T,$$

with $\underline{1}^T = (1, 1, \dots, 1)$, $\underline{D} = \text{diag}\,(-k_1(r_1+1), \dots, -k_n(r_n+1))$ and:

$$\underline{a} = \begin{bmatrix} k_1r_1h_1' \\ k_2r_2h_2' \\ \dots \\ k_nr_nh_n' \end{bmatrix}.$$

It is realistic to assume that both the true and the assessed price functions are strictly decreasing, so h_i strictly increases, implying that $h_i' > 0$ for all i. Similarly, from Relation 13 it is known that $r_i < 0$ and $r_i + 1 > 0$. The characteristic polynomial of $\underline{J}$ can be written as follows:

$$\begin{aligned} \det\,(\underline{D}+\underline{a}\underline{1}^T-\lambda\underline{I}) &= \det(\underline{D}-\lambda\underline{I})\det(\underline{I}+(\underline{D}-\lambda\underline{I})^{-1}\underline{a}\underline{1}^T) \\ &= \det(\underline{D}-\lambda\underline{I})[1+\underline{1}^T(\underline{D}-\lambda\underline{I})^{-1}\underline{a}] \\ &= \pi_{i=1}^{n}(-k_i(r_i+1)-\lambda)\left[1+\sum_{i=1}^{n}\frac{k_ir_ih_i'}{-k_i(r_i+1)-\lambda}\right], \end{aligned} \tag{24}$$

where the fact was utilized that with any $\underline{u}$, $\underline{\nu} \in R^n$,

$$\det\,(\underline{I} + \underline{u}\underline{\nu}^T) = 1 + \underline{\nu}^T\underline{u},$$

which can be proved by finite induction with respect to n.

The main result of this section is as follows.

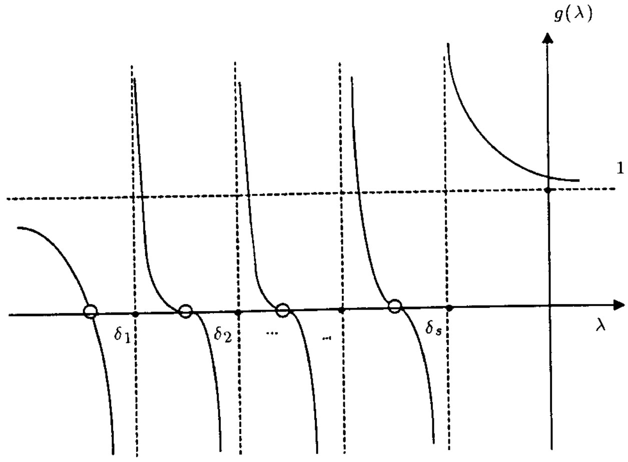

Figure 1. The graph of function g.

Theorem

Under the conditions given above, any positive steady-state of System 14 is always locally asymptotically stable.

Proof

It is sufficient to prove that all eigenvalues of J are real and negative. Notice first that for all i, $-k_i(r_i+1) < 0$, so the roots of the first factor are all negative. The bracketed factor can be rewritten as:

$$g(\lambda) = 1 + \sum_{j=1}^{s} \frac{w_j}{\delta_j - \lambda},$$

where $\delta_1 < \delta_2 < \dots < \delta_s$ denote the different $-k_i(r_i+1)$ values and $w_j = \sum_{\{i|-k_i(r_i+1)=\delta_j\}} k_ir_ih_i' < 0$. Clearly, $\lim_{\lambda\to\pm\infty} g(\lambda) = 1, \lim_{\lambda\to\delta_j\pm 0} g(\lambda) = \pm\infty$ and:

$$g'(\lambda) = \sum_{j=1}^{s} \frac{w_j}{(\delta_j - \lambda)^2} < 0.$$

The graph of function g is shown in Figure 1. Equation $g(\lambda) = 0$ is equivalent to a polynomial equation of degree s and there is a root before δ_1 and a root between each δ_j and $\delta_{j+1}, j = 1, 2, \dots, s-1$. Hence, s negative roots were found. Since there are no more roots, the proof is complete.

Remark

The believed price function f_i can be a very inaccurate approximation of the true price function. If f and f_i $(i = 1, 2, \dots, n)$ satisfy Conditions A and B and are strictly decreasing, the assertion of the theorem remains true.

