Scientia Iranica, Vol. 10, No. 1, pp 116–121

SCIENTIA
IRANICA

Research Note

Heat Transfer in an Axisymmetric Stagnation Flow at High Reynolds Numbers on a Cylinder Using Perturbation Techniques

A.B. Rahimi[1]

Although there are many papers on the subject of heat transfer in an axisymmetric stagnation flow on a cylinder, the available knowledge is mainly for low Reynolds numbers and not much information exists for the same problem at large Reynolds numbers. In this work, the problem of heat transfer in an axisymmetric stagnation flow on a cylinder is solved at large Reynolds numbers using perturbation techniques. Starting from Navier-Stokes equations within a boundary layer approximation and using similarity transformations, the governing equations are obtained in the form of differential equations. The inverse of the Reynolds number is introduced as the perturbation parameter. This parameter appears in front of the highest-order terms and, as it tends to zero, reduces the order of the governing equations and produces singularities. In this paper, the flow field is divided into two regions; rapid changes in the region near wall and slow changes away from the wall. Thus, the flow is found to have dual-layer structure characteristics. Using inner and outer expansion produces uniform values of the relevant quantities.

INTRODUCTION

Although current knowledge regarding heat transfer problems in an axisymmetric stagnation flow on a cylinder is admittedly mature, it should be conceded that not much information is available when the Reynolds numbers are large. For example, in 1976, Gorla [1] numerically solved problems related to heat transfer in an axisymmetric stagnation flow on a cylinder and recently, Davey [2] presented an exact solution for the axisymmetric stagnation flow on an infinite circular cylinder.

Although these studies have thoroughly investigated important flow and heat transfer phenomena, they were limited to low Reynolds numbers ($\mathrm{Re} < 1$) and, thus, provide very little information about the systematic behavior of such flows at high Reynolds numbers.

Engineering applications of the problem of axisymmetric stagnation flow on a cylinder with high Reynolds numbers are found in many industrial cooling processes. Although the above-mentioned numerical work can, in principle, be extended to high Reynolds number flow, the results are not meaningful, because, at large Reynolds numbers, the governing differential equations become singular, and reasonable answers cannot be expected. This is due to multiplication of the highest-order derivative term in governing equations by the small perturbation parameter and its singular reduction to a lower degree equation as the perturbation parameter tends to zero.

In recent analysis, Navier-Stokes equations are used in Cartesian coordinates for the flow on an infinite circular cylinder combined with a similarity transformation. The governing equations are then obtained in the form of differential equations. By introducing the inverse of the Reynolds number as the perturbation parameter, perturbation techniques are utilized to solve this problem. The quantities with rapid changes are analyzed in the thin region near the wall, the so-called inner region, and are then matched to the result in the outer region, avoiding any singularities (see [3]).

A shooting method is used to solve the governing equations in each region and at the same time, the matching process is implemented. Uniform solutions for the temperature field are obtained for isothermal and uniform heat flux wall conditions for different values of Reynolds numbers.

1. *Department of Mechanical Engineering, Ferdowsi University of Mashad, P.O. Box 91775-1111, Mashad, I.R. Iran.*

FORMULATION OF PROBLEM

A steady, laminar, incompressible flow at an axisymmetric stagnation point on an infinite circular cylinder is considered. A model of the flow with the coordinate system is shown in Figure 1. The flow is axisymmetric about the z-axis and also symmetric to the $z = 0$ plane. The stagnation line is at $z = 0$ and $r = a$. The temperature of the free stream fluid is considered as T_∞.

The equations which have to be solved are the two-dimensional equations for continuity, momentum and conservation of energy. Neglecting variable property effects and adapting the well-known boundary layer approximations, it is found that:

$$r\frac{\partial w}{\partial Z} + \frac{\partial}{\partial r}(ru) = 0,$$

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial Z} = -\frac{1}{\rho}\frac{\partial \rho}{\partial r} + v\Big[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\Big],$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial Z} = -\frac{1}{\rho}\frac{\partial p}{\partial Z} + v\Big[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\Big],$$

$$u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial Z} = \alpha\Big[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r})\Big], \quad (1)$$

$$r = a : u = w = 0,$$

$$r \to \infty : u = -A(r - \frac{a^2}{r}),$$

$$w = 2AZ. \quad (2)$$

For the temperature field there is:

