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Research Note

Robust Stability Analysis of Singularly Perturbed Systems Using the Structured Singular Values Approach

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In this paper, the robust stability analysis and stability bound improvement of perturbed parameter (ε) in singularly perturbed systems are considered, using linear fractional transformations and structured singular values (μ) approach. In this direction, by introducing the parametric and dynamic uncertainty in the singularly perturbed systems, the mentioned system is rewritten as a standard μ-interconnection framework by using linear fractional transformations. Also, a set of new stability conditions for the system is derived in the frequency domain. The exact solution of ε-bound is characterized. It is shown that the ε-bound obtained through this approach is larger compared to that of [1], in which only parametric uncertainty is considered. Simulation results show the efficiency of the approach.

INTRODUCTION

The linear time-invariant singularly perturbed systems under consideration have the following standard form:

$$\dot{x}_s(t) = a_s x_s(t) + a_{sf} x_f(t) + b_s u(t), \tag{1}$$

$$\varepsilon \dot{x}_f(t) = a_{fs} x_s(t) + a_f x_f(t) + b_f u(t), \tag{2}$$

$$y(t) = C_s x_s(t) + C_f x_f(t). \tag{3}$$

where $x_s = [x_{s1}, x_{s2}, \cdots, x_{sn}]^T \in R^n$, $x_f = [x_{f1}, x_{f2}, \cdots, x_{fm}]^T \in R^m$, $y(t) \in R^r$ and $u(t) \in R^k$ represent the state vector of the slow-modes and fast-modes, measured output and control input, respectively. The perturbed parameter, ε, is nonnegative and always represents the response time of the fast modes. The initial conditions of the slow and fast dynamics are equal to zero.

Singularly perturbed systems often occur naturally because of the presence of small parasitic parameters multiplying the time derivatives of some of the system states. Singularly perturbed control systems have been intensively studied for the past three decades [2]. A popular approach adopted to handle these systems is based on the so-called reduced technique [3]. The composite design, based on separate designs for slow and fast subsystems, has been systematically reviewed in [3,4]. The techniques, such as the method of singular perturbations and order reduction based on system balancing, have the same robustness accuracy evaluated with respect to the H_∞ norm of the reduced-order system. The authors in [5] proposed how to perform order-reduction of a balanced system using the theory of singular perturbations that can produce very good accuracy at high frequencies, particularly for systems that have lightly damped and highly oscillatory modes. Recently, the robust stability and disturbance attenuation for a class of uncertain singularly perturbed systems has also been investigated [6]. Also, new results on control synthesis for robust stabilization and robust disturbance attenuation of linear state-delayed singularly perturbed systems with norm-bounded nonlinear uncertainties have been considered [7].

The stability problem (ε-bound problem) in singularly perturbed systems differs from what is usually posed in conventional linear systems. It can be formulated as: Characterize an upper bound, ε^*, of the positive perturbing scalar, ε, such that the stability of a reduced-order system would guarantee the stability of the original full-order system for all $\varepsilon \in (0, \varepsilon^*)$ [8]. It is known, by the lemma of Klimushchev and Krasovskii [1,2], that if the reduced-order system is asymptotically stable, then this upper bound, ε^*,

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always exists. Researchers have tried various ways to find either the stability bound, ε^*, or a less conservative lower bound, as described in [1,2,8]. Although numerous ways have been presented to compute bound ε^*, unfortunately, only some of the conservative bounds of ε were achieved.

Recently, in [9,10], a new modeling approach was developed for singularly perturbed systems under the assumption of norm-boundedness of the fast dynamics. In this approach, a portion of the fast dynamics is treated as norm-bounded dynamic uncertainty and the remaining part is augmented to the slow dynamics. In this view, Relations 1 to 3 are expressed in the following form:

$$\begin{cases} \begin{bmatrix} I_{n\times n} & 0 \\ 0 & \varepsilon I_{i-1\times i-1} \end{bmatrix} \dot{X}(t) = A_X X(t) + A_{Xw} w(t) + B_u u(t) \\ y(t) = C_X X(t) + C_w w(t), \end{cases} \quad (4)$$

$$\varepsilon \dot{w}(t) = A_{wX} X(t) + A_w w(t) + B_w u(t)$$
$$\triangleq A_w w(t) + \begin{bmatrix} A_{wX} & B_w \end{bmatrix} Z(t), \quad (5)$$

where $A_X = \begin{bmatrix} A_s & A_{sf} \\ A_{fs} & A_f \end{bmatrix}$, with $A_s \in R^{n\times n}$, $A_f \in R^{(i-1)\times(i-1)}$, $A_{sf} \in R^{n\times(i-1)}$, $A_{fs} \in R^{(i-1)\times n}$ and $A_{Xw} = \begin{bmatrix} A_{sw} \\ A_{fw} \end{bmatrix}$, with $A_{sw} \in R^{n\times(m-i+1)}$, $A_{fw} \in R^{(i-1)\times(m-i+1)}$ and $B_u = \begin{bmatrix} B_s \\ B_f \end{bmatrix}$, with $B_s \in R^{n\times k}$, $B_f \in R^{(i-1)\times k}$ and $C_X = \begin{bmatrix} C_s & C_f \end{bmatrix}$, with $C_s \in R^{r\times n}$, $C_f \in R^{r\times(i-1)}$ and $A_{wX} = \begin{bmatrix} A_{ws} & A_{wf} \end{bmatrix}$, with $A_{ws} \in R^{(m-i+1)\times n}$, $A_{wf} \in R^{(m-i+1)\times(i-1)}$. Also, $X = \begin{bmatrix} X_s^T & X_f^T \end{bmatrix}^T \in R^{n+i-1}$ is the vector of certain dynamics, in which $X_s = [x_{s_1}, x_{s_2}, \cdots, x_{s_n}]^T \in R^n$ and $X_f = [x_{f_1}, x_{f_2}, \cdots, x_{f_{(i-1)}}]^T \in R^{i-1}$ and $w = [x_{f_i}, x_{f_{(i+1)}}, \cdots, x_{f_m}]^T$ is the vector of fast dynamics, which is to be treated as a norm-bounded uncertainty, where i is the index of the first state of the "uncertain" dynamics. It is clear that the smaller i is the size of the nominal system, i.e., the dimension of $X(t)$ in Relation 4 (see, e.g. [9,10]).

