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Research Note

A Simple Method for Exact Evaluation of Element Integrals in Axisymmetric FEM

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In this paper, an efficient analytical method is described for evaluation of the element integrals, which occur in the variational axisymmetric finite element method with first order triangular elements. This method permits exact evaluation of such integrals without any need for numerical integration schemes. The key feature of this technique is that it permits finding the value of these integrals through solution of a simple linear algebraic system of equations.

INTRODUCTION

The Finite Element Method (FEM) [1] is capable of directly handling axisymmetric systems in either 3D or the equivalent dimensionally reduced 2D. In many reported works, the axisymmetric FEM has been used to analyze various problems including fluid flow [2], Poisson Equation [3], stress distribution [4] and rotational molding [5]. However, some difficulties arise in the corresponding 2D formulation. The major problem in converting 3D problems to 2D ones lies in the occurrence of some double element integrals that involve the radial coordinate r. These integrals have been, usually, either simplified with the assumption of constant integrand over the element area, which results in a sacrifice of accuracy, or evaluated numerically, which is complicated and relatively inefficient. Some discussion on these axisymmetric volume and surface integrals can be found in [6]. It is also possible to expand the integrand in terms of radial and axial coordinates and then perform the integration, however, this method is too complicated for most practical purposes. In fact, it has been previously proved that for axisymmetric FEM within the limit of very small elements, the solution does not converge to the accurate one, if exact integration is carried out [7].

In this paper, a simple and exact analytical approach for evaluation of the above integrals, based on linear interpolation functions (first order elements) is derived. As shown below, it is sufficient to solve a simple algebraic equation using a few floating point operations, instead of a complicated and time consuming numerical evaluation of these integrals.

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DEFINITION OF INTEGRALS

Potential problems of the mixed Neumann and Dirichlet type may be described by the following standard mathematical model:

$$\nabla.(\overleftrightarrow{\mathbf{K}}.\nabla\Phi) = \overleftrightarrow{\mathbf{f}}.\Phi + \mathbf{h} \quad \text{in } V, \tag{1a}$$

$$\nabla\Phi.\mathbf{n} = \mathbf{g}_0 + \overleftrightarrow{\mathbf{g}}_1.\Phi \quad \text{over } S \text{ (boundary of } V), \tag{1b}$$

$$\Phi = \Phi_0, \quad \text{in a subdomain of } V, \tag{1c}$$

where $\Phi = \Phi(r, z)$ is an array of unknown functions. $\overleftrightarrow{\mathbf{K}}$, $\overleftrightarrow{\mathbf{f}}$ and $\overleftrightarrow{\mathbf{g}}_1$ are given linear tensors and $\mathbf{h}, \mathbf{g}_0$ and Φ_0 are linear vector, all being explicit functions only of the coordinate variables (n stands for the normal vector to the boundary S). With the aid of Green theorem, the corresponding functional to the variational integral formulation [8] of Equations 1 may be shown as:

$$\mathbf{I}(\Phi) = \iiint_V \frac{1}{2}\nabla\Phi.\overleftrightarrow{\mathbf{K}}.\nabla\Phi + \frac{1}{2}\Phi.\overleftrightarrow{\mathbf{f}}.\Phi + \mathbf{h}.\Phi dv$$

$$- \oint_S \int \left(\frac{1}{2}\Phi.\overleftrightarrow{\mathbf{g}}_1 + \overleftrightarrow{\mathbf{g}}_0\right).\overleftrightarrow{\mathbf{K}}.\Phi q ds. \tag{2}$$

Imposing the axisymmetry condition ($\partial/\partial\theta \equiv 0$)

simplifies Equation 2 into:

$$\mathbf{I}(\Phi) = \iint_{\Omega} \Big(-\frac{1}{2}\nabla\Phi.\overleftrightarrow{\mathbf{K}}.\nabla\Phi + \frac{1}{2}\Phi.\overleftrightarrow{\mathbf{f}}.\Phi + \mathbf{h}.\Phi\Big) r dr dz - \oint_{\Gamma} \Big(\frac{1}{2}\Phi.\overleftrightarrow{\mathbf{g}}_1 + \mathbf{g}_0\Big).\overleftrightarrow{\mathbf{K}}.\Phi r dl. \tag{3}$$