$$r = a :$$

$$i) T = T_w \quad \text{for constant wall temperature,}$$

$$ii) \frac{\partial T}{\partial y} = -\frac{q_w}{k} \quad \text{for constant wall heat flux,}$$

$$r \to \infty : T \to T_\infty. \quad (3)$$

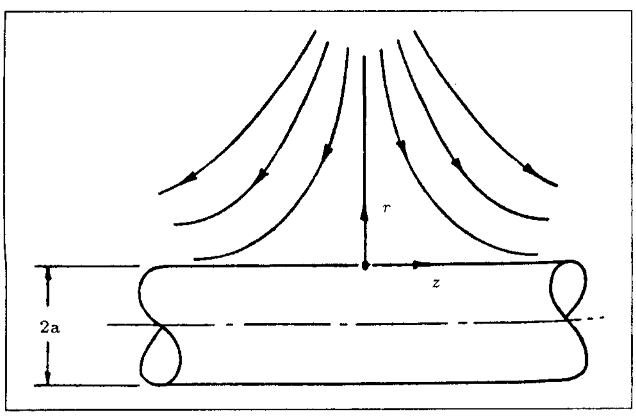

Figure 1. Coordinate system and flow development along the stagnation line.

Using the coordinate transformations as:

$$\eta = (\frac{r}{a})^2,$$

$$u = -Aa\eta^{-\frac{1}{2}}.f(\eta),$$

$$w = 2A\ f'(\eta).Z, \quad (4)$$

satisfying the continuity equation and substituting into momentum and energy equations, one would get:

$$\eta\ f''' + f'' + \text{Re}.[1 + ff'' - (f')]^2 = 0, \quad (5)$$

$$P = P_0 - \rho\Big[\frac{A^2a^2}{2}\frac{f^2}{\eta} + 2\nu Af' + 2A^2Z^2\Big], \quad (6)$$

$$\eta\theta'' + [1 + (\text{Re.Pr})f]\theta' = 0, \quad (7)$$

where the nondimensional form:

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \quad \text{for constant wall temperature,}$$

$$\theta = \frac{T - T_\infty}{(q_w a/2k)} \quad \text{for constant wall heat flux,} \quad (8)$$

has been used. The boundary conditions are given by:

$$f(1) = f'(1) = 0 \ \text{ and } \ f'(\infty) = 1,$$

$$\theta(1) = 1, \theta(\infty) = 0 \quad \text{for constant wall temperature,}$$

$$\theta'(1) = -1, \theta(\infty) = 0 \quad \text{for constant wall heat flux.} \quad (9)$$

PERTURBATION EQUATIONS

Since the Reynolds number is high, its inverse, i.e., 1/Re, can be used as the perturbation parameter, ε. The governing equations then become:

$$\varepsilon(\eta f''' + f'') + [1 + ff'' - f'^2] = 0,$$

$$\varepsilon\eta\theta'' + (\varepsilon + \text{Pr}f)\theta' = 0. \quad (10)$$

In these equations, ε appears in front of the highest order terms, thus, the problem can be categorized as a "singular perturbation case". Therefore, as, $\varepsilon \to 0$, the field must be divided into outer and inner regions, respectively. Obviously rapid changes take place at the inner region, i.e., near the wall and its results should be matched, asymptotically, to the results obtained in the outer region, in order to get uniformly valid solutions throughout the field of study. This solution is valid both very close to the wall and away from it. This procedure explains the existance of any unexpected results one may encounter at the vicinity of the wall using numerical methods. Next, the governing equations in each of these two regions are found.

Inner Region

Since this is a very thin region near the wall, it is stretched in order to make the quantities of the order of one. The stretching variables are:

$$\xi = \frac{1-\eta}{\varepsilon^\alpha}, \quad F(\xi) = \frac{f(\eta)}{\varepsilon^\beta}, \quad \Phi(\xi) = \theta(\eta). \tag{11}$$

