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The Effect of Code Distribution and Parameters on the LPD Feature of Phase-Code Radar Signals

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In this paper, a phase-code radar signal is considered in its most general form, in which the used code has an arbitrary distribution. The suboptimal Electronic Support Measure (ESM) detector, for which the code is not known, is found for this kind of signal and the performance of the detector is evaluated. The deflection measure of this detector shows that the Low Probability of Detection (LPD) feature of the signal is effectively related to a defined quantity called certainty parameter. This new parameter depends on the first and the second order statistic of the code. Using this parameter, it is shown that a zero mean code is the best code to make LPD waveform. Then, the zero mean codes are considered and the code distribution effect is investigated. It is proven that distributions which have smaller Kurtosis show better LPD features in practical cases. It is, therefore, concluded that zero mean binary code is the best code.

INTRODUCTION

Assume that there is a radar site and a target which carries an ESM receiver. This receiver tries to detect the presence of any radar signal in the environment and find the location of its source. Therefore, a radar must be designed in such a way that it brings down the detection probability of an ESM receiver as much as possible. In other words, a radar should have the LPD property. A proper measure for evaluation of the LPD feature is the ratio of ESM range (R_E) to radar range (R_R). The smaller the value of R_E/R_R, the better the LPD feature of a radar signal [1].

Using the system parameters, the following equation is obtained [1]:

$$\frac{R_E}{R_R} = R_R \left[\frac{4\pi}{\delta'} \cdot \frac{1}{\sigma} \cdot \frac{G_{TE} G_{ET}}{G_T G_R} \cdot \frac{L_E}{L_R} \right]^{1/2}, \qquad (1)$$

where:

δ' = ratio of the power required at the ESM receiver (S_E) to detect a signal to the power required by the radar receiver (S_R) to detect a signal,

σ = radar cross section of the target,

G_{TE} = gain of transmit antenna in the direction of the ESM receiver,

G_{ET} = gain of ESM antenna in the direction of the transmitter,

G_T = gain of radar transmit antenna towards the target,

G_R = gain of radar receiver antenna towards the target,

L_R, L_E = losses in the radar and ESM receivers.

Among the above parameters, δ' is the one which is related to signal design and so this is what has been concentrated upon in this paper.

It is known that for a system having F as the noise figure, and B as the noise bandwidth, the relation between signal power (S) and signal to noise ratio (SNR) is given by:

$$S = (SNR)(KTFB), \qquad (2)$$

where K is the Bultzman constant and T is the temperature (usually assumed 290°K). By a logical assumption that $T_R = T_E$ and $F_R = F_E$, it will be obtained that:

$$\delta' = \frac{SNR_E}{SNR_R} \cdot \frac{B_E}{B_R}. \qquad (3)$$

To decrease R_E/R_R, δ' should be increased and, for this purpose, either B_E/B_R and/or $\delta = \frac{SNR_E}{SNR_R}$ could

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be increased. In this paper, δ is considered a proper measure for the LPD feature of signals.

One of the familiar methods for increasing the value of δ, is the phase-coded pulse compression method [2,3]. In this method, a long pulse of duration T is divided into N subpulses, each of width τ. The phase of each subpulse is chosen to be either 0 or π radians. In other words, the pulse is multiplied by a code string which takes ± 1 values.

Recently, it has been suggested that for increasing the LPD degree of the signal against the rate-line detectors, an M-level code (instead of a binary code) could be used [4]. On the other hand, some references discuss the use of a code with analog values, which has a special distribution [5]. Although these kinds of signals are implicitly discussed in some papers [6], this work has two main differences. The first is that the used code is considered in a more general form, in which it can have any arbitrary distribution, and the second is that the suboptimal detectors are obtained for the signals. The latter yields more reliable results, as if a non-optimal ESM detector is considered (like rate-line detector [6]) and a judgment is made about the LPD degree of a signal, it is possible to discover another ESM detector which has a better performance level than that of the considered one. However, if the optimal (or at least suboptimal) ESM detector for a signal is obtained and δ is computed using its performance, one can be certain that this δ will be the lower bound of δ and, so, more reliable for the designers to judge [7,8].

Based on the above explanation, the radar and ESM receivers are studied using the detection-theoretic approach. In the next section, therefore, the detector for an authorized receiver (radar receiver) is found and it is proven that its performance depends only on the SNR and is not related to the code parameters and distribution. Then, in the third section, the suboptimal ESM detector is found and it is shown that its performance is close to the optimal detector. In the next two main sections, the effect of code parameters and distribution is shown on the LPD feature. In the last section a summary along with some conclusions and remarks is presented.

