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Decision Making with the Analytic Hierarchy Process

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In this paper, an introduction is given to the decision-making theory, the Analytic Hierarchy Process (AHP). In the AHP, a decision hierarchy is constructed with a goal, criteria and alternatives. The criteria are pairwise compared for their importance with respect to the goal to derive a scale of relative importance and the alternatives are pairwise compared with respect to each criterion to derive relative scales. The relative scales are synthesized using a weighting and adding process to show which is the best alternative. Judgment is used to make the pairwise comparisons. Data can also be used directly in the pairwise comparisons if there is no need to assess or interpret the value represented by the numbers. This process works well for intangibles with no existing scales of measurement. Even when measurement scales and data using these scales exist, it is often advantageous to use judgment to interpret it. Two concepts integral to the AHP, measurement and consistency, are explained. Finally, the ideas and the method are illustrated through an example.

INTRODUCTION

The scientific literature of hierarchies has demonstrated, to physical behavioral and system scientists and particularly to people interested in organization theory, the lesson that a hierarchy is the single most powerful mental construct for studying complex systems [1,2]. Whether one is simply interested in understanding the actual structure and flow of a system or whether one is concerned with the functional interactions of its components, a hierarchical model of that system must inevitably be examined. Hierarchic organization is crucial to the synthesis and survival of large systems. Hierarchic systems have common properties that are independent of their specific content.

This paper concentrates on providing an informal report on research leading to a useful theory for analyzing the impacts of different levels, representing subsystems, of a system, characterized as a hierarchy, on the entire hierarchy and conversely. The theory also enables one to study the stability of a hierarchy with respect to perturbations in both structure and function.

An intrinsic useful by-product of the theory is an index of consistency that provides information on how serious are violations of numerical (cardinal) and transitive (ordinal) consistency. The result could be to seek additional information and reexamine the data used in constructing the scale in order to improve consistency.

Such an index is not available in other procedures for the construction of ratio scales. The method also utilizes, for reasons suggested by consistency requirements, reciprocal entries $a_{ij} = 1/a_{ji}$ in pairwise comparison matrices instead of the traditional $a_{ij} = -a_{ji}$ used for construction of interval scales. This is both an intuitively reasonable assumption and turns out to be an asset in using positive matrices to establish the existence of a non-negative unique solution as a ratio scale.

The approach in this paper, reduces to the study of two related problems: Measurement and consistency. The problem is to do pairwise comparisons and then turn these into a single list of relative values. This process has the advantage of focusing on two objects at a time and on how they relate to each other. It also uses redundant information since each object is methodically compared with every other.

For problems where there is no scale to validate the result, the pairwise comparison process can prove to be an asset because, in a sense, it is simpler in each of its steps than the first.

All measurements including those that make use of instruments are subject to experimental error and to

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error in the measuring instrument. A serious effect of error is that it can and often does lead to inconsistent conclusions. A simple example of the consequence of error in weighing objects is to find that A is heavier than B and B is heavier than C but C is heavier than A. This can happen particularly when the weights of A, B and C are close and the instrument is not fine enough to distinguish between them. Lack of consistency may be serious for some problems.

Since consistency is a central question in concrete measurement, in judgments, and in the thinking process, an effective measure of consistency in applying numerical ratio scales, is an incentive for developing scales in areas where there are no instruments of measurement. In passing, it is noted that measuring instruments are not and cannot be means of absolute measurement but are themselves the object of scientific research. If these instruments are for any reason inadequate (and one can always devise an experiment for which there is no satisfactory instrument for measurement) then inventing new instruments must be kept. It is not difficult to imagine some important experiment for which no sufficiently fine instrument can ever be found from which consistent answers can always be obtained. In that case the entire problem is shifted to the study of consistency and evaluating the seriousness of inconsistency.

The process of measurement by pairwise comparisons leads in a natural way to the problem of consistency, enabling one to characterize or say exactly what is meant by consistency and to develop a measure for how serious inconsistency is [3,4].

In the social sciences, properties change not only in time and space, but also (and far more seriously) in conjunction with other properties, their meaning also changes. Universal scales cannot be improvised for social events. Social phenomena are even more complicated than physical phenomena because they are harder to replicate in abundance. Too much control must be imposed and controls in themselves often destroy the very social behavior one is trying to measure. Judgments must be sufficiently flexible to take into consideration the contextual setting of the property being measured.

Consider the problem of measuring achievement and happiness. Both may be called relative properties in that the unit of measurement may have to be adjusted to compare, for example, the degree of happiness in one setting with that in another and as will be seen it is possible to do this with the pairwise comparison technique. What should be recognized is that an instrument that varies its scale with the relativity of the circumstance can be the human mind itself, particularly, if it turns out that its measurement is sufficiently consistent to satisfy the requirements of the particular problem. The intensity of feelings serves as a scale-adjustment-device to place the measurement of some objects on a commensurate scale with that of other objects. In fact as the mind improves its precision it becomes the required tool for relative measurement as no instrument except a very personally designed one (one's own mind) can be made to suit one's particular experience and viewpoint. A group must coordinate their outlook to produce results acceptable (in some sense) to them.

However the problem does not end at this stage. By way of validation, whatever method of scaling is developed must so far as it is possible, reproduce results consistent with those known in the physical and any other area where there are scales of measurement. This should serve to encourage extending the method to the "softer" areas of the social sciences.

Now the second problem is concerned with providing greater stability and invariance to social measurement is considered. Granted that the dimensions or properties are variable, how to measure the impact of this variability on still other higher level properties and in turn these on still higher ones is of concern. It seems that for a very wide class of problems the overall properties (or one property) can usually be identified which remain the same for sufficiently long duration of an experiment. This approach leads one to the measurement and analysis of impacts in hierarchies.

The theory of hierarchies says that complex systems problems are hierarchic in nature with their elements occupying different levels of the hierarchy and that the contributions or impacts of these elements on the hierarchy can be studied through the use of measurement. The invariance of the measurement can also be studied by changing the levels of the hierarchy. The results of the measurement may be used to enhance the stability of the system or to design new goal oriented systems. They can also be used (as priorities) to allocate resources.

As mentioned previously, the theory of hierarchies provides models for studying interactions between elements and components (levels) of a system. The assumption here is that measurement derives from judgments based on observation and understanding. These observations enable measurement from comparisons (not from absolutes). Also the comparisons result in separating objects or phenomena into comparability classes that are levels of the hierarchy. Measurement is used to study interactions among comparability classes. Furthermore the objects in any class are compared according to their impact on each element of the immediately higher level. This is equivalent to measuring how strongly each element possesses a property or attribute belonging to a different level. For each property, the comparison shows the relative strength of an object or property (as a generic term) in a level. The hierarchical structure makes it possible to perpetrate the impact study up and down the hierarchy.

MEASUREMENT

It is inevitable, in the study of hierarchies that the problem of measurement should arise. If one is interested in measuring the impact of a set of elements in a level of the hierarchy on each of the elements of the next higher level, he must be able to measure the behavior of the elements or their contribution to the elements of the higher level. Here, it should be recognized that all measurement that leads to a standard scale must begin with observation; in fact observations. When the measurement derive from preferences based on abstract knowledge or feeling, the latter must serve in some reliable fashion, the role of direct observation. Thus, the analytical approach must have an appeal to a wide variety of needs in measurement. Its validation in areas, where measurement is available, would serve to increase confidence in its use in new areas. It is clear that in order to develop a scale of measurement, comparing must be learnt. Comparison provides a relative scale for the objects being compared. When a sufficiently large number of objects have been considered in the comparison process and analytical or judgmental agreement reached on the values in the comparison, the resulting scale acquires greater universality and autonomy.