Also, the controlled output $Z(t)$ for the nominal system is defined as:

$$Z(t) = \begin{bmatrix} X^T(t) & u^T(t) \end{bmatrix}^T$$
$$\triangleq C_1 X_s(t) + C_2 X_f(t) + C_3 u(t). \quad (6)$$

Assumption 1

The structured dynamic uncertainty $\Delta_d(s)$ is assumed to be internally asymptotically stable whose H_∞ norm is less than, or equal to, γ_1, i.e., $\|\Delta_d(s)\|_\infty \le \gamma_1$. In the frequency-domain one has:

$$\Delta_d(s) = (\varepsilon s I - A_w)^{-1} \begin{bmatrix} A_{wf} & A_{ws} & B_w \end{bmatrix}. \quad (7)$$

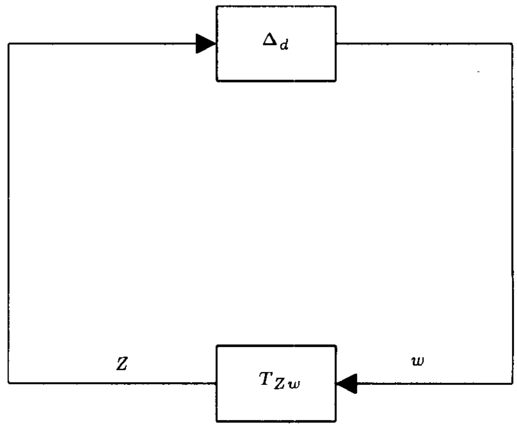

Figure 1. General block diagram for analysis with small gain theorem.

where $\Delta_d(s)$ denotes the open-loop transfer function from $Z(t)$ to $w(t)$ in Relation 5 (see Figure 1).

In [9,10], the structure of the H_∞ controller is determined for the nominal system (Relation 4), such that the sufficient condition of small gain theorem (see Appendix) is satisfied, i.e.:

$$\|T_{Zw}\|_\infty . \|\Delta_d\|_\infty < 1, \quad (8)$$

where $\|T_{Zw}\|_\infty = \sup_{w\in L_2} \frac{\|Z\|_2}{\|w\|_2}$ and T_{Zw} denotes the closed-loop transfer function from $w(t)$ to $Z(t)$, as shown in Figure 1.

Continuing in the same fashion as presented in [1,9-11], this paper considers a new modeling approach for singularly perturbed systems and treats the perturbed parameter, ε, as a parametric uncertainty and uses the structured singular value for robustness analysis to characterize the stability bound. By using linear fractional transformation and applying the small μ theorem, a set of new stability conditions are derived, for which the obtained ε-bound is larger compared to that obtained in [1], where only the parametric uncertainty is considered.

Notation

R^n denotes the n-dimensional Euclidean space, $R^{n\times m}$ is the set of $n\times m$ real matrices, $C^{n\times m}$ is the set of $n\times m$ complex matrices, $I_{n\times n}$ is the $n\times n$ identity matrix, L_2 is the space of square integrable functions on $[0,\infty]$ and $\|.\|_2$ denotes the L_2-norm. An operator on L_2 is a map $\Delta : L_2 \to L_2$; the operator gain is given by the H_∞ norm $\|\Delta\|_\infty = \sup_{z\in L_2} \frac{\|\Delta z\|_2}{\|z\|_2}$ and $\|\Delta\|_\infty$ denotes any H_∞ norm satisfying $\|AB\|_\infty \le \|A\|_\infty.\|B\|_\infty$. $\overline{\sigma}(M)$ denotes the largest singular value of M and $\mu_{+\Delta_P}(M(j\omega))$ is the largest positive real eigenvalue of $M(j\omega)$ if $\Delta_P = \{\Delta | \Delta = \delta I, \delta \in R^+\}$ where R^+ is the positive real space.

LFT REPRESENTATION OF A SINGULARLY PERTURBED SYSTEM

Linear fractional transformations are currently used in the areas of robust control theory and they represent a means of standardizing a wide variety of feedback arrangements. Here, linear fractional transformations are also used and Figures 2 and 3 are referred to.

First, consider the open-loop case, then, the upper feedback loop can be used to define the upper fractional transformation:

$$\underset{u}{LFT}\left(\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \Delta\right) \equiv P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12},$$

whenever $\det(I - P_{11}\Delta) \neq 0$. $\underset{u}{LFT}$ denotes the transfer function from $u(t)$ to $y(t)$ as shown in Figure 2.