By substituting these variables into Equations 10 and comparing each term with other terms in each equation and taking the limits, (see [3]), one gets $\alpha = 1/2$ and $\beta = 3/2$. The governing equations in this region then become:

$$F''' - FF'' + F'^2 - 1 - \varepsilon^{\frac{1}{2}}(\xi F''' + F'') = 0,$$

$$\Phi'' - \mathrm{Pr}F\Phi' - \varepsilon^{\frac{1}{2}}(\xi\Phi'' + \Phi') = 0, \tag{12}$$

with the boundary conditions as:

$$F(0) = F'(0) = 0,$$

$$\Phi(0) = 1 \quad \text{for constant wall temperature},$$

$$\Phi'(0) = -1 \quad \text{for constant wall heat flux}. \tag{13}$$

The following perturbation expansions are assumed for the quantities inside the inner region:

$$F(\xi,\varepsilon) = F_0(\xi) + \varepsilon^{\frac{1}{2}}F_1(\xi) + \varepsilon F_2(\xi) + \cdots,$$

$$\Phi(\xi,\varepsilon) = \Phi_0(\xi) + \varepsilon^{\frac{1}{2}}\Phi_1(\xi) + \varepsilon\Phi_2(\xi) + \cdots \tag{14}$$

Substituting these expansions into the governing equations, collecting the powers of ε and setting their coefficients equal to zero, it is found that:

ε° :

$$F_0''' - F_0F_0'' + F_0'^2 = 1,$$

$$\Phi_0'' - \mathrm{Pr}F_0\Phi_0' = 0, \tag{15}$$

$\varepsilon^{\frac{1}{2}}$:

$$F_1''' - F_0F_1'' - F_0''F_1 + 2F_0'F_1 = \xi F_0''' + F_0'',$$

$$\Phi_1'' - \Phi_0F_1'' - \Phi_0''\Phi_1 + 2\Phi_0'\Phi_1 = \xi\Phi_0'' + F_0'' \tag{16}$$

ε :

$$F_2''' - F_0F_2'' - F_0''F_2 + 2F_0'F_2'$$

$$= F_1F_1'' + F_1'^2 + \xi F_1''' + F_1'',$$

$$\Phi_2'' - \mathrm{Pr}F_0\Phi_2' - \mathrm{Pr}F_2\Phi_0'$$

$$= \mathrm{Pr}F_1\Phi_1' + \xi\Phi_1'' + \Phi_1', \tag{17}$$

and the corresponding boundary conditions are:

$$F_0(0) = F_1(0) = F_2(0) = F_0'(0)$$

$$= F_1'(0) = F_2'(0) = 0,$$

$$\Phi_0(0) = 1, \quad \Phi_1(0) = \Phi_2(0) = 0,$$

for constant wall temperature,

$$\Phi_0'(0) = -1, \quad \Phi_1'(0) = \Phi_2'(0) = 0$$

for constant wall heat flux. (18)

Equations 15 to 17 along with the boundary Conditions 18, govern the thin region next to the wall, which is a correction to the solutions of others. The thickness of this inner region is, generally, the same order as the perturbation parameter chosen. Here, as the Reynolds number gets larger, the corresponding value of η, representing the thickness of the inner region, becomes smaller.

Outer Region

Since the changes in this region are gradual, there is no need for any transformation and Equations 10 govern this region, which is away from the wall. These equations, along with the corresponding boundary conditions, are:

$$\varepsilon(\eta f''' + f'') + [1 + ff'' - f'^2] = 0,$$

$$\varepsilon\eta\theta'' + (\varepsilon + \mathrm{Pr}f)\theta' = 0,$$

$$f'(\infty) = 1,$$

$$\theta(\infty) = 0 \quad \text{for constant wall temperature},$$

$$\theta(\infty) = 0 \quad \text{for constant wall heat flux}. \tag{19}$$

The following perturbation expansions are assumed for the quantities away from the wall:

$$f(\eta,\varepsilon) = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + O(\varepsilon^3),$$

$$\theta(\eta,\varepsilon) = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + O(\varepsilon^3). \tag{20}$$

Substituting these expansions into the above governing equations, collecting the coefficients of the powers of ε and setting them equal to zero, one obtains:

ε^0 :

$$1 + f_0f_0'' - f_0'^2 = 0,$$

$$\theta_0' = 0, \tag{21}$$

ε :

$$\eta f_0''' + f_0'' + f_0 f_1'' + f_1 f_0'' - 2f_0' f_1' = 0,$$

$$\theta_1' = 0, \tag{22}$$

ε^2 :

$$\eta f_1''' + f_1'' + f_0 f_2'' + f_1 f_1'' + f_2 f_0'' - 2f_0' f_2' - f_1'^2 = 0,$$

$$\theta_2' = 0. \tag{23}$$

The boundary conditions are:

$$f_0'(\infty) = 1, \quad f_1'(\infty) = f_2'(\infty) = 0,$$

$$\theta_0(\infty) = \theta_1(\infty) = \theta_2(\infty) = 0. \tag{24}$$

These equations are to be solved and matched to the solution of the inner region governing equations. In this process of matching, the unknown integration constants are evaluated.