TARGET DETECTION BY RADAR RECEIVER

Detection theory is used for signal detection against interference. The input of the detector consists of interference and it may also include a signal from a target.

The well-known Neyman-Pearson test is usually applied in the optimum detection procedure. If $\underline{y}, \underline{n}$ and $\underline{s}$ are the received signal, interference signal, and desired (target) signal vectors, respectively, the problem of detection can be stated in the following hypothesis:

$$\begin{cases} H_0 : \underline{y} = \underline{n} \\ H_1 : \underline{y} = \underline{s} + \underline{n} \end{cases}, \tag{4}$$

where H_1 and H_0 represent the hypothesis of presence or absence of the target signal, respectively. Using Neyman-Pearson test, it is obtained that [9]:

$$L(\underline{y}) = \frac{f_y(\underline{y}|H_1)}{f_y(\underline{y}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta, \tag{5}$$

where $L(\underline{y})$ is the likelihood ratio (LR) function, $f_y(\underline{y}|.)$ is the conditional probability density function (pdf) of $\underline{y}$ and η is a threshold which depends on the desired false alarm probability (P_{fa}).

For optimum detection of a signal vector which contains random parameters, the conditional LR should be averaged over these parameters. This detector is called ALR (Average Likelihood Ratio) detector [10].

In this problem, it is assumed that the radar sends one pulse to the target and receives its reflection. Since each pulse consists of N chips, the samples of the desired signal have the following form:

$$s_k = v_k e^{j\Phi_k} c_k, \quad k = 0, 1, \cdots, N-1, \tag{6}$$

where v_k and ϕ_k are the amplitude and phase of the kth sample, respectively, and c_k is the real code value in this chip. So, it can be defined as:

$$\underline{s} = [s_0\ s_1\ \cdots\ s_{N-1}]^T, \tag{7}$$

$$\underline{n} = [n_0\ n_1\ \cdots\ n_{N-1}]^T, \tag{8}$$

$$\underline{c} = [c_0\ c_1\ \cdots\ c_{N-1}]^T, \tag{9}$$

$$\underline{y} = [y_0\ y_1\ \cdots\ y_{N-1}]^T, \tag{10}$$

where $\underline{n}$ is the noise vector and $\underline{y}$ is the received signal vector so that:

$$\underline{y} = \underline{s} + \underline{n}. \tag{11}$$

Meanwhile, it is assumed that noise is a white and complex Gaussian interference, with the following pdf:

$$f_{\underline{n}}(\underline{n}) = \prod_{k=0}^{N-1} f_{n_k}(n_k), \tag{12}$$

$$f_{n_k}(n_k) = \frac{1}{\pi\sigma^2} e^{-|n_k|^2/\sigma^2}, \tag{13}$$

Notice that this assumption is not realistic, because the main interference in a radar problem is clutter, which cannot usually be assumed either white and/or

Gaussian. However, it is shown that the code distribution and parameters have no effect on the performance of the radar receiver, as it knows the used code. Therefore, AWGN has been used as an example of interference and the final result can be generalized for any other interferences.

So, if the signal vector, $\underline{s}$, is completely known, the likelihood ratio will be [11]:

$$L(\underline{y}|\underline{s}) = \exp\left\{ -\frac{1}{\sigma^2}\underline{s}^H.\underline{s} + \frac{2}{\sigma^2}Re(\underline{y}^H.\underline{s}) \right\}. \quad (14)$$

A practical case is considered where ϕ_k in Equation 6 is the same for all chips (i.e., chip coherent) and is modeled as a uniformly distributed random variable in $[0, 2\pi)$. Also, because of the slow variations of the target in a pulse duration width, it can be assumed that:

$$v_k = v, \quad k = 0, 1, \cdots, N-1, \quad (15)$$

where v is usually assumed to have Rayleigh distribution [12] with parameter a (i.e., $f_v(v) = \frac{v}{a}e^{-v^2/2a}$). By these assumptions, it is obtained that:

$$\underline{s}^H.\underline{s} = v^2 \sum_{k=0}^{N-1} c_k^2, \quad (16)$$

$$Re(\underline{y}^H.\underline{s}) = vRe(e^{j\phi}\underline{y}^H.\underline{c}). \quad (17)$$