One is able to distinguish between objects or between phenomena because they have different properties or occupy different positions in space and time. The distinction is usually made with respect to several properties at the same time. In fact, one needs to know the relative standing of each object according to all the properties; it is necessary to distinguish between the properties themselves. This process of distinguishing between things is a process of comparison. If the objects are independent, all comparisons can be reduced to pairwise comparisons. If they are dependent, then the dependence itself must be taken into consideration in the final weighting.

When interested in causal explanation, it is found that phenomena can be arranged according to precedence: Something must happen before something else can happen. Ordering leads to hierarchical type of structures in which first causes occupy higher levels of the hierarchy.

The ordering is a first step in the process of measuring variations among the objects being compared according to each of several properties. What is usually desired is stronger than simple order. In scientific measurement one seek measurement on ratio scales. This is what will be discussed here.

When one deals with phenomena for which there are no known or widely agreed upon scales and instruments of measurement, it becomes a matter of judgment to estimate numerical values for comparison. As more people interact and agree on these judgments, a scale (implicit or explicit) gradually evolves and eventually acquires universality. Examples are the scales for measuring distance, time, weight and economic value. In some areas of social interaction it is useful to expedite the process of measurement by making available fundamental tools which by way of validation yield the same results in areas where measurement is already available.

Attention is focussed on a single level of a hierarchy and the relation of dominance of its elements is first studied with respect to a single property. Then the more general problem of the impact of all elements of a level on the entire hierarchy will be studied. The introduction already alluded to a large number of other results and interactions that can be studied this way. It seems adequate to only show how this fundamental step can be carried out. What will be done here, is to motivate the method of generating a ratio scale from pairwise comparisons and generalize the result to a hierarchy. The full justification of the approach may be found in books written by authors on the subject.

PARADIGM CASE OF CONSISTENCY

Suppose that n activities are being considered by a group of interested people. The group's goals are assumed to be:

(a) To provide judgments on the relative importance of these activities,

(b) To ensure that the judgments are quantified to an extent that also permits a quantitative interpretation of the judgments among all activities.

Clearly, goal (b) will require appropriate technical assistance. A method will now be described for deriving, from the group's quantified judgments (i.e., from the relative values associated with pairs of activities), a set of weights to be associated with the individual activities; in a sense defined below, these weights should reflect the group's quantified judgments. What this approach achieves is to put the information resulting from (a) and (b) into usable form without deleting information residing in the qualitative judgments.

Let $C_1, C_2, ..., C_n$ be the set of activities or contingencies. The quantified judgments on pairs of objectives C_I, C_j are represented by an n-by-n matrix:

$$\mathbf{A} = (a_{ij}), (i, j = 1, 2, ..., n).$$

Having recorded the quantified judgments on pairs (C_I, C_j) as numerical entries a_{ij} in the matrix

A, the problem now is to assign to the n contingencies $C_1, C_2, ..., C_n$ a set of numerical weights $w_1, w_2, ..., w_n$ that would "reflect the recorded judgments".

In order to do so, the vaguely formulated problem must first be transformed into a precise mathematical one. This essential and apparently harmless step, is the most crucial one in any problem that requires the representation of a real-life situation in terms of an abstract mathematical structure. It is particularly crucial in the present problem where the representation involves a number of transitions that are not immediately discernible. Therefore, it appears desirable in the present problem to identify the major steps in the process of representation and to make each step as explicit as possible in order to enable the potential user to form his own judgment on the meaning and value of the method in relation to his problem and his goal [5].

It is convenient to first get a simple question out of the way. The matrix **A** of quantified judgments a_{ij} may have several, or only a few, non-zero entries.

The question arises: How many non-zero entries (i.e., how many quantifiable judgments) are necessary in order to ensure the existence of a set of weights that is meaningful in the context of the problem? The obvious answer is: It is necessary that there should be a set of nonzero entries that interconnects all activities in the sense that for every two indices, i, j, there should be some chain of non-zero entries connecting i with j. Note that a_{ij} itself is such a chain of length 1. Such a matrix **A**$= (a_{ij})$ corresponds to what is known as a strongly connected graph. This gives precise content to the formulation of goal (b).

The major question is the one concerned with the meaning of the vaguely formulated condition in the statement of our goal: "these weights should reflect the group's quantified judgments." This presents the need to describe in precise, arithmetic terms, how the weights should relate to the judgments a_{ij} or, in other words, the problem of specifying the desirable conditions to be imposed on the weights sought in relation to the judgments obtained. The desired description is developed in steps, proceeding from the simplest special case to the general one.

To measure intangibles, comparisons are needed which have been provided by a person who has experienced them. Since scales are not available yet, these comparisons must be made in relative terms. Comparison and experience are an integral part of measurement. To see how this can be done and that its results are credible and valid, a tangible attribute must first be used, the area of several geometric figures. In Figure 1, five geometric figures that are required for comparison are presented according to area.

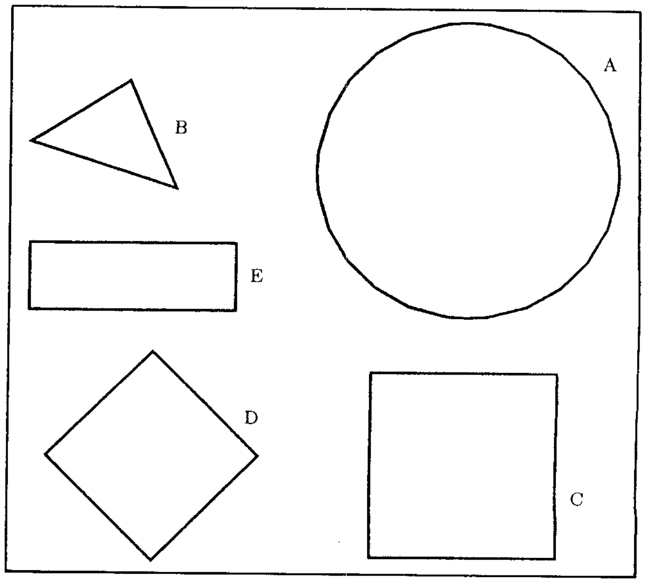

Figure 1. Five geometric objects.

To compare two figures according to their area, one determines which of the two is larger and then estimates how many times it is larger than the smaller of the two. The result of these comparisons is arranged in a matrix $\mathbf{A} = \{a_{ij}\}$. If the area of figure i is a_{ij} times larger than the area of object j, then the area of object j is $a_{ij} = 1/a_{ij}$ times larger than the area of figure i, a reciprocal relationship. Here is the matrix for these comparisons, the derived scale of relative values and the relative values obtained from actual measurement:

	Five Figures					Estimated Relative Areas	Actual Relative Areas
	A	B	C	D	E		
A	1	9	2.5	3.5	5	0.490	0.471
B	1/9	1	1/5	1/2.5	1/2	0.050	0.050
C	1/2.5	5	1	2	2.5	0.235	0.234
D	1/3.5	2.5	1/2	1	1.5	0.131	0.149
E	1/5	2	1/2.5	1/1.5	1	0.094	0.096

A more abstract form of comparison would involve tangibles that one can think about but cannot perceive through the senses. Here is the matrix for estimating the relative amount of protein in seven food items:

Which food has more protein?

Protein in Food	A	B	C	D	E	F	G
A: Steak	1	9	9	6	4	5	1
B: Potatoes		1	1	1/2	1/4	1/3	1/4
C: Apples			1	1/3	1/3	1/5	1/9
D: Soybean				1	1/2	1	1/6
E: Whole wheat bread					1	3	1/3
F: Tasty cake						1	1/5
G: Fish							1

The derived scale and actual values are:

Steak	Potatoes	Apples	Soybean	W. bread	T. cake	Fish
0.345	0.031	0.030	0.065	0.124	0.078	0.328
0.370	0.040	0.000	0.070	0.110	0.090	0.320

with a consistency ratio of 0.028.