ASYMPTOTIC BEHAVIOR WITH TIME LAGS

In the case of the model of the previous section it was assumed that at each time period each firm knows the instantaneous price. This assumption is not realistic, since price information is given to the manufacturers by the retailers and other sellers with some delay. The exact value of the delay is usually not known, so continuously distributed lags were assumed. The firms do not observe Q_i directly, since they do not know the exact price function. Instead of Q_i the values of Q_i^e, given in Equation 10, are only observed with delay. As in [5] a weighted average of past values is assumed:

$$Q_i^{eD}(t) = \int_0^t w(t-s, T_i, m_i) Q_i^e(s) ds, \tag{25}$$

where the weighting function is assumed to have the form:

$$w(t-s, T, m) = \begin{cases} \frac{1}{T} e^{-\frac{t-s}{T}} & \text{if } m = 0 \\ \frac{1}{m!} (\frac{m}{T})^{m+1} (t-s)^m e^{-\frac{m(t-s)}{T}} & \text{if } m \geq 1. \end{cases} \tag{26}$$

Here, T is a positive real parameter and $m \geqq 0$ is an integer. Notice that this weighting function has the following properties:

i) The area under the weighting function is unity for all T and m;

ii) For $m = 0$, weights are exponentially declining with the most weight given to the most current value. If $m \geqq 1$ then zero weight is given to the most current value, rising to maximum at $t-s = T$ and decreasing exponentially afterwards;

iii) As m increases, the weighting function becomes more peaked around $t - s = T$. As $m \to \infty$, the weighting function converges to the Dirac delta function centered at $t - s = T$;

iv) As $T \to \infty$, the weighting function tends to the Dirac delta function for all $m \geqq 0$.

With delayed information, dynamic System 14 is modified in the following way:

$$\dot{x}_i(t) = k_i(R_i^B(Q_i^{eD}(t)) - x_i(t)),$$

$$i = 1, 2, \dots, n, \tag{27}$$

where $Q_i^{eD}(t)$ is given in Equation 25. This is a system of nonlinear integro-differential equations, the asymptotical behavior of which can be examined by linearization. The linearized equation can be written as:

$$\dot{x}_{i\delta}(t) = k_i\Big(r_i \cdot \int_0^t w(t-s, T_i, m_i)\Big(h_i' \cdot \Big(\sum_{j=1}^n x_{j\delta}(s)\Big) - x_{i\delta}(s)\Big) ds - x_{i\delta}(t)\Big), \tag{28}$$

where r_i and h_i' are the same as in the previous section and $x_{j\delta}$ is the deviation of x_j from its steady-state level. As in [6], the solution is sought in the form:

$$x_{j\delta}(t) = \nu_j e^{\lambda t}, \; j = 1, 2, \dots, n. \tag{29}$$

Substitute this solution into Equation 28 and let $t \to \infty$ to have:

$$\left[\lambda + k_i\left(1 - r_i(h' - 1)\left(\frac{\lambda T_i}{q_i} + 1\right)^{-(m_i+1)}\right)\right] v_i,$$

$$-k_i r_i h_i' \left(\frac{\lambda T_i}{q_i} + 1\right)^{-(m_i+1)} \sum_{j \neq i} \nu_j = 0, \tag{30}$$

where:

$$q_i = \begin{cases} 1 & \text{if } m_i = 0 \\ m_i & \text{if } m_i \geqq 1. \end{cases}$$

Notice first that this equation can be rewritten in the following way:

$$\left[(\lambda + k_i)\left(\frac{\lambda T_i}{q_i} + 1\right)^{m_i+1} - k_i r_i (h_i' - 1)\right] \nu_i$$

$$- k_i r_i h_i' \sum_{j \neq i} \nu_j = 0,$$

which is equivalent to the determinant equation:

$$\det \begin{bmatrix} A_1(\lambda) & B_1(\lambda) & \dots & B_1(\lambda) \\ B_2(\lambda) & A_2(\lambda) & \dots & B_2(\lambda) \\ \vdots & \vdots & & \vdots \\ B_n(\lambda) & B_n(\lambda) & \dots & A_n(\lambda) \end{bmatrix} = 0, \tag{31}$$

with:

$$A_i(\lambda) = (\lambda + k_i)\left(\frac{\lambda T_i}{q_i} + 1\right)^{m_i+1} - k_i r_i (h_i' - 1),$$