Next, the lower feedback loop is used to define the lower linear fractional transformations:

$$\underset{l}{LFT}\left(\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, k\right) \equiv P_{11} + P_{12}k(I - P_{22}k)^{-1}P_{21}$$

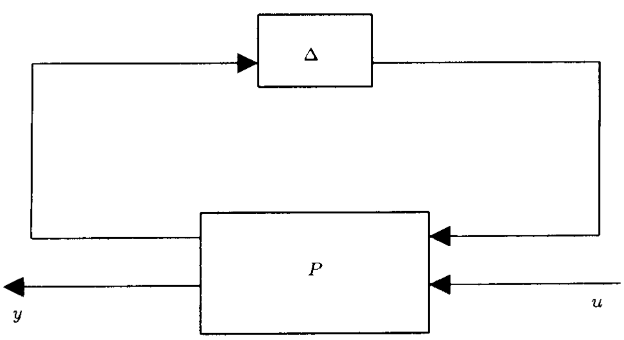

Figure 2. Block diagram for upper LFT in terms of Δ.

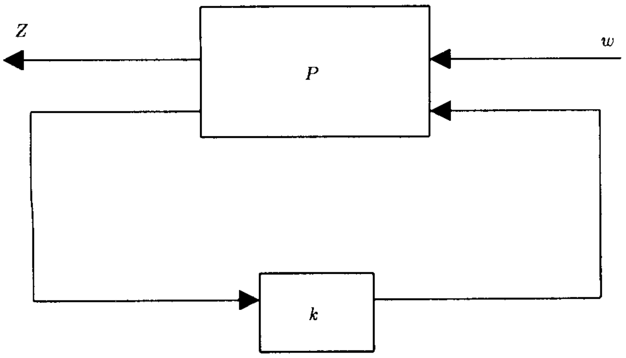

Figure 3. Block diagram for lower LFT in terms of the controller k.

whenever $\det(I - P_{22}k) \neq 0$. $\underset{l}{LFT}$ denotes the transfer function from $w(t)$ to $Z(t)$ as shown in Figure 3.

The upper and lower LFTs provide a convenient framework and a very general means of describing (uncertain) systems (see [12,13]).

Taking the Laplace transform of Relations 4 and 5, an LFT representation of the nominal system can be obtained (Relation 4), as shown in Figure 4:

$$N = \begin{bmatrix} A_f & A_{fw} & B_f & A_{fs} \\ C_2 & 0 & C_3 & C_1 \\ C_f & C_w & 0 & C_s \\ A_{sf} & A_{sw} & B_s & A_s \end{bmatrix}. \quad (9)$$

By letting $M_s(s) = \underset{l}{LFT}(N, s^{-1})$ and $M_s(s) = \underset{u}{LFT}(N, (\varepsilon s)^{-1})$, two alternative descriptions can also be obtained, as shown in Figure 5:

$$M_s(s) = \left[\begin{array}{c:c:c} A_f + A_{fs} & A_{fw} + A_{fs} & B_f + A_{fs} \\ \times(sI - A_s)^{-1}A_{sf} & \times(sI - A_s)^{-1}A_{sw} & \times(sI - A_s)^{-1}B_s \\ \hline C_2 + C_1 & C_1(sI - A_s)^{-1} & C_3 + C_1 \\ \times(sI - A_s)^{-1}A_{sf} & \times A_{sw} & \times(sI - A_s)^{-1}B_s \\ \hline C_f + C_s & C_w + C_s & C_s(sI - A_s)^{-1} \\ \times(sI - A_s)^{-1}A_{sf} & \times(sI - A_s)^{-1}A_{sw} & \times B_s \end{array}\right]$$

$$\equiv \left[\begin{array}{c|c|c} M_{11}(s) & M_{12}(s) & M_{13}(s) \\ \hline M_{21}(s) & M_{22}(s) & M_{23}(s) \\ \hline M_{31}(s) & M_{32}(s) & M_{33}(s) \end{array}\right], \quad (10)$$

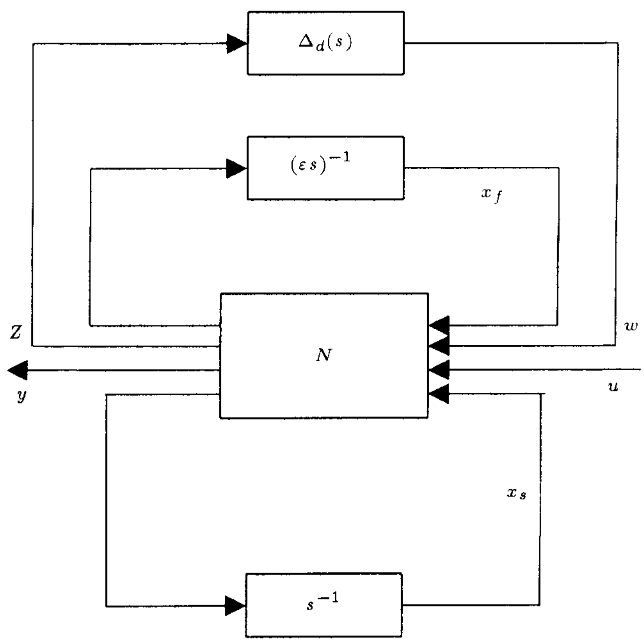

Figure 4. General block diagram for nominal system.