Both examples show that an experienced person can provide informed numerical judgments from which relativeing good estimates are derived. Before proceeding to deal with intangibles, a sketch of the theoretical concepts is given underlying the way scales are derived from paired comparisons.

Consider n stocks, $A_1, ..., A_n$, with known worth $w_1, ..., w_n$, respectively and suppose that a matrix of pairwise ratios is formed whose rows give the ratios of the worth of each stock with respect to all others as follows:

$$\begin{array}{c} A_1 \\ \vdots \\ A_n \end{array} \begin{bmatrix} A_1 & \cdots & A_n \\ \frac{w_1}{w_1} & \cdots & \frac{w_1}{w_n} \\ \vdots & \ddots & \vdots \\ \frac{w_n}{w_1} & \cdots & \frac{w_n}{w_n} \end{bmatrix}.$$

The following equation is obtained:

$$\mathbf{Aw} = \begin{bmatrix} \frac{w_1}{w_1} & \cdots & \frac{w_1}{w_n} \\ \vdots & \ddots & \vdots \\ \frac{w_n}{w_1} & \vdots & \frac{w_n}{w_n} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = n \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = n\mathbf{w}$$

where $\mathbf{A}$ has been multiplied on the right by the vector of weights $\mathbf{w}$. The result of this multiplication is $n\mathbf{w}$. Thus, to recover the scale $\mathbf{w}$ from the matrix of ratios $\mathbf{A}$, one must solve the eigenvalue problem $\mathbf{Aw} = n\mathbf{w}$ or $(\mathbf{A} - n\mathbf{I})\mathbf{w} = 0$. This is a system of homogeneous linear equations. It has a nontrivial solution if and only if the determinant of $\mathbf{A} - n\mathbf{I}$ vanishes, that is, n is an eigenvalue of $\mathbf{A}$. Now $\mathbf{A}$ has unit rank, since every row is a constant multiple of the first row. Thus, all its eigenvalues except one are zero. The sum of the eigenvalues of a matrix is equal to its trace, the sum of its diagonal elements and, in this case, the trace of $\mathbf{A}$ is equal to n. Thus n is the principal eigenvalue of $\mathbf{A}$ and one has a nonzero solution $\mathbf{w}$. The solution consists of positive entries and is unique within a multiplicative constant.

To make $\mathbf{w}$ unique, one can normalize its entries by dividing by their sum. Note that if one divides two readings from a ratio scale one obtains an absolute number. Normalization transforms a ratio scale to an absolute scale. Thus, given the comparison matrix, one can recover the original scale in relative terms. In this case, the solution is any column of $\mathbf{A}$ normalized. The matrix $\mathbf{A}$ is not only reciprocal, but also consistent. Its entries satisfy the condition $a_{jk} = a_{ik}/a_{ij}$. It follows that the entire matrix can be constructed from a set of n elements that form a spanning tree across the rows and columns. It has been shown that if values from a standard scale are used to make the comparisons, the principal eigenvector recovers these values in normalized form [6].

In the general case when only judgment but not the numbers themselves are available, the precise value of w_i/w_j is not known, but instead only an estimate of it can be given as a numerical judgment. For the moment, consider an estimate of these values by an expert who is assumed to make small perturbations of the ratio w_i/w_j. This implies small perturbations of the eigenvalues. The problem now becomes $\mathbf{A}'\mathbf{w}' = \lambda_{\max}\mathbf{w}'$ where $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{A}'$. To determine how good the derived estimate $\mathbf{w}$ is, it is noted that, if $\mathbf{w}$ is obtained by solving $\mathbf{A}'\mathbf{w}' = \lambda_{\max}\mathbf{w}'$, the matrix $\mathbf{A}$ whose entries are w_i'/w_j' is a consistent matrix. It is a consistent estimate of the matrix $\mathbf{A}'$ which need not be consistent. In fact, the entries of A′ need not even be transitive; that is, A_1 may be preferred to A_2 and A_2 to A_3 but A_3 may be preferred to A_1. What is desired is a measure of the error due to inconsistency. It turns out that $\mathbf{A}'$ is consistent if and only if $\lambda_{\max} = n$ and that there is always $\lambda_{\max} \geq n$.

Since small changes in a_{ij} imply a small change in $\lambda_{\max}$, the deviation of the latter from n is a deviation from consistency and can be represented by $(\lambda_{\max} - n)/(n-1)$, which is called the Consistency Index (C.I.). If $a_{ij} = \frac{w_i}{w_j}\varepsilon_{ij}$ is written then:

$$\mu \equiv \frac{\lambda_{\max} - n}{n-1} = -1 + \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} \left[\varepsilon_{ij} + \frac{1}{\varepsilon_{ij}}\right],$$

which represents the average cumulative inconsistency of the matrix. When the consistency has been calculated, the result is compared with that of the average shown in Table 1 of a large number of reciprocal matrices of the nth order whose entries are randomly selected from the 1/9,...,9 scale. This index is called the Random Index (R.I.). The following provides the order of the matrix n (first row) and the average R.I. (second row).

The ratio of C.I. to the average R.I. for the same order matrix is called the Consistency Ratio (C.R.). The consistency ratio needs to be kept "small", e.g., less than 10 percent, indicating deviations from non-random entries (informed judgments) of less than an order of magnitude.

Factors that contribute to the consistency of a judgment are: (1) Homogeneity of the elements in a group, that is, one must compare elements that are within an order of magnitude of each other and not, for example, try to directly compare a cherry tomato with a watermelon, (2) Sparseness of the elements in the group; that is, one must not put too many elements in a group and (3) The individuals making the judgments must know enough about the elements being compared to ensure a valid outcome.

Let it be noted that while in the first example above the eye perceives different size areas, in the second example the mind through wide experience derives an understanding of the relative presence of proteins in the different foods. Feelings are ususally distinguished

Table 1. Average inconsistency of 50,000 randomly generated reciprocal matrices for each order from the 1/9,...,9 scale.

n	1	2	3	4	5	6	7	8	9	10
Average R.I.	0	0	.52	.89	1.11	1.25	1.35	1.40	1.45	1.49

Table 2. Association of numerical ratings and verbal ratings for pairwise comparison judgments.

Intensity of Importance	Definition	Explanation
1	Equal importance	Two activities contribute equally to the objective
2	Weak	
3	Moderate importance	Experience and judgment slightly favor one activity over another
4	Moderate plus	
5	Strong importance	Experience and judgment strongly favor one activity over another
6	Strong plus	
7	Very strong or demonstrated importance	An activity is favored very strongly over another; its dominance is demonstrated in practice
8	Very, very strong	
9	Extreme importance	The evidence favoring one activity over another is of the highest possible order of affirmation
Reciprocals of above	If activity i has one of the above nonzero numbers assigned to it when compared with activity j, then j has the reciprocal value when compared with I	A reasonable assumption

qualitatively and associated with numerical values. In practice, the entries a_{ij} are estimated using the scale given in Table 2. This scale has been validated in numerous applications.

In situations where the scale 1 to 9 is inadequate to cover the spectrum of comparisons needed, that is, the elements compared are inhomogeneous, as the cherry tomato and the watermelon below, one uses a process of clustering with a pivot from one cluster to an adjacent cluster that is one order of magnitude larger or smaller than the given cluster and continues to use the 1 to 9 scale within each cluster. In doing that, the scale is extended as far out as one desires, as illustrated in Figure 2. This means that 34.14/.07 = 487.7 unripe cherry tomatoes are equal to the oblong watermelon.