Here, Ω and Γ are the surface and contour equivalents to V and S, respectively and a trivial 2π coefficient has been dropped. Notice that it is allowed to close the line integral over Γ, even for domains containing the symmetry axis. This fact is due to the disappearance of the integrand at $r = 0$, i.e., the z-axis.

Following the standard procedure for minimizing Equation 3 by linear elements, in its most general form, a system of equations results as:

$$\sum_{e=1}^{E} \iint_{S^e} (\mathbf{N}^{e^T}\mathbf{Q}^e\mathbf{N}^e + \mathbf{X}^e) r ds^e \Phi^e + \sum_{b=1}^{B} \int_{L^b} \mathbf{M}^{b^T}\mathbf{Y}^b\mathbf{M}^b r dl^b \Phi^b = \sum_{e=1}^{E} \iint_{S^e} (\mathbf{N}^{e^T}\mathbf{N}^e\mathbf{Z}^e + \mathbf{P}^e\mathbf{N}^{e^T}) r ds^e + \sum_{b=1}^{B} \int_{L^b} \mathbf{M}^{b^T}\mathbf{T}^b r dl^b, \tag{4}$$

in which $\mathbf{Q}^e, \mathbf{X}^e, \mathbf{Y}^b, \mathbf{Z}^e, \mathbf{P}^e$ and $\mathbf{T}^b$ are constant matrices which depend on the geometry of elements and edges and on the form of prescribed functions $\overleftrightarrow{\mathbf{K}}$, $\overleftrightarrow{\mathbf{f}}$, $\overleftrightarrow{\mathbf{g}}_1, \mathbf{h}, \mathbf{g}_0$ and Φ_0 in Equations 1. The first summation is carried over all elements with a total number of E and of area S^e and the second one over all boundary edges of elements with a total number of B and of length L^b. Φ^e and Φ^b are arrays of the values of Φ over the nodes (triangle vertices) of element e and edge b, respectively. Also, $\mathbf{N}^e$ and $\mathbf{M}^b$ are the shape function arrays given by:

$$\mathbf{N}^e = \begin{bmatrix}1 & r & z\end{bmatrix} \begin{bmatrix}1 & r_i^e & z_i^e \\ 1 & r_j^e & z_j^e \\ 1 & r_k^e & z_k^e\end{bmatrix} \equiv \begin{bmatrix}1 & r & z\end{bmatrix} \mathbf{D}^e \equiv \begin{bmatrix}N_i^e & N_j^e & N_k^e\end{bmatrix}, \tag{5a}$$

$$\mathbf{M}^b = \Big[\tfrac{l}{L^b} \quad \tfrac{L^b - l}{L^b}\Big], \tag{5b}$$

in which (r_n^e, z_n^e) are the coordinates of the nth node belonging to the element e, L^b is the length of the edge b and l is the length of the path element on the boundary.

The system of Equation 4 is seen to be composed of the following integrals:

$$R_{nm}^e \equiv \iint_{S^e} r N_n^e N_m^e ds^e, \tag{6a}$$

$$R_n^e \equiv \iint_{S^e} r N_n^e ds^e, \tag{6b}$$

$$U^e \equiv \iint_{S^e} r ds^e, \tag{6c}$$

$$W_k^b \equiv \int_{L^b} r M_k^b dl^b, \tag{6d}$$

$$V_{kl}^b \equiv \int_{L^b} r M_k^b M_l^b dl^b. \tag{6e}$$

In the next section, methods for the exact evaluation of the above integrals are discussed.