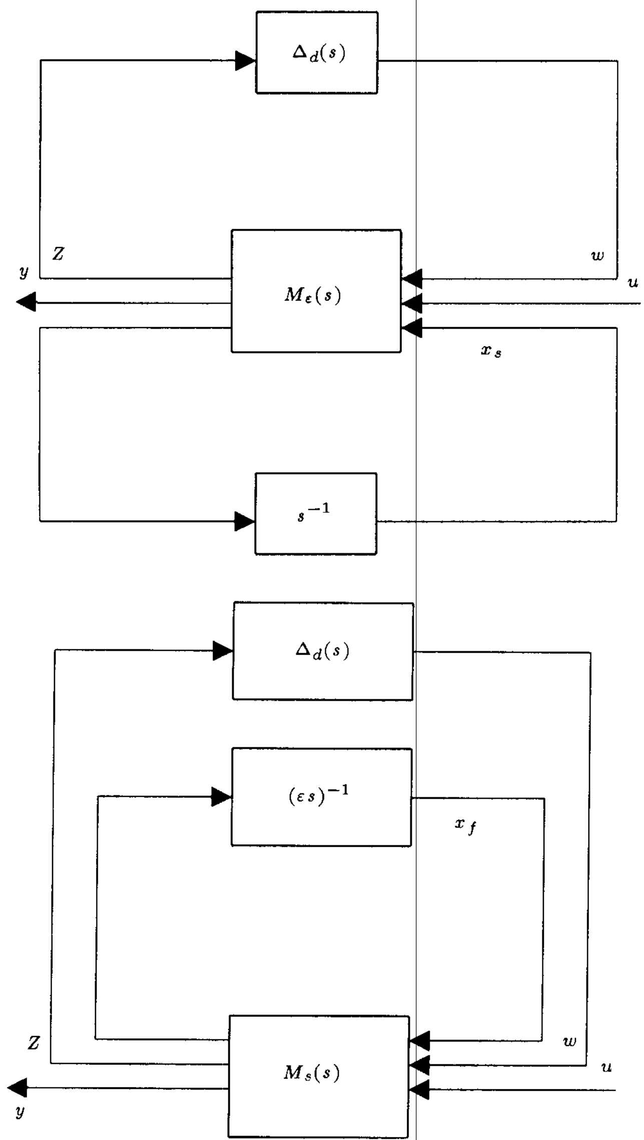

Figure 5. Block diagram representation for nominal system in alternative descriptions.

and:

$$M_\varepsilon(s) = \left[\begin{array}{c|c|c} C_2(\varepsilon sI - A_f)^{-1} \times A_{fw} & C_3 + C_2 \times(\varepsilon sI - A_f)^{-1} \times B_f & C_1 + C_2 \times(\varepsilon sI - A_f)^{-1} \times A_{fs} \\ \hline C_w + C_f \times(\varepsilon sI - A_f)^{-1} \times A_{fw} & c_f(\varepsilon sI - A_f)^{-1} \times B_f & C_s + C_f \times(\varepsilon sI - A_f)^{-1} \times A_{fs} \\ \hline A_{sw} + A_{sf} \times(\varepsilon sI - A_f)^{-1} \times A_{fw} & B_s + A_{sf} \times(\varepsilon sI - A_f)^{-1} \times B_f & A_s + A_{sf} \times(\varepsilon sI - A_f)^{-1} \times A_{fs} \end{array}\right] \tag{11}$$

M_0 is defined from Relation 11, as follows:

$$M_0 \equiv M_\varepsilon(s)|_{\varepsilon=0}$$

$$= \begin{bmatrix} -C_2 A_f^{-1} A_{fw} & C_3 - C_2 A_f^{-1} B_f & C_1 - C_2 A_f^{-1} A_{fs} \\ C_w - C_f A_f^{-1} A_{fw} & -C_f A_f^{-1} B_f & C_s - C_f A_f^{-1} A_{fs} \\ A_{sw} - A_{sf} A_f^{-1} A_{fw} & B_s - A_{sf} A_f^{-1} B_f & A_s - A_{sf} A_f^{-1} A_{fs} \end{bmatrix}$$

$$\equiv \begin{bmatrix} D_0 & C_0 \\ B_0 & A_0 \end{bmatrix}. \tag{12}$$

Now, the certain part of the main system Relations 1 to 3, namely, the nominal system, can be represented by:

$$P_\varepsilon(s) = \underset{u}{LFT}(M_s(s), (\varepsilon s)^{-1})$$

$$= \underset{l}{LFT}(M_\varepsilon(s), s^{-1}). \tag{13}$$

With $\varepsilon = 0$, a reduced order system is then given by:

$$P_0(s) = LFT_l(M_0(s), s^{-1})$$

$$= D_0 + C_0(sI - A_0)^{-1}B_0. \tag{14}$$

From Relations 10 and 13, it can be found that:

$$P_\varepsilon(s)$$

$$= \left[\begin{array}{c|c} M_{22} + M_{21}(\varepsilon sI - M_{11})^{-1} \times M_{12} & M_{23} + M_{12} \times(\varepsilon sI - M_{11})^{-1} M_{13} \\ \hline M_{32} + M_{31}(\varepsilon sI - M_{11})^{-1} \times M_{12} & M_{33} + M_{31} \times(\varepsilon sI - M_{11})^{-1} M_{31} \end{array}\right], \tag{15}$$

and:

$$P_0(s) =$$

$$\begin{bmatrix} M_{22} - M_{21}M_{11}^{-1}M_{12} & M_{23} - M_{21}M_{11}^{-1}M_{13} \\ M_{32} - M_{31}M_{11}^{-1}M_{12} & M_{33} - M_{31}M_{11}^{-1}M_{13} \end{bmatrix}, \tag{16}$$

Let $\delta P(s) = P_\varepsilon(s) - P_0(s)$. Then,

$$\delta P(s) = \begin{bmatrix} M_{21} & M_{11}^{-1} \\ M_{31} & M_{11}^{-1} \end{bmatrix} \left(I - (I - \varepsilon s M_{11}^{-1})^{-1}\right) \begin{bmatrix} M_{12} & M_{13} \end{bmatrix} \tag{17}$$

As a result, a block diagram of $P_\varepsilon(s) = P_0(s) + \delta P(s)$ can be found, which is shown in Figure 6. Moreover, a particularly useful LFT representation of the main system (Relations 1 to 3) is given in Figure 7.