PERTURBATION SOLUTION

In this section the solutions of the perturbation equations of Systems 15 to 17 along with boundary Conditions 18 and 24 and Systems 21 to 23 are presented.

From the second equation of Systems 21 to 23, it is readily seen that:

$$\theta_0(\eta) = \theta_1(\eta) = \theta_2(\eta) = 0,$$

and, therefore, in the outer region:

$$\theta(\eta, \varepsilon) = 0. \tag{25}$$

Other equations in the above systems are solved numerically. The results are then matched together in order to achieve a uniformly valid solution throughout the entire inner and outer regions. This is done by using a general shooting method [4].

PRESENTATION OF THE RESULTS

System Equations 15 to 17 are three equations and two unknowns which can be solved numerically. These equations constitute the inner solutions which are related to points very close to the cylinder wall. These values are used as boundary conditions for the outer region. Then, the first equation in Systems 21 to 23 is solved using a shooting method. In this manner, the matching process takes place and a uniformly valid solution throughout the region of interest is obtained. In this way, corrected values of quantities are obtained for large values of Reynolds numbers (see [5]).

Now the heat transfer solution is considered. For a constant wall temperature case, the local wall heat flux can be written using Fourier's law as:

$$q_w(z) = -k\left(\frac{\partial T}{\partial r}\right)_{r=a}$$

$$= -\frac{2k}{a}(T_w - T_\infty)\theta'(1). \tag{26}$$

The local heat transfer coefficient is given by:

$$h(z) = \frac{q_w}{T_w - T_\infty} = -\frac{2k}{a}\theta'(1). \tag{27}$$

The local Nusselt number then becomes:

$$\mathrm{Nu}_z = \frac{h(z).z}{k} = -2\left(\frac{\mathrm{Re}_z}{\mathrm{Re}}\right)^{\frac{1}{2}}.\theta'(1), \tag{28}$$

where:

$$\mathrm{Re}_z = \frac{Az^2}{2v}. \tag{29}$$

For a constant wall heat flux case, the local heat transfer coefficient is given by:

$$h(z) = \frac{q_w}{T_w - T_\infty} = \frac{2k}{a}.\frac{1}{\theta(1)}. \tag{30}$$

The local Nusselt number becomes:

$$\mathrm{Nu}_z = 2\left(\frac{\mathrm{Re}_2}{\mathrm{Re}}\right)^{\frac{1}{2}}.\frac{1}{\theta(1)}. \tag{31}$$

Figures 2 to 4 show the matched (uniformly valid) values of the variations of temperature, temperature gradient and velocity profile function, respectively, for different values of Reynolds numbers. These curves are the presentation of the intersections of inner and outer solutions in each case for different values of Reynolds numbers.

SIGNIFICANCE OF THE RESULTS

In this paper, the systematic behavior of heat transfer in an axisymmetric stagnation flow at a high Reynolds number on a cylinder has been investigated. Rapid changes in the region near the wall and slow changes away from the wall were considered. Uniform values of the relevant quantities were produced by use of inner and outer expansion. This dual structure of flow permits evaluation of the uniformly valid solution of temperature and temperature gradient throughout the field, which is a fundamental contribution. In this way, unexpected results, due to the singularity of the governing equations in cases of high Reynolds numbers, do not occur and the actual results provide a more complete picture of the manner in which the heat