By definition of $A = \underline{y}^H.\underline{c}$, Equation 17 becomes:

$$Re(\underline{y}^H.\underline{s}) = v|A|.\cos(\phi + \measuredangle A) \quad (18)$$

and therefore:

$$L(\underline{y}|\underline{s}) = L(\underline{y}|v, \phi) = \exp\left\{ -\frac{v^2u}{\sigma^2} + \frac{2v}{\sigma^2}|A|\cos(\phi + \measuredangle A) \right\}, \quad (19)$$

where $u = \sum_{k=0}^{N-1} c_k^2$. Now, there should be an average of Equation 19 over v and φ. Averaging over φ will result in:

$$L(\underline{y}|v) = \frac{1}{2\pi}\int_0^{2\pi} L(\underline{y}|v, \phi)d\phi = e^{-v^2u/\sigma^2} I_0(\frac{2v}{\sigma^2}|A|). \quad (20)$$

Also, by averaging Equation 20 over v, it will be seen that:

$$L(\underline{y}) = \int_0^\infty e^{-v^2u/\sigma^2} I_0(\frac{2v}{\sigma^2}|A|).\frac{v}{a}e^{-v^2/2a}dv. \quad (21)$$

This integral is a special kind of the general form of the Watson integral [12]; Consequently,

$$L(\underline{y}) = \frac{1}{1 + \frac{2au}{\sigma^2}} e^{2a|A|^2/\sigma^2(1+2au)}. \quad (22)$$

Since exp(.) and square functions are strictly increasing functions, the following test statistic can be used:

$$|A| \underset{H_0}{\overset{H_1}{\gtrless}} \eta, \quad (23)$$

or:

$$L_1 = |\underline{y}^H.\underline{c}| = \left| \sum_{k=0}^{N-1} y_k^* c_k \right| \underset{H_0}{\overset{H_1}{\gtrless}} \eta. \quad (24)$$

The above detector is the well-known matched filter, which is used in practical receivers [12].

For evaluating the performance of the derived detector, an effort will be made to find the relation between P_{fa} (Probability of false alarm) and P_d (Probability of detection). Note that in the H_0 case, there is:

$$y_k = n_k = x_k + jz_k. \quad (25)$$

Considering the distribution of n_k, similar normal distribution for the following quantities can be proven:

$$\alpha \triangleq Re\left[\sum_k n_k^* c_k\right] = \sum_k c_k x_k, \quad (26)$$

$$\beta \triangleq Im\left[\sum_k n_k^* c_k\right] = -\sum_k c_k z_k. \quad (27)$$

Since, under the H_0 hypothesis, L_1 is equal to $\sqrt{\alpha^2 + \beta^2}$, L_1 has Rayleigh distribution [13] with parameter $u\sigma^2/2$ and:

$$P_{fa} = \int_\eta^\infty f_{L_1}(L_1|H_0)dL_1 = e^{-\eta^2/u\sigma^2}. \quad (28)$$

On the other hand, under the H_1 hypothesis, it can be seen that:

$$y_k = vc_k e^{j\phi} + n_k = (vc_k \cos\phi + x_k) + j(vc_k \sin\phi + z_k). \quad (29)$$

Consequently, the distribution of L_1, given v and φ, will be:

$$f_{L_1}(L_1|H_1, v, \phi) = f_{L_1}(L_1|H_1, v)$$
$$= \frac{2L_1}{u\sigma^2}e^{-(L_1^2+u^2v^2)/u\sigma^2} I_0\left[\frac{2L_1 v}{\sigma^2}\right], \quad (30)$$

and the following is obtained:

$$P_d = \int_\eta^\infty f_L(L_1|H_1)d_{L_1}$$
$$= \int_\eta^\infty \int_0^\infty f_{L_1}(L_1|H_1, v)f_v(v)dvdL_1$$
$$= e^{-\frac{1}{u(\sigma^2+2au)}\eta^2}. \quad (31)$$

The combination of Equations 28 and 31 yields:

$$P_d = P_{fa}^{\frac{1}{1+2au/\sigma^2}}. \quad (32)$$

By definition of the SNR for the received signal, Equation 32 can be rewritten as:

$$P_d = P_{fa}^{\frac{1}{1+(SNR_R)^{-N}}}, \quad (33)$$

where:

$$SNR_R \triangleq \frac{E(\underline{s}^H.\underline{s})}{E(\underline{n}^H.\underline{n})} = \frac{E(v^2)\sum_k c_k^2}{\sum_k E(|n_k|^2)} = \frac{2au}{N\sigma^2}. \quad (34)$$

Equation 33 shows that the ROC (Receiver Operating Characteristics) of the detector only depends on SNR_R, and if one wishes to compute SNR_R for fixed P_d and P_{fa}, the code distribution and parameters would have no effect on performance. This can also be generalized for any other interferences. As a result, to investigate the effect of code distribution and parameters on δ, it is sufficient to evaluate its effect on SNR_E which is studied in the next section.