Remark 1

If $A_0 = A_s - A_{sf}A_f^{-1}A_{fs}$ is a Hurwitz matrix, then it can be verified by direct computation from Relations 14 and 16 that $M_{11}^{-1}, P_0, M_{11}^{-1}\begin{bmatrix} M_{12} & M_{13} \end{bmatrix}$ and $\begin{bmatrix} M_{21} \\ M_{31} \end{bmatrix} M_{11}^{-1}$ are all Hurwitz.

As can be seen from Relations 11, 13 and 14, by the continuity property, the necessary and sufficient conditions for existence of $\varepsilon^* > 0$ without destabilizing $P_\varepsilon(s)$ are A_0 and A_f being Hurwitz. This result is concluded through the following Lemma.

Lemma 1

Given the nominal system (Relation 4), if the matrices A_0 and A_f are Hurwitz, then there exists an $\varepsilon^* > 0$, such that $P_\varepsilon(s)$ (or equivalently $\delta P(s)$) is stable for all $\varepsilon \in [0, \varepsilon^*)$.

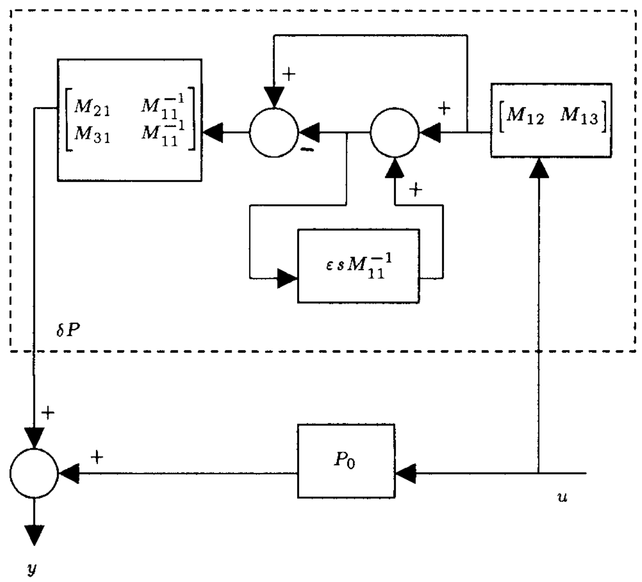

Figure 6. Block diagram of $P_\varepsilon(s)$.

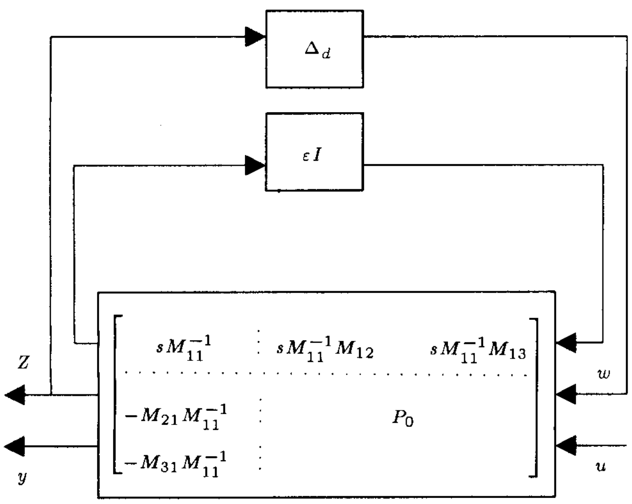

Figure 7. Block diagram representation for the main system.

ROBUST STABILITY ANALYSIS

The structured singular value has been proven to be a powerful tool for the robustness analysis of linear systems. Some concepts and results used in the sequel are outlined in [12,13].

Definition 1

The uncertainty structures Δ that will be used in the sequel are [12]:

$$\Delta_d(s) = \{\Delta | \Delta = \text{block diag } [\Delta_{d_1}, \Delta_{d_2}, \cdots, \Delta_{d_m}],$$

$$\Delta_{d_i} \in C^{r_i \times r_i}, \overline{\sigma}(\Delta_{d_i}) \leq \gamma_d, i = 1, 2, \cdots, m\},$$

called structured dynamic uncertainty and:

$$\Delta_P(s) = \{\Delta | \Delta = \text{block diag } [\Delta_{P_1}, \Delta_{P_2}, \cdots, \Delta_{P_m}],$$

$$\Delta_{Pi} \in R^{r_i \times r_i}, |\Delta_{Pi}| \leq \gamma_P, i = 1, 2, \cdots, m\},$$

called parametric uncertainty. In the case of having simultaneously both types of uncertainties, i.e., parametric and dynamic uncertainty, another type will be used, namely, mixed uncertainty, which may be represented as follows:

$$\Delta_m(s) = \left\{ \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \middle| \Delta_1 \in \Delta_d, \Delta_2 \in \Delta_P \right\}.$$

Definition 2

For a given matrix $M \in C^{n \times n}$, the structured singular value of M with respect to $\Delta \in \Delta_m$, is defined by [12]:

$$\mu_\Delta[M(j\omega)] = \begin{cases} (\underset{\Delta \in \Delta_m}{Inf} \{\overline{\sigma}(\Delta) | \det(I - M(j\omega)\Delta) = 0)^{-1} \\ 0 \quad \text{if} \det(I - M(j\omega)\Delta) \neq 0, \ \forall \Delta \in \Delta_m, \end{cases} \tag{18}$$

Now, two theorems on the robust stability are first reviewed.