AN EXAMPLE OF HIERARCHIC DECISIONS

The ideas will be illustrated with a simple example of a decision problem. The example uses relative measurement for the criteria, but standards are established for each criterion and the alternatives are rated based on these standards. That is called absolute measurement (for more details see [7-10]).

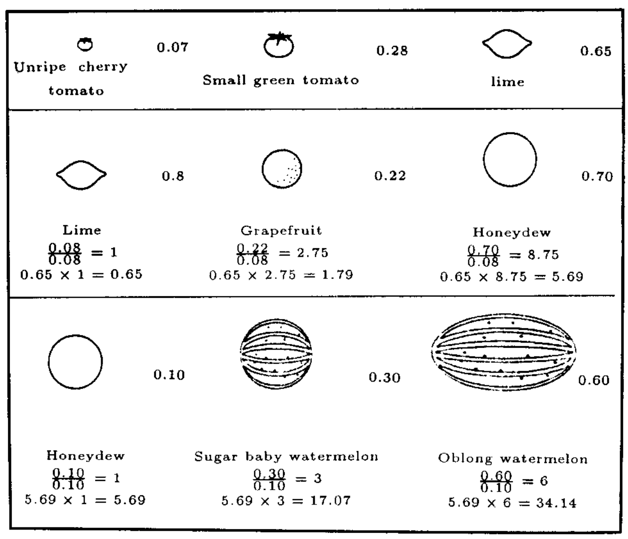

Figure 2. The clustering process.

Choosing the Best House - An Example of Relative Measurement

Consider the following example: A family wishing to purchase a house identifies eight criteria that are important to them. The problem is to select one of three candidate houses. The first step is to structure the problem into a hierarchy (see Figure 3). On the

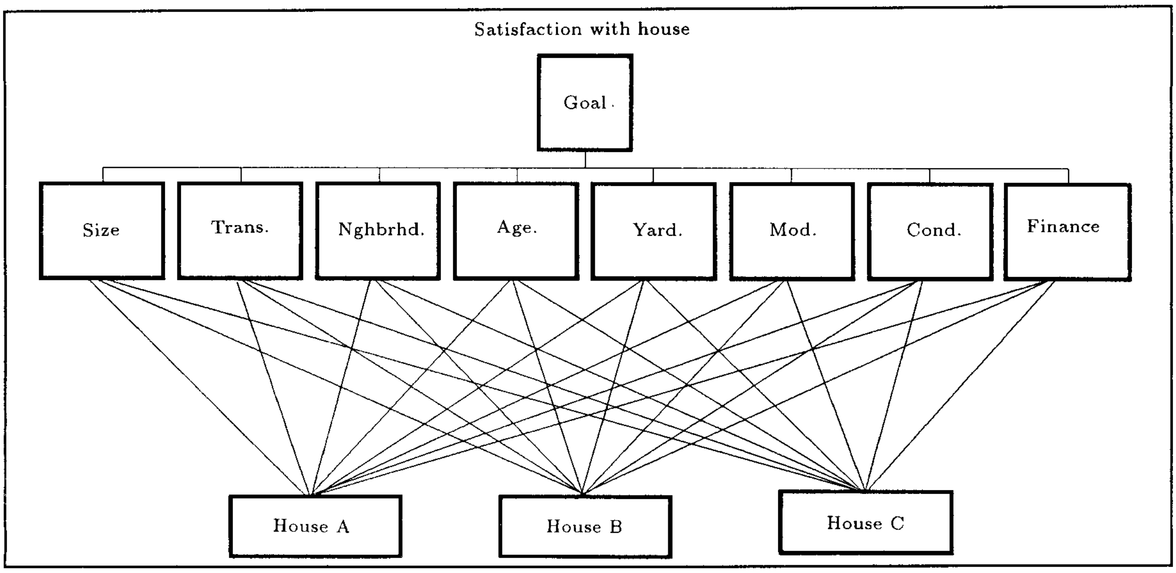

Figure 3. Decomposition of the problem into a hierarchy.

first (top) level is the overall goal of satisfaction with house. On the second level are the eight criteria that contribute to the goal and on the third (bottom) level are the three candidate houses that are to be evaluated by considering the criteria on the second level.

The criteria important to the family are:

1. Size of house: Storage space, size of rooms, number of rooms, total area of house,
2. Transportation: Convenience and proximity of bus service,
3. Neighborhood: Degree of traffic, security, taxes, physical condition of surrounding buildings,
4. Age of house,
5. Yard space: Front, back and side space and space shared with neighbors,
6. Modern facilities: Dishwasher, garbage disposal, air conditioning, alarm system and other such items,
7. General condition: Extent to which repairs are needed, condition of walls, carpet, drapes, wiring, cleanliness,
8. Financing: Availability of assumable mortgage, seller financing, or bank financing.

The next step is to make comparative judgments. The family buying the house assesses the relative importance of all possible pairs of criteria with respect to the overall goal, satisfaction with house, and their judgments are arranged into a matrix.

The question to ask when comparing two criteria is: which is more important and how much more important is it with respect to satisfaction with a house?

The matrix of pairwise comparison judgments on the criteria given by the home buyers in this case is shown in Table 3. The judgments are entered using the fundamental scale of the AHP: A criterion compared with itself is always assigned the value 1, so the main diagonal entries of the pairwise comparison matrix are all 1. The numbers 3, 5, 7 and 9

Table 3. Pairwise comparison matrix for the criteria.

	Size	**Trans.**	**Nbrhd.**	**Age**	**Yard**	**Modern**	**Cond.**	**Finance**	**Priority Vector**
Size	1	5	3	7	6	6	1/3	1/4	0.173
Trans.	1/5	1	1/3	5	3	3	1/5	1/7	0.054
Nbrhd.	1/3	3	1	6	3	4	6	1/5	0.188
Age	1/7	1/5	1/6	1	1/3	1/4	1/7	1/8	0.018
Yard	1/6	1/3	1/3	3	1	1/2	1/5	1/6	0.031
Modern	1/6	1/3	1/4	4	2	1	1/5	1/6	0.036
Cond.	3	5	1/6	7	5	5	1	1/2	0.167
Finance	4	7	5	8	6	6	2	1	0.333
$\lambda_{max} = 9.669$									

Table 4. Pairwise comparison matrices for the alternatives.

Size of House	A	B	C	Normalized Priorities	Idealized Priorities
A	1	6	8	0.754	1.000
B	1/6	1	4	0.181	0.240
C	1/8	1/4	1	0.065	0.086
$\lambda_{\max} = 3.136$, C.I. = 0.068, C.R. = 0.117					

Yard Space	A	B	C	Normalized Priorities	Idealized Priorities
A	1	5	4	0.674	1.000
B	1/5	1	1/3	0.101	0.150
C	1/4	3	1	0.226	0.335
$\lambda_{\max} = 3.086$, C.I. = 0.043, C.R. = 0.074					

Transportation	A	B	C	Normalized Priorities	Idealized Priorities
A	1	7	1/5	0.233	0.327
B	1/7	1	1/8	0.005	0.007
C	5	8	1	0.713	1.000
$\lambda_{\max} = 3.247$, C.I. = 0.124, C.R.= 0.213					

Modern Facilities	A	B	C	Normalized Priorities	Idealized Priorities
A	1	8	6	0.747	1.000
B	1/8	1	1/5	0.060	0.080
C	1/6	5	1	0.193	0.258
$\lambda_{\max} = 3.197$, C.I. = 0.099, C.R. = 0.170					

Neighborhood	A	B	C	Normalized Priorities	Idealized Priorities
A	1	8	6	0.745	1.000
B	1/8	1	1/4	0.065	0.086
C	1/6	4	1	0.181	0.240
$\lambda_{\max} = 3.130$, C.I. = 0.068, C.R. = 0.117					