RADAR DETECTION BY ESM RECEIVER

ESM receiver, which is assumed to be assembled on the target, receives the pulse which is sent by the radar (not its echo). Therefore, the ESM received signal can be modelled as:

$$\underline{y} = \underline{s} + \underline{n}, \quad (35)$$

where:

$$s_k = ve^{j\phi}c_k, \quad (36)$$

and the definition of $\underline{n}$ is like before. However, c_k's are not known for ESM and this is the main difference between the ESM and radar receivers. In addition, v and φ are not known and should be properly modeled for an ESM receiver. It is reasonable to assume that φ is a uniformly distributed random variable in $[0, 2\pi)$. However, v is assumed to be a random variable with $f_v(v)$ as its pdf. It will be seen that this pdf will not affect the results.

Based on the above assumptions, the following is obtained:

$$\underline{s}^H.\underline{s} = v^2 \sum_k c_k^2, \quad (37)$$

$$Re(\underline{y}^H.\underline{s}) = v \sum_k c_k |y_k| \cos(\phi - \measuredangle y_k), \quad (38)$$

$$L(\underline{y}|\underline{c}, v, \phi) = \exp\left\{ -\frac{v^2}{\sigma^2} \sum_k c_k^2 + \frac{2v}{\sigma} \sum_k c_k |y_k| \cos(\phi - \measuredangle y_k) \right\}. \quad (39)$$

Note that c_k's are assumed to be randomly chosen, based on a pdf like $f_c(c)$. Also, it is defined that:

$$m_1 \triangleq E(c_k), \quad (40)$$

$$\sigma_c^2 \triangleq E\left\{(c_k - m_1)^2\right\}, \quad (41)$$

$$m_2 \triangleq E(c_k^2) = \sigma_c^2 + m_1^2, \quad (42)$$

and it is assumed that m_1 and m_2 are known for the ESM receiver. Before averaging Equation 39 over the unknown parameters, note that for sufficiently large values of $N, \sum_k c_k^2$ is approximately an observation-independent constant, because:

$$E(c_k^2) \simeq \frac{1}{N} \sum_{k=0}^{N-1} c_k^2 \Rightarrow \sum_k c_k^2 \simeq Nm_2. \quad (43)$$

Therefore, Equation 39 can be expressed as:

$$L(\underline{y}|\underline{c}, v, \phi) \simeq e^{-v^2 N m_2/\sigma^2} e^{\frac{2v}{\sigma^2} \sum_k c_k |y_k| \cos(\phi - \measuredangle y_k)}. \quad (44)$$

Firstly, Equation 44 is averaged over ϕ, which follows that:

$$L(\underline{y}|\underline{c}, v) \simeq \frac{1}{2\pi} \int_0^{2\pi} e^{-v^2 N m_2/\sigma^2} e^{\frac{2v}{\sigma^2} \sum_k c_k |y_k| \cos(\phi - <y_k)} d\phi = e^{-v^2 N m_2/\sigma^2} I_0 \left[\frac{2v}{\sigma^2} \left| \sum_k y_k^* c_k \right| \right]. \quad (45)$$

Then, the above equation should be averaged over v. However, before doing this, it should be noted that the argument of the Bessel function in Equation 45 will be very small for low SNR's. Therefore, the following approximation can be used:

$$I_0(x) \simeq 1 + \frac{x^2}{4} \text{ for } x << 1, \quad (46)$$

and then:

$$L(\underline{y}|\underline{c}, v) \simeq e^{-v^2 N m_2/\sigma^2} \left[1 + \frac{v^2}{\sigma^4} \left| \sum_k y_k^* c_k \right|^2 \right]. \quad (47)$$

If Equation 47 is averaged over v, regardless of v distribution, it will be obtained that:

$$L(\underline{y}|\underline{c}) \simeq A_1 + A_2 \left| \sum_k y_k^* c_k \right|^2, \quad (48)$$

where A_1 and A_2 are constant values and do not depend on $\underline{y}$ or $\underline{c}$. Therefore:

$$L(\underline{y}) \simeq A_1 + A_2 E_{\underline{c}} \left\{ \left| \sum_k y_k^* c_k \right|^2 \right\}. \quad (49)$$