Theorem 1 (Small μ Theorem)

Let $P(s)$ be stable. Then $\underset{u}{LFT}(P(s), \Delta)$ is stable for all $\Delta \in \Delta_m$ if, and only if [12],

$$\mu_\Delta(P_{11}(j\omega)) < \frac{1}{\gamma}, \quad \overline{\sigma}(\Delta) < \gamma, \tag{19}$$

i.e., $\det(I - P_{11}(j\omega)\Delta) \neq 0$ for all $\Delta \in \Delta_m$ and all ω.

Theorem 2

As indicated in Figure 8, the system $M(s)$ is robustly stable for the mixed uncertainty Δ_m if, and only if [12],

1. $\tilde{M}(s, \Delta_P) = \underset{u}{LFT}(M(s), \Delta_P)$ is stable for all $|\Delta_P| < \gamma_P$.
2. $\mu_{\Delta_d}(\tilde{M}(j\omega, \Delta_P)) \leq \frac{1}{\gamma_d}$ for all $\overline{\sigma}(\Delta_d) < \gamma_d$ and all ω.

Theorem 1, Theorem 2 and Lemma 1 are now applied to characterize the stability bound.

Remark 2

A comparison between Figures 6 and 7 reveals that:

$$M(s) = \left[\begin{array}{c|cc} sM_{11}^{-1} & sM_{11}^{-1}M_{12} & sM_{11}^{-1}M_{13} \\ \hline -M_{21}M_{11}^{-1} & \multicolumn{2}{c}{\multirow{2}{*}{P_0}} \\ -M_{31}M_{11}^{-1} & & \end{array} \right]$$

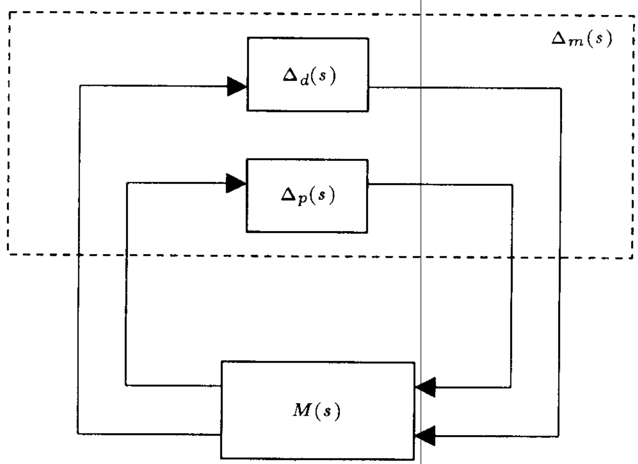

Figure 8. General block diagram for mixed uncertainty descriptions.

and:

$$\tilde{M}(s, \Delta_P) = P_\varepsilon(s).$$

Now, by utilization of Lemma 1 and the above mentioned theorems, one can determine the stability bound of perturbed parameter in the following theorem.

Theorem 3

Consider the main system (Relations 1 to 3) and suppose that A_0 and A_f are Hurwitz matrices. Then, the largest positive scalar ε^* that may be obtained (without destabilizing the main system for $\varepsilon < \varepsilon^*$) is given by:

$$\varepsilon^* = \{\sup_\omega(\mu_{+\Delta_P}(j\omega M_{11}^{-1}(j\omega)))\}^{-1}, \tag{20}$$

where $\Delta_P = \varepsilon I$.

Proof

First, by using the proposed approach in [9,10], the main system is modeled as the nominal system and the controller for the nominal system (Relation 4) is designed such that Condition 8 is satisfied. From Condition 8, one can find that:

$$\bar{\sigma}(T_{zw}\Delta_d) < 1. \tag{21}$$

Also, from Relation 21 and Lemma 1, it is clear that:

$$\det(I - T_{zw}\Delta_d) \neq 0, \tag{22}$$

where $T_{zw} = \tilde{M}(s, \Delta_P)$.

According to Definition 2, one has:

$$\mu_{\Delta_d}(T_{zw}) \leq \frac{1}{\gamma_d} \quad \text{for } \bar{\sigma}(\Delta_d) < \gamma_d, \tag{23}$$

which verifies Expression 2 of Theorem 2. Now, the correctness of expression 1 in Theorem 2 (by using the small μ theorem) depends on the following inequality:

$$\mu_{+\Delta_P}(jwM_{11}^{-1}(j\omega)) < \frac{1}{\gamma_P} \quad \text{for } |\Delta_P| < \gamma_P. \tag{24}$$

Letting $\Delta_P = \varepsilon I$, it is concluded that $|\varepsilon| < \gamma_P$.