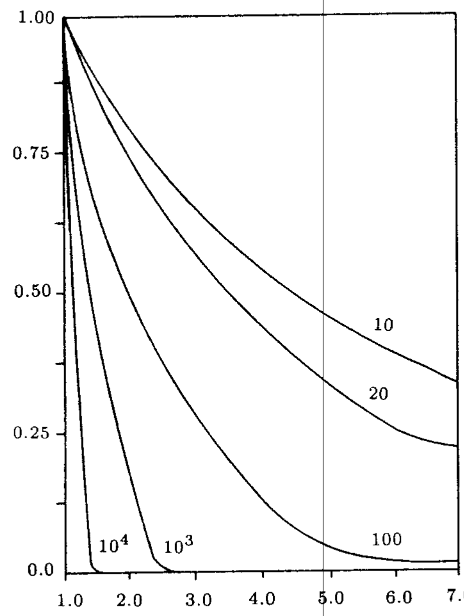

Figure 2a. Temperature distribution for various values of Re in the case of isothermal wall boundary condition.

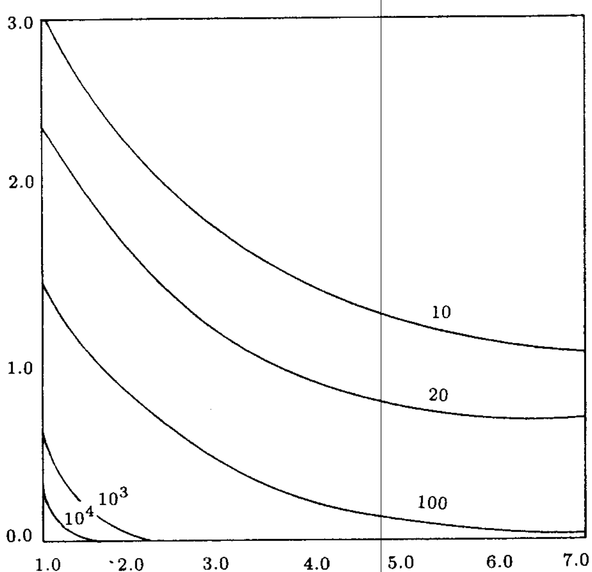

Figure 2b. Temperature distribution for various values of Re in the case of constant heat flux.

transfer and flow behavior changes with the Reynolds number.

It is expected that the results for large values of Reynolds numbers would correspond to the two dimensional stagnation flow on a flat plate. Using the transformation:

$$f(\eta) = \mathrm{Re}^{-\frac{1}{2}}\phi(\zeta),$$

where:

$$\zeta = \mathrm{Re}^{\frac{1}{2}}(\eta - 1), \tag{32}$$

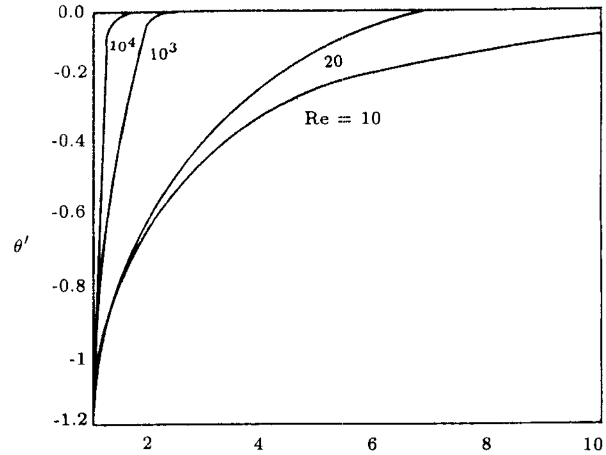

Figure 3. Temperature first derivative for various values of Re in the case of isothermal wall boundary condition (Pr = 100).

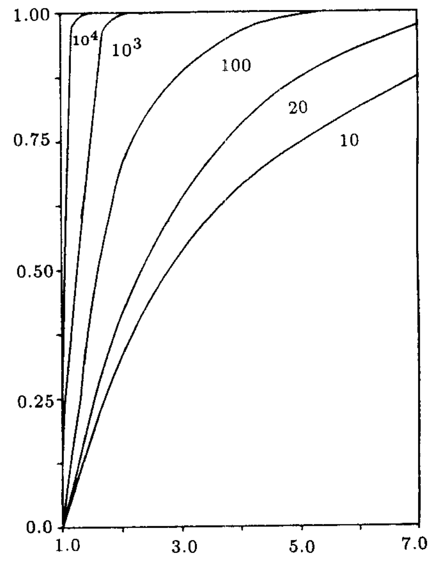

Figure 4. velocity distribution for various values of Re.