As a result, it is sufficient to simplify the term $E_{\underline{c}}\left\{|\sum_k y_k^* c_k|^2\right\}$, which clearly follows that:

$$E_{\underline{c}}\left\{\left|\sum_k y_k^* c_k\right|^2\right\} =$$

$$E_{\underline{c}}\left\{Re^2\left[\sum_k y_k^* c_k\right] + Im^2\left[\sum_k y_k^* c_k\right]\right\}. \quad (50)$$

The first term is equal to:

$$E_{\underline{c}}\left\{Re^2\left[\sum_k y_k^* c_k\right]\right\} = E_{\underline{c}}\left\{\sum_k c_k^2 Re^2(y_k) + 2\sum_k \sum_{\substack{l \\ l<k}} c_k c_l Re(y_k) Re(y_l)\right\}$$

$$= m_2 \sum_k Re^2(y_k) + 2m_1^2 \sum_k \sum_{\substack{l \\ l<k}} Re(y_k) Re(y_l). \quad (51)$$

In a similar manner, the second term of Equation 50 can be written as:

$$E_{\underline{c}}\left\{Im^2\left[\sum_k y_k^* c_k\right]\right\} = m_2 \sum_k Im^2(y_k^*) + 2m_1^2 \sum_k \sum_{\substack{l \\ l<k}} Im(y_k^*) Im(y_l^*). \quad (52)$$

Notice that in the derivation of Equations 51 and 52, the independence of c_k's has been used.

Using Equations 49 to 52 and removing the constant values, the following test statistic has been derived [7]:

$$L_2 \triangleq m_2 \sum_k |y_k|^2 + 2m_1^2 \sum_k \sum_{\substack{l \\ l<k}} Re(y_k^* y_l). \quad (53)$$

Therefore, if the ESM receiver does not have any knowledge about the signal, except the mean and variance of the used code, the suboptimum detector will be:

$$L_2 \underset{H_0}{\overset{H_1}{\gtrless}} \eta. \quad (54)$$

Note that the performance of the above detector will be close to the optimal one, provided that the following reasonable conditions are established:

- N is sufficiently large for the validity of Condition 43
- SNR_E is sufficiently small so that the employed approximation in Equation 47 is assured.

THE EFFECT OF CODE PARAMETERS

Because of the complexity of the detector (Equation 53), analytic calculation of its ROC is very complicated. However, the deflection measure can be found, which is a proper measure in most cases. This approach assists in investigating the effect of code parameters on the SNR_E and, therefore, on the LPD feature.

The deflection of a detector with a given test statistic, say L_2, is defined as [11]:

$$d \triangleq \frac{E(L_2|H_1) - E(L_2|H_0)}{var(L_2|H_0)^{1/2}}. \quad (55)$$

The greater the value of deflection, the better will be the performance of the detector. Therefore, as far as possible, in this problem, the radar signal designer would like to decrease the value of d for an unauthorized receiver.

Under the H_0 hypothesis, it follows that:

$$y_k = n_k, \quad (56)$$

and n_k's are independent complex Gaussian random variables which have the following properties [13]:

$$\begin{cases} E(n_k) = 0 \\ E(|n_k|^2) = \sigma^2 \\ E(|n_k|^4) = 2\sigma^4. \end{cases} \quad (57)$$

Using these properties, it is obtained that:

$$E(L_2|H_0) = E\left\{m_2 \sum_k |n_k|^2 + 2m_1^2 \sum_k \sum_{\substack{l \\ l<k}} Re(n_k^* n_l)\right\}$$

$$= m_2 \sum_k E(|n_k|^2) + 2m_1^2 \sum_k \sum_{\substack{l \\ l<k}} Re\left\{E(n_k^*)E(n_l)\right\}$$

$$= m_2 N \sigma^2, \quad (58)$$

and also:

$$Var(L_2|H_0)$$

$$= E\left\{\left[m_2 \sum_k |n_k|^2 + 2m_1^2 \sum_k \sum_{\substack{l \\ l<k}} Re(n_k^* n_l)\right]^2\right\}$$

$$- m_2^2 N^2 \sigma^4$$

$$
=m_2^2 E\left\{\left[\sum_k |n_k|^2\right]^2\right\} + 4m_1^4 E\left\{\left[\sum_k \sum_{\substack{l \\ l<k}} Re(n_k^* n_l)\right]^2\right\}
$$