and:

$$B_i(\lambda) = -k_i r_i h_i'.$$

The structure of Determinant 31 is similar to the structure of the Jacobian given earlier in Equation 23. Therefore, by using the same idea as in deriving the closed form Representation 24 of its characteristic polynomial, it can be shown that Equation 31 is equivalent to a nonlinear algebraic equation:

$$\pi_{i=1}^{n}\left(A_i(\lambda) - B_i(\lambda)\right)\left[1 + \sum_{j=1}^{n} \frac{B_j(\lambda)}{A_j(\lambda) - B_j(\lambda)}\right] = 0. \qquad (32)$$

Notice, in addition, that in the special case of $T_i = 0$ (without time delay) for all i, Equation 31 formally reduces to the characteristic equation of the Jacobian (Equation 23).

Equation 32 is satisfied if either:

$$A_i(\lambda) - B_i(\lambda) = 0, \qquad (33)$$

or:

$$1 + \sum_{j=1}^{n} \frac{B_j(\lambda)}{A_j(\lambda) - B_j(\lambda)} = 0. \qquad (34)$$

Since $B_i(\lambda)$ is a constant and $A_i(\lambda)$ is a polynomial, both equations are polynomial. The roots of these equations can be obtained by using standard numerical techniques. For an introduction to the solution of nonlinear equations see, for example [7]. In general cases, analysis of the locations of the roots needs numerical methods, however, in certain special cases, it is possibly analytical.

Now the special case is considered where the firms are identical,

$$\begin{array}{ll} k_1 = k_2 = \cdots = k_n = k, & r_1 = \cdots = r_n = r, \\ h'_1 = \cdots = h'_n = h', & T_1 = \cdots = T_n = T \\ m_1 = \cdots = m_n = m. & \end{array}$$

Then $q_1 = \cdots = q_n = q$ and Equations 33 and 34 are simplified as:

$$(\lambda + k)\Big(\frac{\lambda T}{q} + 1\Big)^{m+1} + kr = 0, \qquad (35)$$

and:

$$(\lambda + k)\Big(\frac{\lambda T}{q} + 1\Big)^{m+1} + kr(1 - nh') = 0. \qquad (36)$$

Consider first the case of $m = 0$. Then there are two quadratic equations:

$$\lambda^2 T + \lambda(1 + kT) + k(1 + r) = 0, \qquad (37)$$

and:

$$\lambda^2 T + \lambda(1 + kT) + k(1 + r - rnh') = 0. \qquad (38)$$

Notice that under the assumptions given earlier, all coefficients are positive, so all eigenvalues have negative real parts implying the local asymptotical stability of the steady-state.

Assume next that $m = 1$. Then two cubic equations are obtained:

$$\lambda^3 T^2 + \lambda^2(2T + kT^2) + \lambda(1 + 2kT) + k(1 + r) = 0, \qquad (39)$$

$$\lambda^3 T^2 + \lambda^2(2T + kT^2) + \lambda(1 + 2kT) + k(1 + r - rnh') = 0. \qquad (40)$$

All coefficients are positive, however, the local asymptotical stability of the steady-state is not guaranteed, in general, since the Routh-Hurwitz criterion is not necessarily satisfied (see, e.g. [8]). In case of instability cyclic behavior of the solution is possible. This phenomenon is known as limit cycles. The Hopf bifurcation theorem gives sufficient conditions for the birth of limit cycles around the steady-state (see, e.g. [9]). Limit cycles exist if there is a pure complex eigenvalue $\lambda^* = i\alpha^*$ with some $\alpha^* \neq 0$ and the derivative of this eigenvalue, with respect to a bifurcation parameter (what is selected as r), is nonzero at λ^*. A number $\lambda = i\alpha$ is an eigenvalue if either:

$$-i\alpha^3 T^2 - \alpha^2(2T + kT^2) + i\alpha(1 + 2kT) + k(1 + r) = 0,$$

or:

$$-i\alpha^3 T^2 - \alpha^2(2T + kT^2) + i\alpha(1 + 2kT) + k(1 + r - rnh') = 0.$$