EVALUATION OF INTEGRALS

Calculation of Double Integral (Equation 6a)

From Equation 5a it can be observed that $N_s^e = D_{is}^e + D_{js}^e r + D_{ks}^e z$, where D_{rs}^e stands for the elements of $\mathbf{D}^e$ matrix in Equation 5a. Therefore:

$$\iint_{S^e} N_n^{e^2} N_m^e ds^e = D_{in}^e \iint_{S^e} N_n^e N_m^e ds^e + D_{jn}^e \iint_{S^e} r N_n^e N_m^e ds^e + D_{kn}^e \iint_{S^e} z N_n^e N_m^e ds^e, \tag{7a}$$

$$\iint_{S^e} N_n^e N_m^{e^2} ds^e = D_{im}^e \iint_{S^e} N_n^e N_m^e ds^e + D_{jm}^e \iint_{S^e} r N_n^e N_m^e ds^e + D_{km}^e \iint_{S^e} z N_n^e N_m^e ds^e, \tag{7b}$$

in which N_n^e and N_m^e in the integrands of the left-hand-sides of Equations 7a and 7b, respectively, have been expanded according to Equation 5a. With the aid of the following well-known identity [9]:

$$\iint_{S^e} N_i^{e^a} N_j^{e^b} N_k^{e^c} ds^e = \frac{2a!b!c!}{(a+b+c+2)!} S^e, \tag{8}$$

in which a, b and c are integers, Equations 7 transforms into either:

$$\begin{bmatrix} D_{jn} & D_{kn} \\ D_{jm} & D_{km} \end{bmatrix} \begin{bmatrix} R_{mn} \\ Z_{mn} \end{bmatrix} = \begin{bmatrix} 1/30 - D_{in}/12 \\ 1/30 - D_{im}/12 \end{bmatrix} S^e, \quad \text{if } m \neq n, \tag{9a}$$

or:

$$\begin{bmatrix} D_{jm} & D_{km} \\ D_{jp} & D_{kp} \end{bmatrix} \begin{bmatrix} R_{nn} \\ Z_{nn} \end{bmatrix} = \begin{bmatrix} 1/30 - D_{im}/6 \\ 1/30 - D_{ip}/6 \end{bmatrix} S^e, \quad \text{if } m,p \neq n, \tag{9b}$$

Here, the e superscript is dropped and (m, n, p) are indices for the vertices of element e; Z_{rs} represents the last integrals in Equations 7. Also, for evaluation of R_{nn} and Z_{nn} in Equation 9b, m and p must be chosen so that they are not equal to n, e.g. if $n = j$ then either $m = i$ and $p = k$, or $m = k$ and $p = j$. (The choice is immaterial as shown below.) Thus, R_{rs} and Z_{rs} can be exactly computed by solving the system of linear equations given in Equations 9 as:

$$\begin{bmatrix} R_{mn} \\ Z_{mn} \end{bmatrix} = \frac{S^e}{60(D_{jn}D_{km} - D_{jm}D_{kn})} \times \begin{bmatrix} 2(D_{km} - D_{kn}) + 5(D_{im}D_{kn} - D_{in}D_{km}) \\ 2(D_{jn} - D_{jm}) + 5(D_{in}D_{jm} - D_{im}D_{jn}) \end{bmatrix}, \quad \text{if } m \neq n, \tag{10a}$$

or:

$$\begin{bmatrix} R_{nn} \\ Z_{nn} \end{bmatrix} = \frac{S^e}{30(D_{jm}D_{kp} - D_{jp}D_{km})} \times \begin{bmatrix} D_{kp} - D_{km} + 5(D_{ip}D_{km} - D_{im}D_{kp}) \\ D_{jm} - D_{jp} + 5(D_{im}D_{jp} - D_{ip}D_{jm}) \end{bmatrix}, \quad \text{if } m,p \neq n. \tag{10b}$$

In Equation 10b it can be verified that upon interchanging m and p, the final values of R_{nn} and Z_{nn} remain the same. It is, thus, seen that the element integral (Equation 6a) can be evaluated exactly without need of numerical integration. The computation of relations presented in Equations 10, which requires few floating point operations, is obviously much faster in comparison to the most efficient numerical integration schemes. In addition, the element integral Z_{rs} is simultaneously evaluated, however, its value is not used in the current formulation of the first-order axisymmetric finite elements.