$$
+4m_1^2 m_2 E\left\{\left[\sum_i |n_i|^2\right]\left[\sum_k \sum_{\substack{l \\ l<k}} Re(n_k^* n_l)\right]\right\}
$$

$$
- m_2^2 N^2 \sigma^4. \tag{59}
$$

It can easily be shown that [7]:

$$
var(L_2|H_0) = m_2^2 N\sigma^4 + m_1^4 N(N-1)\sigma^4. \tag{60}
$$

Similarly, $E(L_2|H_1)$ can be computed. Note that the expected value should be applied over v, n_k's and also c_k's. Doing this, it will follow that:

$$
E(L_2|H_1) = m_2^2 N E(v^2) + m_2 N\sigma^2 + m_1^4 N(N-1)E(v^2). \tag{61}
$$

Substituting Equations 58, 60 and 61 in Equation 55 yields:

$$
d = \frac{m_1^4 N(N-1)E(v^2) + m_2^2 N E(v^2)}{\sqrt{m_2^2 N\sigma^4 + m_1^4 N(N-1)\sigma^4}}. \tag{62}
$$

By some manipulations, the above equation can be expressed as [7]:

$$
d = \lambda\sqrt{N}.\ \sqrt{1 + h^2(N-1)}, \tag{63}
$$

where:

$$
\lambda = SNR_E = \frac{E(\underline{s}^H.\underline{s})}{E(\underline{n}^H.\underline{n})}
$$

$$
= \frac{E(v^2)\sum_k E(c_k^2)}{\sum_k E(|n_k|^2)} = \frac{E(v^2)Nm_2}{N\sigma^2} = \frac{E(v^2)m_2}{\sigma^2}, \tag{64}
$$

$$
h = \frac{m_1^2}{m_2}. \tag{65}
$$

From Equation 63, it is concluded that for constant values of N and λ, increasing the value of h causes improvement in the performance of the ESM detector and consequently decreases δ.

Since $\sigma_c^2 = m_2 - m_1^2$, increasing the value of h is equivalent to decreasing σ_c^2, which is a measure of uncertainty for the enemy (ESM detector). Therefore, h is known as the certainty parameter. The greater the value of h, the more certainty there will be for the enemy and the smaller will be the δ.

Although the exact ROC has not been found and the results are based on the deflection measure, the simulation-derived ROC's confirm those results.

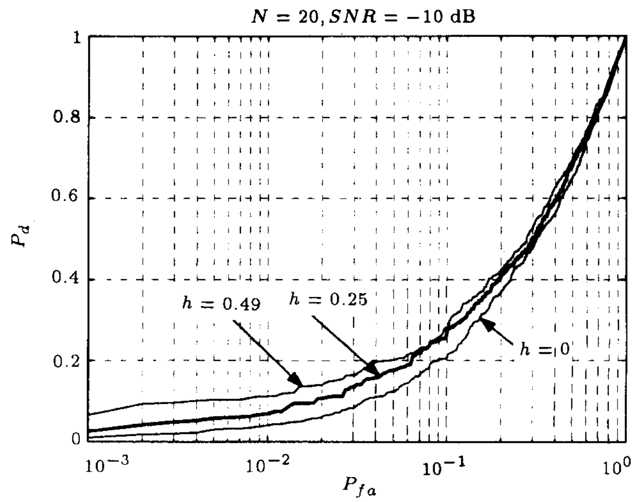

Figure 1. The ROC of the detector for binary code and some h values.

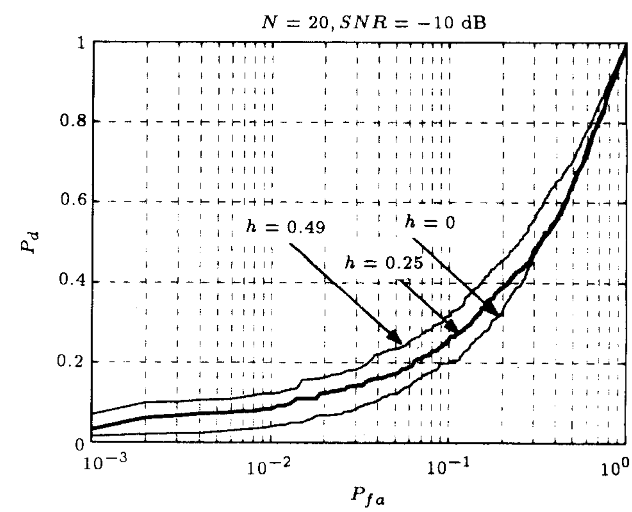

Figure 2. The ROC of the detector for Gaussian code and some h values.