General Condition	A	B	C	Normalized Priorities	Idealized Priorities
A	1	1/2	1/2	0.200	0.500
B	2	1	1	0.400	1.000
C	2	1	1	0.400	1.000
$\lambda_{\max} = 3.000$, C.I. = 0.000, C.R. = 0.000					

Age of House	A	B	C	Normalized Priorities	Idealized Priorities
A	1	1	1	0.333	1.000
B	1	1	1	0.333	1.000
C	1	1	1	0.333	1.000
$\lambda_{\max} = 3.000$, C.I. = 0.000, C.R. = 0.000					

Financing	A	B	C	Normalized Priorities	Idealized Priorities
A	1	1/7	1/5	0.072	0.111
B	7	1	3	0.650	1.000
C	5	1/3	1	0.278	0.428
$\lambda_{\max} = 3.065$, C.I. = 0.032, C.R. = 0.056					

correspond to the verbal judgments "moderately more dominant", "strongly more dominant", "very strongly more dominant" and "extremely more dominant" (with 2, 4, 6 and 8 for compromise between the previous values). Reciprocal values are automatically entered in the transpose position, so the family must make a total of 28 pairwise judgments. It is permitted to interpolate values between the integers, if desired. The vector of priorities for the criteria is obtained by computing the principal eigenvector of the pairwise comparison matrix. Because the pairwise comparison matrix has positive entries, there is a unique positive vector whose entries sum to one, i.e., is an eigenvector of the pairwise comparison matrix which is associated with an eigenvalue of largest modulus. That eigenvalue is also unique and it must be positive; it is denoted by $\lambda_{\max}$.

Table 3 shows that size dominates transportation strongly and the number 5 is entered in the (size, transportation) position. In the (finance, size) position there is number 4, which means that finance is between moderately and strongly more important than size. The priority vector shows that case financing is the most important to the family as it has the highest priority, i.e. 0.333.

The family's next task is to compare the houses pairwise with respect to how much better (more dominant) one is than the other in satisfying each of the eight criteria. There are eight 3 × 3 matrices of judgments since there are eight criteria and three houses are to be compared for each criterion. The matrices in Table 4 contain the judgments of the family. In order to facilitate understanding of the judgments, a brief description of the houses is given:

House A: This house is the largest. It is located in a good neighborhood with little traffic and low taxes. Its yard space is larger than that of either House B or C. However, its general condition is not very good and it needs cleaning and painting. It would have to be bank-financed at high interest;

House B: This house is a little smaller than house A and is not close to a bus route. The neighborhood feels insecure because of traffic conditions. The yard space is fairly small and the house lacks basic modern facilities. On the other hand, its general condition is very good and it has an assumable mortgage, with a rather low interest rate;

House C: House C is very small and has few modern facilities. The neighborhood has high taxes, but is in good condition and seems secure. Its yard is bigger than that of House B, but smaller than House A's spacious surroundings. The general condition of the house is

Table 5. Results shown by synthesizing in two modes.

	1 (0.173)	2 (0.054)	3 (0.188)	4 (0.018)	5 (0.031)	6 (0.036)	7 (0.167)	8 (0.333)	Composite vector
Distributive Mode									
A	0.754	0.233	0.754	0.333	0.674	0.747	0.200	0.072	0.396
B	0.181	0.055	0.065	0.333	0.101	0.060	0.400	0.650	0.341
C	0.065	0.713	0.181	0.333	0.226	0.193	0.400	0.278	0.263
Ideal Mode									
A	1.00	0.327	1.00	1.00	1.00	1.00	0.500	0.111	0.584
B	0.240	0.007	0.086	1.00	0.150	0.080	1.00	1.00	0.782
C	0.086	1.00	0.240	1.00	0.335	0.258	1.00	0.428	0.461

good and it has a pretty carpet and drapes. The financing is better than for A but poorer than for B.

To obtain the composite or global priorities for the houses the (local) priority vectors of the three houses are laid out in Table 5 with respect to each of the criteria, each column is multiplied by the priority of the corresponding criterion and added across each row. This gives the composite or global priority vector of the houses. In this case, House A is preferred to Houses B and C in the ratios 0.396/0.341 and 0.396/0.263, respectively. This method of synthesis is known as the distributive mode.

There are other ways one could synthesize the priorities. For example, each of the priority vectors could first be divided by a positive weight before multiplying by the priority of the corresponding criterion and combining as in the distributive mode. If one chooses that positive weight to be the largest entry in the priority vector, the resulting synthesis is called the ideal mode. The weighted priority vectors involved in the ideal mode synthesis for the three houses are shown in the bottom half of Table 5.

The last column of Table 5 shows the two composite vectors obtained by the two modes. If the family wants the best of just these three houses, then they base their decision on the vector obtained using distributive synthesis and should buy House A. If they want the best house no matter what other houses there may be, they use the ideal method of synthesis and should buy House B.

Evaluating Employees for Salary Raises - An Example of Absolute Measurement

Employees are evaluated for salary raises. The criteria are dependability, education, experience and quality. Each criterion is subdivided into intensities, standards, or subcriteria as shown in Figure 4. Priorities are set

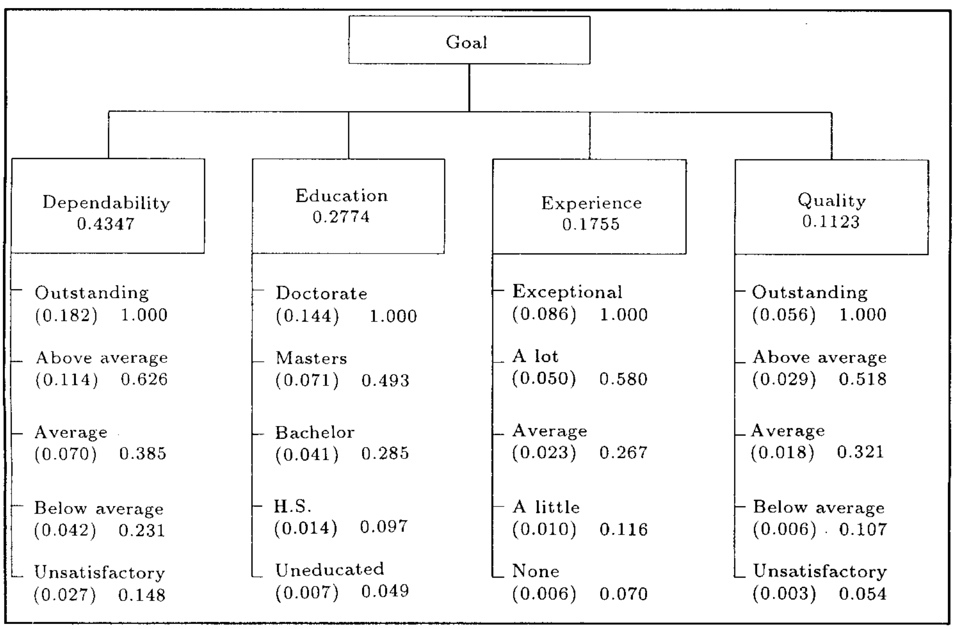

Figure 4. Employee evaluation hierarchy.

Table 6. Ranking the intensities for dependability.

	Outstanding	Above Average	Average	Below Average	Unsatisfactory	Priorities
Outstanding	1	2	3	4	5	0.419
Above Average	1/2	1	2	3	4	0.263
Average	1/3	1/2	1	2	3	0.630
Below Average	1/4	1/3	1/2	1	2	0.097
Unsatisfactory	1/5	1/4	1/3	1/2	1	0.062
Inconsistency ratio = 0.015						

Table 7. Ranking alternatives.