Calculation of Double Integral (Equation 6b)

In a manner similar to the above expansion of N_m and N_p, as defined in Equation 5a, it can be easily shown that:

$$\iint_{S^e} N_m N_n ds = D_{im} \iint_{S^e} N_n ds + D_{jm} \iint_{S^e} r N_n ds + D_{km} \iint_{S^e} z N_n ds, \tag{11a}$$

$$\iint_{S^e} N_p N_n ds = D_{ip} \iint_{S^e} N_n ds + D_{jp} \iint_{S^e} r N_n ds + D_{kp} \iint_{S^e} z N_n ds. \tag{11b}$$

Therefore:

$$\begin{bmatrix} D_{jm} & D_{km} \\ D_{jp} & D_{kp} \end{bmatrix} \begin{bmatrix} R_n \\ Z_n \end{bmatrix} = \begin{bmatrix} 1/6 - D_{im}/3 \\ 1/6 - D_{ip}/3 \end{bmatrix} S^e, \quad m,p \neq n, \tag{12}$$

which can be similarly solved for R_n and Z_n as:

$$\begin{bmatrix} R_n \\ Z_n \end{bmatrix} = \frac{S^e}{6(D_{jm}D_{kp} - D_{jp}D_{km})} \times \begin{bmatrix} D_{kp} - D_{km} + 2(D_{ip}D_{km} - D_{im}D_{kp}) \\ D_{jm} - D_{jp} + 2(D_{im}D_{jp} - D_{ip}D_{jm}) \end{bmatrix}, \quad m,p \neq n. \tag{13}$$

Calculation of Double Integral (Equation 6c)

The identity:

$$\iint_{S^e} (\mathbf{r} - \mathbf{r}^e) ds^e = 0, \tag{14}$$

where $\mathbf{r}^e$ stands for the position vector of the centroid of an element, results in:

$$U^e = \frac{r_i + r_j + r_k}{3} S^e. \tag{15}$$

Calculation of Line Integrals (Equations 6d and 6e)

Explicit expansions of Equations 6d and 6e result in:

$$W_k^b = \frac{2r_k + r_l}{6} L^b, \quad l \neq k, \tag{16a}$$

$$V_{kk}^b = V_{ll}^b = \frac{3r_k + r_l}{12} L^b,$$

$$V_{kl}^b = \frac{r_k + r_l}{12} L^b, \quad l \neq k. \tag{16b}$$

Here, (l, k) represents the indices of two end nodes of the boundary edge b.

NUMERICAL EXAMPLE

To check the performance of the above integration formula, the following axisymmetric test problem is considered:

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = f(r,z), \qquad (17a)$$

where f is the source function, considered here as $10^4 z(1-r^2)$. Equation 17a is subject to the following boundary conditions:

$$V(r,0)=0,\ V(r,1)=1,\ \left.\frac{\partial V}{\partial r}\right|_{r=0,z=2-r}=0. \qquad (17b)$$

As shown in Figure 1, around the equivalent 2D solution region, the original 3D solution region is a conical volume, obtained by revolving the equivalent shaded solution region about the z-axis . The contour plot of the solution is plotted in Figure 2.

In Figure 3, the superior performance of the above proposed scheme (solid line) can be observed, compared with the similar 3D code (dashed line) and 2D axisymmetric code via numerical integration, by a simple interpolation method (dot dashed line). The horizontal axis is the number of divisions n (therefore, the total number of nodes would be about n^2 and n^3 for 2D and 3D schemes, respectively). The 2D codes need less computation time for the number of divisions above 100. However, for less number of divisions, the 3D code is more effective compared with the 2D version with numerical integration (please note that in 3D no numerical integration is needed). As a final notice, this approach has been successfully used to simulate the time domain gas discharges in axisymmetric Nitrogen two-electrode and triggerable spark gaps [10], where the Poisson equation in the equivalent axisymmetric 2D area has been solved numerically by FEM.