Equating the real and imaginary parts to zero, the following relations are obtained:

$$\alpha^2 = \frac{k(1 + r)}{2T + kT^2} = \frac{1 + 2kT}{T^2},$$

or:

$$\alpha^2 = \frac{k(1 + r - rnh')}{2T + kT^2} = \frac{1 + 2kT}{T^2}. \qquad (41)$$

In both cases $\alpha^2 > 0$, so real α exists. The critical value of the bifurcation parameter is obtained by solving these equations for r. The solutions are:

$$r^* = \frac{(1 + 2kT)(2T + kT^2) - kT^2}{kT^2} = 2\Big(\frac{1}{kT} + kT + 2\Big),$$

$$r^* = \frac{(1 + 2kT)(2T + kT^2) - kT^2}{kT^2(1 - nh')} = -\frac{2}{nh' - 1}\Big(\frac{1}{kT} + kT + 2\Big). \qquad (42)$$

Since $-1 < r < 0$, the first solution is not feasible. Figure 2 shows the value of r as a function of kT. In

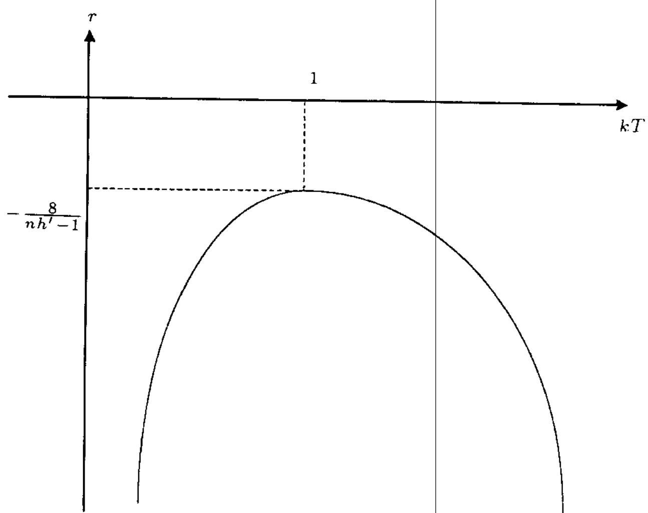

Figure 2. The critical value of r as a function of kT.

order to have feasible solution condition

$$-\frac{8}{nh'-1} > -1,$$

must be satisfied, which is equivalent to $nh' > 9$. That is, if h' is very close to 1, which is its value in the case of the knowledge of the true price function, then $n = 9$ or 10 is the possible smallest value of n, depending on the condition that h' is less or larger than 1. If h' increases, then the value of the smallest feasible n decreases and if h' decreases, then the smallest n increases. The derivative of λ, with respect to r, can be obtained by differentiating Equation 40 implicitly:

$$3\lambda^2\dot{\lambda}T^2+2\lambda\dot{\lambda}(2T+kT^2)+\dot{\lambda}(1+2kT)$$
$$+k(1-nh')=0,$$

where the simplifying notation $\dot{\lambda} = d\lambda/dr$ was used. Consequently,

$$\dot{\lambda} = \frac{k(nh'-1)}{3\lambda^2T^2 + 2\lambda(2T+kT^2) + (1+2kT)},$$

and the real part of its value at $\lambda^* = i\alpha^*$, where α^* is given in Equation 41, is as follows:

$$Re\ \dot{\lambda}|_{\lambda=\lambda^*} = \frac{-k(nh'-1)}{2(2kT+1)+2(2+kT)} \neq 0, \qquad (43)$$

if $h' \neq \frac{1}{n}$. This is a realistic condition, since, in full information case, $h' = 1$. Hence, the Hopf bifurcation theorem implies that there is a limit cycle for r in the neighborhood of the steady-state. This negative derivative indicates that if the value of r decreases from r^*, then the real part of the corresponding eigenvalue becomes positive, showing instability, and if r increases then $Re\ \lambda$ decreases and becomes negative, implying the local asymptotical stability of the steady-state.

CONCLUSIONS

In this paper, dynamic oligopolies were examined without knowledge of the price function. Under realistic assumptions on the exact and approximating price and cost functions, it could be proven that if instantaneous information is available to each firm about the market price, then the steady-state is always locally asymptotically stable. If time lag is assumed in obtaining and implementing price information, then instability might occur. Assuming continuously distributed time lags, the possibility of the birth of limit cycles was examined, based on the Hopf bifurcation theorem.

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