		Dependability 0.4347	Education 0.2774	Experience 0.1775	Quality 0.1123	Total
1.	Adams, V.	Outstanding	Bachelor	A little	Outstanding	0.646
2.	Becker, L.	Average	Bachelor	A little	Outstanding	0.379
3.	Hayat, F.	Average	Masters	A lot	Below average	0.418
4.	Kesselman, S.	Above average	High school	None	Above average	0.369
5.	O'Shea, K.	Average	Doctorate	A lot	Above average	0.605
6.	Peters, T.	Average	Doctorate	A lot	Average	0.583
7.	Tobias, K.	Above average	Bachelor	Average	Above average	0.456

for the criteria by comparing them in pairs and these priorities are then given in a matrix. The intensities are then pairwise compared according to priority with respect to their parent criterion (as in Table 6) and their priorities are divided by the largest intensity for each criterion (second column of priorities in Figure 4). Finally, each individual is rated in Table 7 by assigning the intensity rating that applies to him or her under each criterion. The scores of these subcriteria are weighted by the priority of that criterion and summed to derive a total ratio scale score for the individual. This approach can be used whenever it is possible to set priorities for intensities of criteria, which is usually possible when sufficient experience with a given operation has been accumulated. Salary raises can be made proportionately to the final priorities.

COMMENTS ON COST/BENEFIT ANALYSIS

Often, the alternatives from which a choice must be made in a choice-making situation have both costs and benefits associated with them. In this case, it is useful to construct separate costs and benefits hierarchies, with the same alternatives on the bottom level of each. Thus one obtains both a costs-priority vector and a benefits-priority vector. The benefit/cost vector is obtained by taking the ratio of the benefit priority to the costs priority for each alternative, with the highest such ratio indicating the preferred alternative. In the case where resources are allocated to several projects, such benefit-to-cost ratios or the corresponding marginal ratios prove to be very valuable.

For example, in evaluating three types of copying machines, one represents in the benefits hierarchy the good attributes one is looking for and one represents in the costs hierarchy the pain and economic costs that one would incur in buying or maintaining the three types of machines. Note that the criteria for benefits and the criteria for costs need not be simply opposites of each other but may be totally different. Also note that each criterion may be regarded at a different threshold of intensity and that such thresholds may themselves be prioritized according to desirability, with each alternative evaluated only in terms of its highest priority threshold level. Similarly, three hierarchies can be used to assess a benefit/(cost $\times$ risk) outcome. This is illustrated next with an example.

THE WISDOM OF A TRADE WAR WITH CHINA OVER INTELLECTUAL PROPERTY RIGHTS

This example was developed jointly in an unpublished paper with the author's colleague, Jen S. Shang, in mid February 1995 to understand the issues when the media were voicing strong conflicting concerns prior to the action to be taken in Beijing later in February. Many

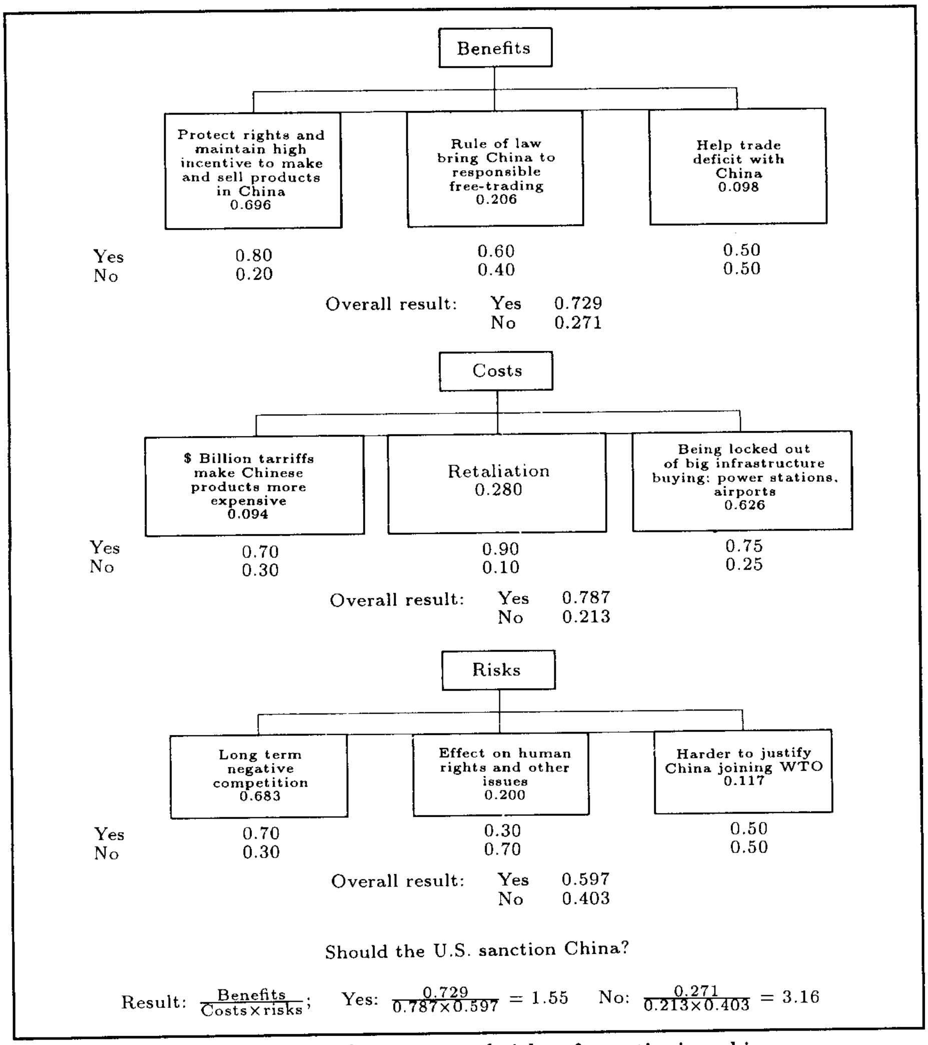

Figure 5. Benefits, costs and risks of sanctioning china.

copies of the analysis were sent to congressmen and senators and to the chief U.S. negotiator in Washington and to several newspapers in the U.S. and in China. No credit is claimed for influence on what happened later, still it is extremely satisfying that the outcome of the decision was along the lines of the presented recommendation. The tense of the writing has been kept as it was when the paper was written to better convey the sense of urgency in which it was written.

There are many and strong conflicting opinions about what to do with Chinese piracy of U.S. technology and management know-how. Should the U.S. sanction China in Beijing on February 26 (1995)? The basic arguments in favor of imposing tariffs derive from the U.S. perceived need of not permitting China to become a runaway nation with an inward oriented closed economy. Some also argue convincingly that a nation whose economy will equal that of the U.S. in three decades must be taught to play by the rules. We have made a brief study of the decision to impose tariffs on Chinese products in the U.S. It is not the immediate small injury to U.S. corporations from such an action that is of major concern, but what might happen in the future. The effect of the tariffs will be decisively more intangible with long-term results that can aggravate trade in the Pacific.

The findings based on benefits, costs and risks (see Figure 5) and on all the factors we could bring to bear on the outcome is a definite and very decisive *No*, which means that it is not in the best overall interest of the U.S. to take strong action against China. Since usually we are not told much about what China says, we also summarize some arguments gleaned from Chinese newspapers. We explain our analysis and offer the reader the opportunity to perform a similar evaluation based on the factors given here plus others we may have overlooked. In our opinion, the costs are too high to treat China in the same style as an outlaw nation even though China can and should do better as a member of the world community.