Figure 1 illustrates the ROC for binary code in a typical case $N = 20$ and $\lambda = -10$ dB for some h values. Also, Figure 2 shows similar curves for a Gaussian code. Both figures confirm that the most effective parameter is the certainty parameter and the probability of detection is increased as this parameter is increased. However, it is clear that $0 \leq h \leq 1$ and, then, for a zero mean code ($h = 0$), there is the smallest P_d. In other words, if a code is used whose mean is zero, the other code parameters have no notable effect. Therefore, the most important conclusion derived from this section is that, as far as possible, the designer of LPD radars must use a zero mean code to increase the uncertainty for the ESM detectors.

THE EFFECT OF CODE DISTRIBUTION

As seen in the previous section, there is no difference between code distributions, provided that the h parameter is the same. However, this result is based on the deflection measure and not on the exact ROC. Even the simulation results, which have been presented in the previous section, only compare the ROC of signals with similar code distributions (in each figure, the code distribution is similar for the three curves). Thus, the reader could be expected to ask about the effects of code distribution on the LPD measure. In other words, is there any difference between binary code, Gaussian code and etc. from the point of view LPD feature?

To answer this important question, the exact ROC's of the different zero mean codes are compared. Clearly, since it has been concluded that the zero mean is the best case for each code distribution, it is sufficient to compare these zero mean codes.

Considering Equation 53, it can easily be found that for $m_1 = 0$, the test statistic will become:

$$L_3 = \sum_{k=0}^{N-1} |y_k|^2. \tag{66}$$

It is, therefore, decided to derive the analytic ROC of the above detector and its relation to the code distribution.

By assumption that N is sufficiently large, the pdf's of L_3 under H_0 and H_1 hypothesis can be properly approximated by Normal distribution. So, it is sufficient to compute their means and variances. Also, for simplicity, note that the $P_{d|v}$ can be found. In other words, it is assumed that v is known and that the probability of detection is found, given by v. Since the final results are valid for each value of v, it can be concluded that the results will be valid for the unconditional P_d.

Using Equations 58, 60, and 61, setting $m_1 = 0$ and removing some constants, it can be proven that [7]:

$$E(L_3|H_0) = N\sigma^2, \tag{67}$$

$$Var(L_3|H_0) = N\sigma^4, \tag{68}$$

$$E(L_3|H_1, v) = v^2 m_2 N + N\sigma^2. \tag{69}$$

The only additional quantity which should be computed is the $Var(L_3|H_1, v, \varphi)$:

$$\begin{aligned} Var(L_3|H_1, v, \phi) &= Var(L_3|H_1, v) \\ &= \sum_k Var[|y_k|^2|H_1] \\ &= \sum_k \left[E\left\{ |vc_k e^{j\phi} + n_k|^4 \right\} \right. \\ &\quad \left. - E^2\left\{ |vc_k e^{j\phi} + n_k|^2 \right\} \right] \\ &= \sum_k \left[\left(v_4 m_4 + 2\sigma^4 + 4v^2 m_2 \sigma^2\right) \right. \\ &\quad \left. - \left(v^4 m_2^2 + \sigma^4 + 2v^2 m_2 \sigma^2\right) \right] \\ &= N\left(v_4 m_4 - v^4 m_2^2 + \sigma^4 + 2v^2 m_2 \sigma^2\right), \end{aligned} \tag{70}$$

where $m_4 \triangleq E(c_k^4)$. Therefore, Equations 67 to 70 could identify the distributions of L_3 under H_0 and H_1. Thus, P_{fa} and P_d can be computed and the following is obtained [7]:

$$P_{d|v} = Q\left[\frac{Q^{-1}(P_{fa}) - \lambda\sqrt{N}}{\lambda^2 K + 2\lambda^2 + 2\lambda + 1}\right], \tag{71}$$

where:

$$Q(x) \triangleq \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx, \tag{72}$$

and K is the Kurtosis of the code distribution, i.e., $K \triangleq \frac{m_4}{m_2^2} - 3$. Therefore, the ROC depends on a parameter which is related to the code distribution. For $Q^{-1}(P_{fa}) - \lambda\sqrt{N} > 0$, the larger value of K causes the larger value for $P_{d|v}$. Note that for the typical values of P_{fa}, λ and N, the mentioned condition is established. Figure 3 illustrates $P_{d|v}$ versus K for a typical case where $N = 20$, $\lambda = -3$ dB and $P_{fa} = 0.01$ $\left(Q^{-1}(P_{fa}) - \lambda\sqrt{N} \simeq 0.09\right)$. Since this result can be obtained for each value of v, it will be valid for the unconditional P_d and, therefore, will be entirely correct.