It can be seen that Equation 5 reduces to the Hiemenz's problem:

$$\phi''' + \phi\phi'' - \phi'^2 + 1 = 0, \tag{33}$$

with boundary conditions:

$$\phi(0) = \phi'(0) = 0, \quad \phi'(\infty) = 1. \tag{34}$$

Using Equation 32, for large values of Re the energy equation can be reduced to the following form, as a first approximation:

$$\theta'' + \mathrm{Pr}\phi\theta' = 0, \tag{35}$$

with boundary conditions:

$$\theta(0) = 1, \quad \theta(\infty) = 0$$

for constant surface temperature case,

Table 1. Values of $\mathrm{Re}^{-\frac{1}{2}}.f''(1)$ versus Re.

$\mathrm{Re}^{-\frac{1}{2}}.f''(1)$		
Re	**[2]**	**Present Result**
20	1.7577	1.7577
100	1.484185	1.484185
1000	1.31643	1.316427
10000		1.259642
∞		1.232588

Table 2. Values of $-\mathrm{Re}^{-\frac{1}{2}}.\theta'$ (1) versus Pr.

Pr	$-\mathrm{Re}^{-\frac{1}{2}}.\theta'(1)$	$\mathrm{Re}^{\frac{1}{2}}.\theta(1)$
0.01	0.07598	13.16136
0.1	0.2195	4.55581
0.7	0.4959	2.01645
1.0	0.5704	1.75316
10.0	1.3389	0.74688
100.0	2.98633	0.33486
1000.0	6.25914	0.15316

and:

$$\theta'(0) = -\mathrm{Re}^{-\frac{1}{2}}, \quad \theta(\infty) = 0$$

for constant surface heat flux case.

The error in deriving Equations 33 and 35 is of the order $\mathrm{Re}^{-\frac{1}{2}}$. Equation 35 corresponds to the heat transfer at a two dimensional stagnation flow on a flat plate. In order to facilitate comparison of the present results for the skin friction coefficient, as well as the Nusselt number, with the corresponding results for the two-dimensional stagnation flow problem, values of $\mathrm{Re}^{-\frac{1}{2}}.f''(1)$ versus Re are shown in Table 1. Similarly, values of $-\mathrm{Re}^{-\frac{1}{2}}.\theta'$ (1) versus Pr for the constant wall temperature case and $\mathrm{Re}^{\frac{1}{2}}\theta(1)$ versus Pr, for $\mathrm{Re} \to \infty$, for the case of constant heat flux, are given in Table 2.

NOMENCLATURE

A	constant
a	radius of cylinder
f	velocity profile function
F	inner velocity
h	local heat transfer coefficient
k	thermal conductivity
L	cylinder length
Nu	Nusselt number (hz/k)
P	pressure
Pr	Prandtl number
q	heat flux at the wall
r	coordinate axis
Re	Reynolds number ($Aa^2/2v$)
T	temperature
u	r-velocity component
v	θ-velocity component
w	z-velocity component
z	coordinate axis

Greek

α	diffusivity and constant
β	constant
η	dimensionless coordinate
θ	dimensionless temperature
v	kinematic viscosity
ε	perturbation parameter
ξ	inner region variable
Φ	temperature in inner region
μ	dynamic viscosity
ρ	fluid density

Indices

0	zeroth-order quantities
1	first-order quantities

REFERENCES

1. Gorla, R.S.R. "Heat transfer in an axisymmetric stagnation flow on a cylinder", *Applied Science Research*, **32**, pp 541-553 (Nov. 1976).
2. Davey, A. "Solution of axisymmetric stagnation flow on an infinite circular cylinder", *J. of Engineering Mathematics*, **33**, pp 113-128 (1998).
3. Nayfeh, A.H., *Introduction to Perturbation Techniques*, John Wiley (1995).
4. Press, W.H., Flannery, B.P., Tenkolsky, S.A. and Vetterling, W.T., *Numerical Receipes, the Art of Scientific Computing*, Cambridge University Press, Cambridge (1993).
5. Carey, V.P. "Analysis of transient natural convection flow at high Prandtl number using a matched asymptotic expansion technique", *Int. J. of Heat and Mass Transfer*, **26**(6), pp 911-918 (1987).