The war of intellectual property is just a reflection of the contention between the two sides. Several factors drive this friction between China and the U.S. Among the issues are human rights, weapon proliferation, the independence of Taiwan and the trade deficit. In the ten years leading to 1994, China's exports to the U.S. grew from \$3.1 billion to \$38 billion, whereas U.S. exports to China only grew from \$3 billion to \$8.5 billion. Due to this deficit, the U.S. has come to believe that China should open its markets in return.

According to the Economist magazine of February 11, 1995, the official trade deficit figures have been overstated. Those figures count Chinese goods re-exported through Hong-Kong to the U.S. as imports from China. On the other hand the value of American goods exported to China through Hong Kong was not added to the American export figures. As a result, U.S. statistics have overestimated its bilateral deficit with China since 1990 by about one-third. Another telling point is that cheap Chinese labor has persuaded firms from Hong Kong and Taiwan to move their labor-intensive production to China. Goods that were once imported from Taiwan and Hong Kong now count as Chinese. The U.S. deficit with the three countries combined rose by less than 10% between 1987 and 1992, but its deficit with China alone grew by a whopping 550% in that period. When capital can cross borders freely, it appears that bilateral deficits are misleading. Despite these facts, the perception of a huge trade deficit makes it difficult for the U.S. to see China's benefits from violating U.S. intellectual property rights as being a small part of the trade.

It is important that China follow global commercial practices. However, many Chinese do not recognize sufficiently the leadership role the U.S. has taken in international trade. They feel that it is arrogant for the U.S. to act as a judge, a policeman or an umpire, to sanction other nations who then would retaliate for loss of dignity. Many Chinese officials believe that if they accept sanctions quietly, China would be perceived to be passive to actions against its own sovereignty and dignity, just as they did in the colonial era. To save face, so far they have cancelled a \$97 million purchase of corn and threaten to disengage from a deal of more than \$2 billion with Boeing. It is evident that they are not going to yield to intimidation.

China's GDP per capita income is \$2946. There is still a huge gap between China and the advanced industrialized nations. For decades, officials and citizens in China have not known what intellectual property rights are. Did Chinese officials overtly indulge in encouraging piracy, or is it one of the many difficulties a newly arrived developing country encounters in the course of development? According to Chinese newspapers, the Chinese government has tried to control piracy but on the surface appears not to have been effective and not to have been given credit for trying.

There are also political entanglements between central and local governments in China and also between the judicial and legislative systems. Usually, local governments frown upon punishing piracy because it would result in less taxable income and higher unemployment. That is one reason why they have not tried hard to implement the law. In turn, the courts are not serious about piracy because their finances and personnel are all constrained by the local government. Many of those who profit from the piracy are sons and daughters of senior comrades, army officers and provincial bureaucrats. Top officials are too embarrassed by their children's behavior to touch that issue. Even if China were to make an agreement with the U.S., the immediate benefits of such an agreement are in doubt. According to The Wall Street Journal of February 6 (1995), few Chinese can actually afford \$14 CDs or \$150 Windows disks, which raises some doubt about the numbers being tossed around for lost sales in China. The pirates are awakening appetites that did not exist a decade ago. However, punishing China will not automatically increase U.S. exports of such items.

To arrive at a rational decision, the above factors were considered as they influence the outcome of the decision and were arranged in three hierarchies: One for the benefits of implementing such a sanction, one for the costs and a third for the risks and uncertainties that can occur. Each hierarchy has a goal followed by the criteria that affect the performance of the goal. The alternatives are listed at the last level of the hierarchy. They are: Yes - to sanction China or No - not to sanction China.

The relative importance of the criteria contributing to the goal were then determined by comparing them in pairs. For example, in the benefits hierarchy the following question was answered: How much more important is protecting American interests than teaching China to follow international business practices? Apparently, from the U.S. standpoint, protecting American rights is more important and there is dependence between these two factors. If China does not follow the international copyright law, more American products will be pirated. However, so long as the U.S. ensures that China does not pirate its technology, the U.S. would benefit even if China does not strictly follow the rules in dealing with other countries. From the U.S. view point, protecting American copyrights is more important than the urgency to teach China to be responsible. A higher priority was assigned to the former in this comparison. The rest of the comparisons are examined in the same manner.

A scale of relative importance is derived for the factors from the pairwise comparison judgments. Proceeding to the third level of the hierarchy, the alternatives are compared under each factor. For example, under the costs hierarchy, *Yes* is judged to be extremely more important in contributing to retaliation than *No* and is given the value 9 when compared with *No*. Similarly for the other values in each of the three hierarchies. *Yes* was gaven the same priority as *No* in terms of the "Harder to justify China joining WTO" factor. This is because from the U.S. viewpoint, whether sanctions are made or not, China does not qualify to receive WTO membership at the current level of protecting copyrights, patents and trademarks.

In each hierarchy, the values for *Yes* and *No* are synthesized by multiplying each alternative's priority with the importance of its parent criterion and adding them to obtain the overall result for *Yes* and *No*. A user-friendly computer software program, Expert Choice, was used to do all the calculations. To combine the results from the three hierarchies, the benefit results for *Yes* are divided by the costs and by the risks for *Yes* to obtain the final outcome. The same is done for *No* and *Yes* or *No* is selected depending on which has the larger value. While *Yes* benefits are high, the corresponding costs and risks are also high. Its ratio is less than that of the *No* decision. *No* dominates *Yes*, both when no risk is considered and when projected risk is taken into account. Including risk by using possible scenarios of the future can be a powerful tool in assessing the decision on the effect of the future.

To ensure that the outcome would not be construed as a result of whimsical judgments, a comprehensive sensitivity analysis was performed. Sensitivity analysis assists the decision maker to discover how changes in the priorities affect the recommended decision. The *Yes* and *No* weights are fixed because they are the best judgments based on the facts. So, the *Yes* and *No* judgments were fixed as shown in Figure 6 and the importance of each factor was varied. There is a wide range of admissible priority value that a policy maker may choose for each factor. The sensitivity analysis covers all the reasonable priorities a politician might choose. Each factor's importance was changed from its value indicated in the hierarchy to the near extreme values 0.2 and 0.8. This gave six variations in each hierarchy, because there are three factors in each. With three hierarchies, 216 (6*6*6) data points were generated. In this simulation, it was found that it is only when long term negative competition is thought to be unimportant that sanctions would be justified. From Figure 6 depicting the 216 possibilities, it can be seen that *No* dominates *Yes* appreciably. Regardless of the weights one assigns to the factors, over 90% of the cases lead to *No*, not to sanction China.

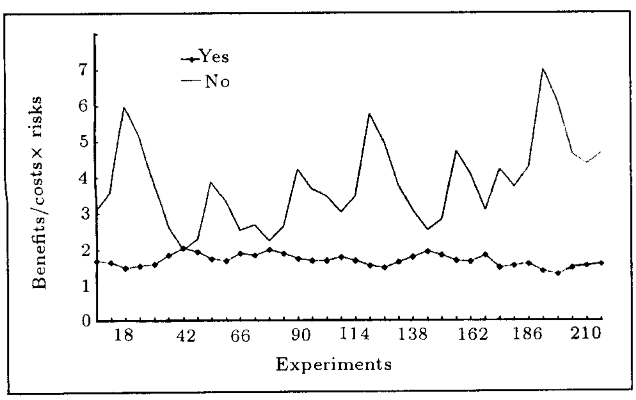

Figure 6. The dominance of *No* over *Yes*.

If a trade war becomes inevitable and the U.S. follows the *Yes* option, both sides would be affected. There is the possibility that the U.S. might then be locked out of major Chinese infrastructure business and China would have a hard time joining GATT and the World Trade Organization (WTO). It is also possible that because both countries share many common interests, should the war start, it may not last long. The U.S. has previously been engaged in trade wars with the Europeans and the Japanese, all of which ended with last-minute bargaining.