It is worth noting that the K parameter has little effect on the P_d values, because K is multiplied by λ^2, which has, typically, a very small value. In other words, the effect of code distribution is very small. For example, Figure 4 shows the ROC of the ESM detector in the case where $N = 5$, $\lambda = -3$ dB

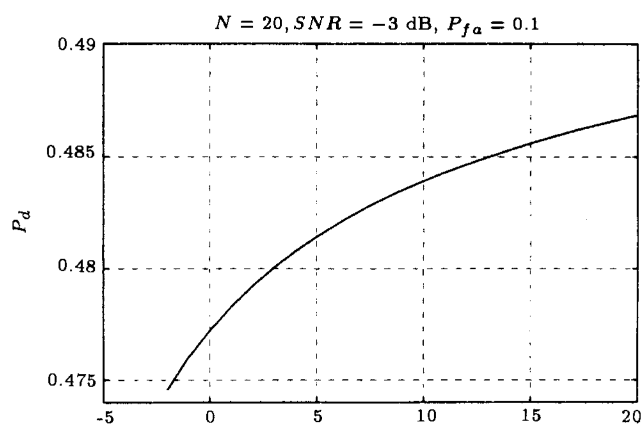

Figure 3. $P_{d|v}$ versus K for a typical case.

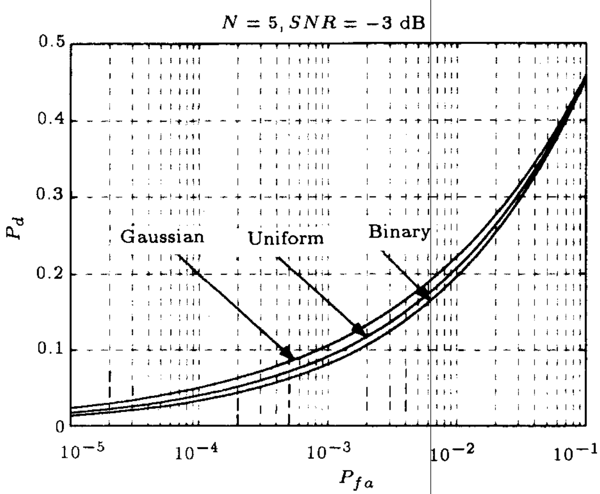

Figure 4. The comparison of three code distributions Binary, Gaussian and Uniform.

and $P_{fa} < 0.1 \left(Q^{-1}(P_{fa}) - \lambda\sqrt{N} > 0\right)$ for three code distributions: Binary, Gaussian and Uniform. Since:

$$K_{BIN} = -2,\ K_{UNI} = -1.2,\ K_{GUS} = 0, \tag{73}$$

the observed differences in Figure 4 are expected.

As a matter of fact, based on Schwartz inequality, it can be seen that:

$$m_4 \geq m_2^2 \Rightarrow K \geq -2 \tag{74}$$

Therefore, the binary code, whose K is equal to -2, has the best LPD feature amongst all the code distributions. This result shows that whatever has been previously suggested about using an M-level code (or a code with any other distribution) is wrong and that the binary code is the best code from the LPD point of view.

CONCLUSIONS

The phase-code modulated radar signals have been considered in its most general form, in which the used code has an arbitrary distribution. For this signal, it was proven that the performance of an authorized (radar) detector does not depend on the code distribution and parameters, but only on the SNR_R and N.

Then, the suboptimal ESM detector was obtained and it was shown that the first and the second order statistics of the code distribution have the main role in the detector structure and performance. Based on the deflection of the derived detector, a certainty parameter ($h = \frac{m_1^2}{m_2}$) was defined and it was shown that the best code, leading to the lowest performance for the enemy, is a code which has zero mean.

Subsequently, the effect of code distribution on the LPD feature was investigated. The ROC of the ESM detector was derived for any arbitrary zero mean code distribution. It was shown that the LPD feature is inversely proportional to the Kurtosis of code distribution. Although this parameter has little effect, the binary code is slightly better than other distributions. Therefore, the zero mean binary code is the best code in practical cases. In other words, it is recommended that the designers use binary code, also considering that the probability of occurrence +1 and -1 is equal, as a result of which the mean of the code distribution is zero.

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