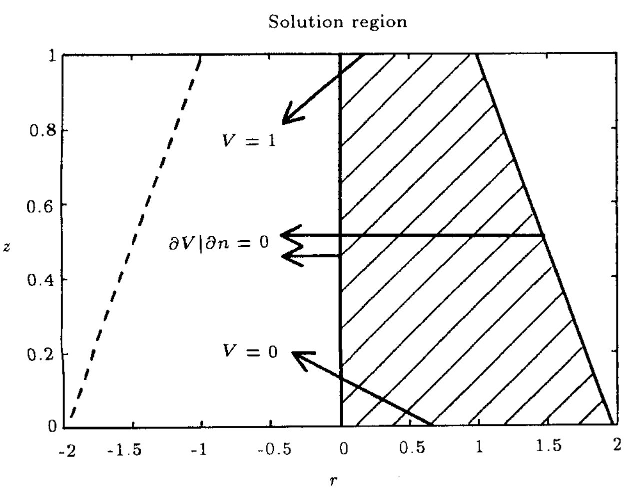

Figure 1. Equivalent 2D solution region (hatched area) of the conical axisymmetric volume and the boundary conditions.

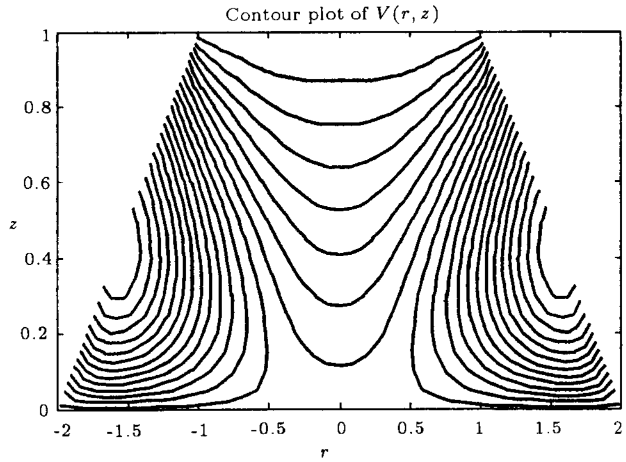

Figure 2. Contour plot of the potential function; the solution is mirrored in the $r < 0$ region.

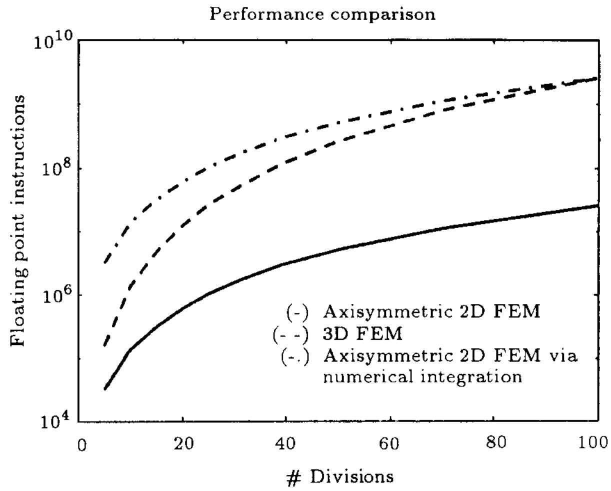

Figure 3. Performance comparison between FEM codes.

CONCLUSIONS

The integrals appearing in axisymmetric variational formulation of the finite element method for classical potential problems by first-order elements are introduced. Simple analytical methods are presented for evaluation of these integrals. The evaluation is done through solution of algebraic equations, which requires few floating point operations, instead of the time-consuming numerical integration methods. This permits solving such 3D axisymmetric problems in 2D with much better efficiency and accuracy.

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