Deng Rong, the daughter of Deng Xiaoping, the most senior elder statesman of China, said recently "sanctions are never the best way to resolve a dispute. One should talk things over and consider the interests of the people". The analysis presented here seems to support this attitude.

THE EIGENVECTOR SOLUTION FOR WEIGHTS AND CONSISTENCY

The solution is obtained by raising the matrix to a sufficiently large power, then summing over the rows and normalizing to obtain the priority vector $w = (w_1, ..., w_n)$. The process is stopped when the difference between components of the priority vector obtained at the kth power and at the $(k+1)$st power is less than some predetermined small value.

An easy way to get an approximation to the priorities is to normalize the geometric means of the rows. This result coincides with the eigenvector for $n = 3$. A second way to obtain an approximation is by normalizing the elements in each column of the judgment matrix and then averaging over each row.

It is cautioned that for important applications one should use only the eigenvector derivation procedure because approximations can lead to rank reversal in spite of the closeness of the result to the eigenvector [10].

A simple way to obtain the exact value (or an estimate) of $\lambda_{\max}$, when the exact value (or an estimate) of w is available in normalized form, is to add the columns of the matrix of judgements and multiply the resulting vector by the priority vector w.

The AHP includes a consistency index for an entire hierarchy. The AHP has also been generalized to the use of a network, rather than a hierarchy, to deal with dependence and feedback. This generalization is called the Analytic Network Process (ANP).

HOW TO STRUCTURE A HIERARCHY

Now, the reader might think that all decision hierarchies in the AHP can only have three levels, however, an example is shown of a multilevel hierarchy and its priorities (see Figure 7) on which the author worked with a hospital staff to choose the best way to the authors treat terminal cancer patients.

Perhaps the most creative part of decision making that has a significant effect on the outcome is the structuring of a decision as a hierarchy. The basic principle to follow in creating this structure is always to see if one can answer the following question: "Can I compare the elements on a lower level, in terms of some or all of the elements on the next higher level?"

A useful way to proceed is to come down from the goal as far as one can and then go up from the alternatives until the levels of the two processes are linked in such a way as to make comparison possible. Here are some suggestions for an elaborate design:

1. Identify overall goal. What are you trying to accomplish? What is the main question?
2. Identify subgoals of overall goal. If relevant, identify time horizons that affect the decision;
3. Identify criteria that must be satisfied to fulfill subgoals of the overall goal;
4. Identify subcriteria under each criterion. Note that criteria or subcriteria may be specified in terms of ranges of values of parameters or in terms of verbal intensities such as high, medium and low;
5. Identify actors involved;
6. Identify actor goals;
7. Identify actor policies;
8. Identify options or outcomes;
9. For *Yes - No* decisions take the most preferred outcome and compare benefits and costs of making the decision with those of not making it;

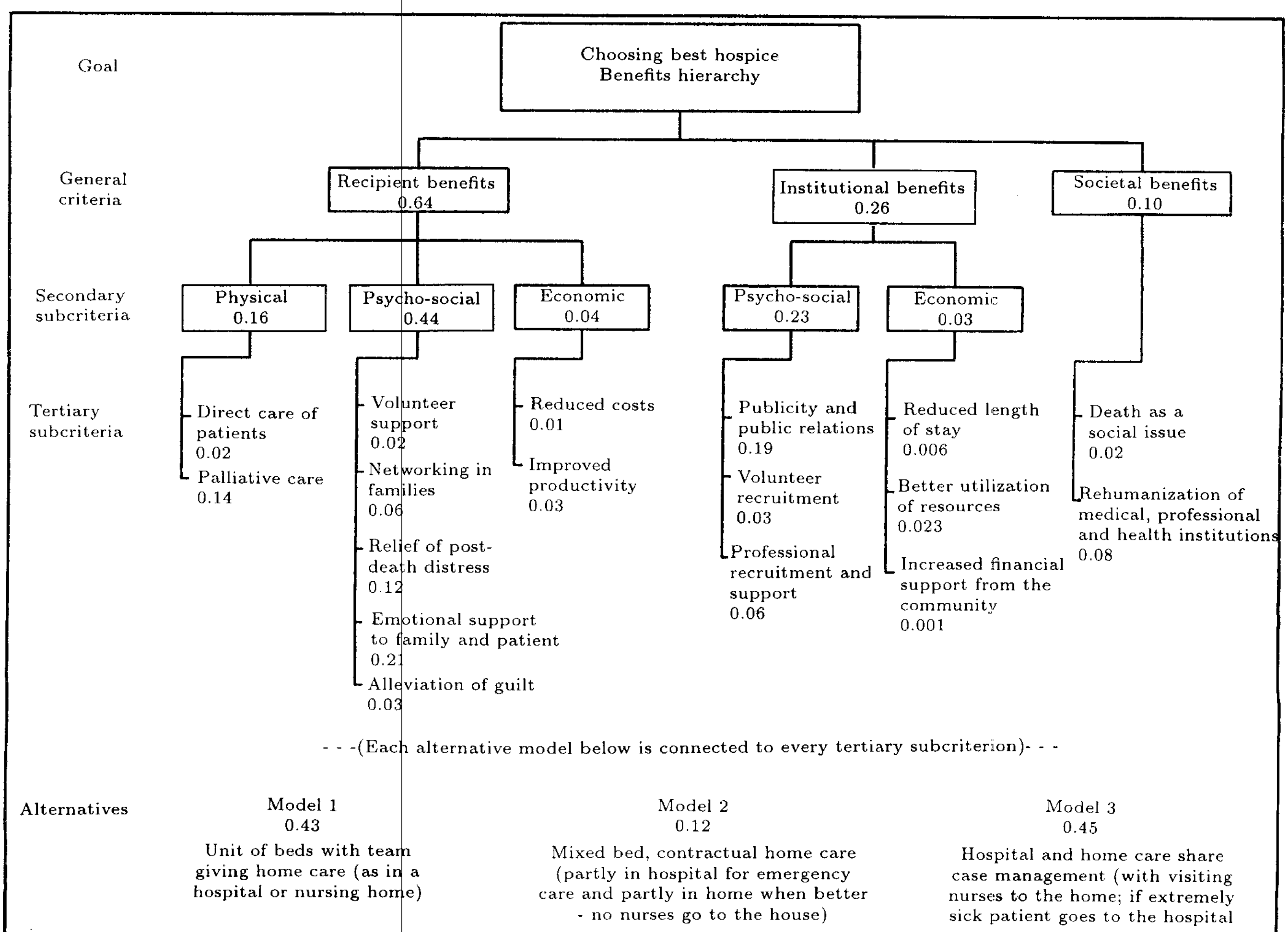

Figure 7. Hierarchy for benefits of choosing the best hospice.

10. Do benefit/cost analysis using marginal values. Because dominance hierarchies are being dealt with, ask which alternative yields the greatest benefit; for costs, ask which alternative costs the most. Do similarly if a risks hierarchy is included. For more details see [11-13].

CONCLUSION

The AHP has been applied in a variety of areas. It has been used extensively in the economics/management area in subjects including auditing, database selection, design, architecture, finance, macro-economic forecasting, marketing (consumer choice, product design and development strategy), planning, portfolio selection, facility location, forecasting, resource allocation (budget, energy, health, project), sequential decisions, policy/strategy, transportation, water research and performance analysis. In political problems, the AHP is used in such areas as arms control, conflicts and negotiation, political candidacy, security assessments, war games and world influence. For social concerns, it is applied in education, behavior in competition, environmental issues, health, law, medicine (drug effectiveness, therapy selection), population dynamics (interregional migration patterns, population size) and public sector. Some technological applications include market selection, portfolio selection and technology transfer.

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