By Lemma 1, Theorem 2 and Inequality 24, it is concluded that system $M(s)$ is robustly stable in the presence of mixed uncertainties. Then, the largest positive scalar, without destabilizing the main system, is given by:

$$\varepsilon^* \triangleq \gamma_P = \{\sup_w(\mu_{+\Delta_P}(jwM_{11}^{-1}(jw)))\}^{-1}. \tag{25}$$

Remark 3

Theorem 3 implies that if the relation $\mu_{+\Delta_P}(jwM_{11}^{-1}(j\omega)) = 0$ is satisfied for all ω, then the stability bound ε^* will be infinite. As can be found from Relations 10 and 15, a necessary condition for the main system to have the infinite stability bound is that A_s must be a Hurwitz matrix. Recall that:

$$P_\varepsilon(s)|_{\varepsilon=\infty} = \begin{bmatrix} M_{22}(s) & M_{23}(s) \\ M_{32}(s) & M_{33}(s) \end{bmatrix}. \tag{26}$$

Remark 4

By utilizing [1], one may find that the result of Theorem 3 is equivalent to evaluating whether or not:

$$\det(I - j\omega M_{11}^{-1}(j\omega)\varepsilon) \neq 0, \quad \forall\varepsilon \in [0, \varepsilon^*), \forall\omega. \tag{27}$$

Now, it is shown that the stability bound of perturbed parameter, as derived in this paper, is better, i.e., larger than that of [1], in which only parametric uncertainty is considered. Then, the ε-bound of the system is obtained by using the approach presented in [1]. Comparing Relations 1 to 3 in the main system with Relations 4 and 5, it is obtained that:

$$a_s = A_s, \; a_{sf} = \begin{bmatrix} A_{sf} & A_{sw} \end{bmatrix},$$

$$a_f = \begin{bmatrix} A_f & A_{fw} \\ A_{wf} & A_w \end{bmatrix}, \; a_{fs} = \begin{bmatrix} A_{fs} \\ A_{ws} \end{bmatrix}. \tag{28}$$

Lemma 2

Define $Z(s)$ as:

$$Z(s) = \begin{bmatrix} A_f + A_{fs}(sI - A_s)^{-1}A_{sf} & A_{fw} + A_{fs}(sI - A_s)^{-1}A_{sw} \\ A_{wf} + A_{ws}(sI - A_s)^{-1}A_{sf} & A_w + A_{ws}(sI - A_s)^{-1}A_{sw} \end{bmatrix}$$

$$\triangleq \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}. \tag{29}$$

Then, the following relation can be concluded:

$$\det(I - j\omega\varepsilon Z^{-1}(j\omega)) = \det(Z^{-1}(j\omega))\det(Z_{11}(j\omega)) \times \det(\overline{Z}_{22}(j\omega))\det(I - j\omega\varepsilon Z_{11}^{-1}(j\omega)) \times \det(I - j\omega\varepsilon \overline{Z}_{22}^{-1}(j\omega)), \quad (30)$$

where:

$$\overline{Z}_{22}(s) \triangleq Z_{22} - Z_{21}(Z_{11} - \varepsilon sI)^{-1}Z_{12}. \quad (31)$$

Proof

Using Schur formula, regarding determinant of a partitioned matrix, Relation 30 can be concluded.

Now, according to [1], the stability bound of the perturbed parameter can be obtained by using the structured singular values approach, such that:

$$\varepsilon_1^* = \{\sup_\omega(\mu_{+\Delta_P}(j\omega Z^{-1}(j\omega)))\}^{-1}. \quad (32)$$

From Definition 2 and Relation 32, the following result is obtained:

$$\det(I - j\omega\varepsilon Z^{-1}(j\omega)) \neq 0, \quad \forall\varepsilon \in [0, \varepsilon_1^*), \forall\omega. \quad (33)$$

According to Relations 30 and 33, for robust stability of the main system (Relations 1 to 3), the following relations should be satisfied simultaneously.

$$\begin{cases}\det(I - j\omega\varepsilon Z_{11}^{-1}(j\omega)) \neq 0, & \forall\omega, \forall\varepsilon \in [0, \varepsilon_1^*) \\ \det(I - j\omega\varepsilon \overline{Z}_{22}^{-1}(j\omega)) \neq 0, & \forall\omega, \forall\varepsilon \in [0, \varepsilon_1^*)\end{cases}. \quad (34)$$

A comparison between Relations 34 and 27 shows that ε_1^*-bound is equal to, or less than, ε^*-bound. This is due to the fact that the ε_1^*-bound in restricted Relation 34 satisfies two constraints but ε^*-bound is limited only to Relation 27. Then, by introducing the new modeling approach and analysis for the singularly perturbed systems, an improvement on the stability bound of the perturbed parameter may be obtained, in the sense that it is less conservative.

EXAMPLE

As an example, the robust stability analysis of a singularly perturbed system is considered consisting of two fast states, coupled with a slow state such that the open-loop system is unstable and the state-space realization for this system reads as follows:

$$\begin{bmatrix}\dot{x}_s(t)\\ \varepsilon\dot{x}_{f_1}(t)\\ \varepsilon\dot{x}_{f_2}(t)\end{bmatrix} = \begin{bmatrix}5 & 2 & 3\\ 2 & -5 & 1\\ 4 & 1 & -15\end{bmatrix}\begin{bmatrix}x_s(t)\\ x_{f_1}(t)\\ x_{f_2}(t)\end{bmatrix} + \begin{bmatrix}5\\2\\1\end{bmatrix}u(t), \quad (35)$$

$$y(t) = \begin{bmatrix}1 & 1 & 0\end{bmatrix}\begin{bmatrix}x_s(t)\\ x_{f_1}(t)\\ x_{f_2}(t)\end{bmatrix}.$$

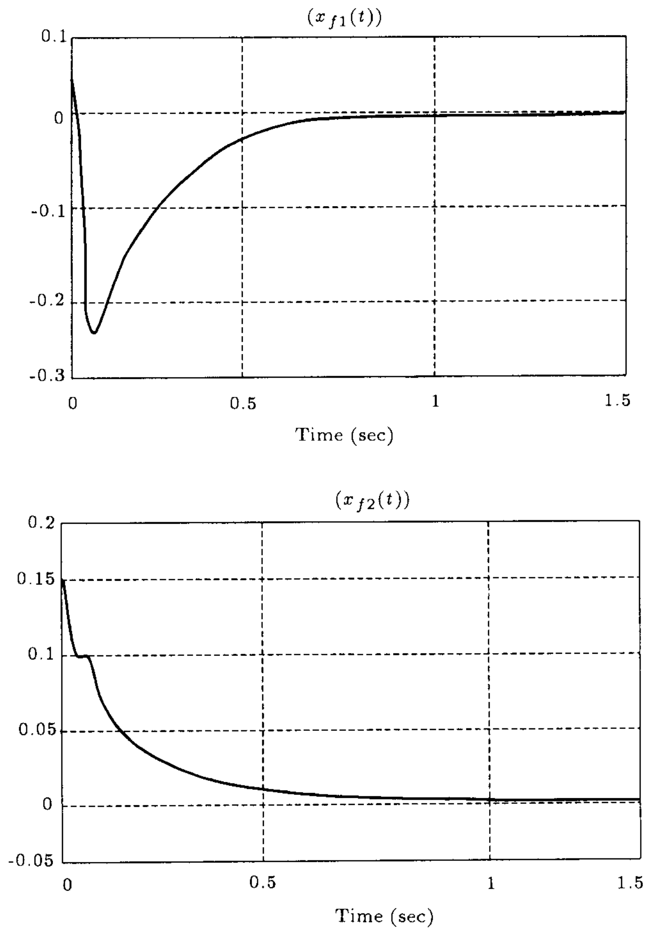

Figure 9. Response of uncertain dynamics under output feedback.

Since the fast sub-system (Relation 35) is stable, according to the approach proposed in [9,10] for separating the system dynamics into certain and uncertain dynamics, it is found that dynamics $x_{f_1}(t)$ and $x_{f_2}(t)$ can be modeled as a norm-bounded uncertainty, whose time response is depicted in Figure 9. Also, the time response of certain dynamic ($x_s(t)$) has been shown in Figure 10 and the H_∞ control signal is depicted in Figure 11. In this example, it is found that $M_{11}(s)$ is equal to zero, then, according to Remark 3, stability bound ε^* will be infinite.

CONCLUSIONS

In this paper, robust stability analysis and stability bound improvement of the perturbed parameter in the singularly perturbed systems are considered by using the linear fractional transformations and structured singular values (μ) approach.

In this direction, by introducing the parametric and dynamic uncertainty in the singularly perturbed systems, the problem was formulated as a standard μ-interconnection framework by using linear fractional transformations. A set of new stability conditions

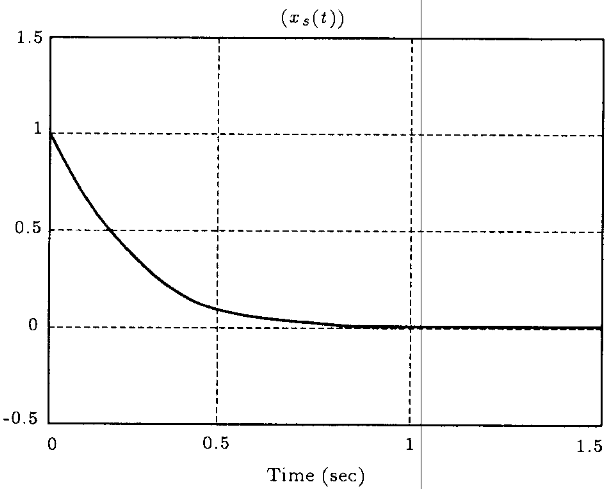

Figure 10. Response of certain dynamic under output feedback.

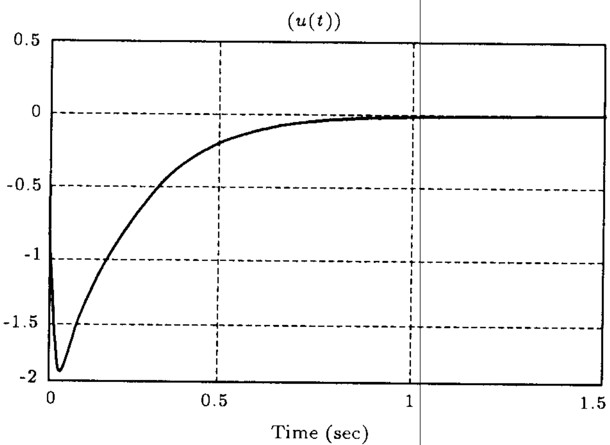

Figure 11. H_∞ controller.

for the system were derived based on the frequency domain representation, while the exact solution of ε-bound was characterized. It was shown that the obtained ε-bound is larger compared to that obtained in [1], where only the parametric uncertainty is considered.

The extension of the results to the multiple time-scale systems is straightforward. The proposed approach may provide a great deal of potential for robust controller synthesis and the robust performance analysis of output feedback singularly perturbed systems.

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APPENDIX

Small Gain Theorem

Consider a system with a stable loop transfer function $L(s)$. Then the closed-loop system is stable if [12]:

$$||L(j\omega)||_\infty < 1 \quad \forall\omega, \tag{36}$$

where $||L||_\infty$ denotes any matrix norm satisfying $||AB||_\infty \leq ||A||_\infty.||B||_\infty$.

The small gain theorem can be extended to include more than one block in the loop, e.g., $L = L_1L_2$. In this case, the system is stable if $||L_1||_\infty.||L_2||_\infty < 1$, $\forall\omega$ (a conservative result).