3-D Thin Layer Navier-Stokes Solution of Supersonic Turbulent Flow

M.M. Alishahi*, H. Emdad1 and O. Abouali1

In this research, a 3-D Thin Layer Navier-Stokes (TLNS) code is developed. This code consists of several numerical algorithms for space and time discretization, together with appropriate turbulence modeling. The Roe method is used for the discretization of inviscid terms and the central scheme for viscous terms. The explicit time marching technique is applied, based on finite volume space discretization. This code can be employed in the range of laminar and turbulent flow. It is validated for a supersonic flow with Mach number 3 around a tangent-ogive with incidence angles of $6^\circ$ and a secant-ogive with incidence angles of $10^\circ$. The circumferential pressure distribution is compared with experimental and Euler code results and the results of TLNS are acceptable. The cross-sectional Mach number contours are also presented. In addition to an outer shock, a cross-flow shock wave is captured in the case of a $10^\circ$ angle of incidence.

INTRODUCTION

One of the most important aims of CFD research is to determine the complicated 3-D flow fields around flying objects. Before progress of digital computers, analytical and experimental methods were the only means available for resolving fluid dynamic problems.

It is obvious that the Euler or Navier-Stokes equations do not have analytical solutions except for very simple geometries. Fortunately, the progress of digital computers throughout the last decades, has brought about the opportunity of solving these equations numerically.

The methods for the solution of Euler equations are categorized in two main groups. The first one is the central differencing scheme. Lax-Wendrof [1] and MacCormac [2] methods belong to this group. Using this method, a programming task is made easy but one needs dissipative terms to capture shock waves. These artificial viscous terms are case dependent and determined via trial and error. Therefore, the generality of these methods is limited. The second category is based on an upwinding approach. Upwinding methods model the physical phenomenon of flow in a better way compared with the central differencing method. The type of discretization is selected depending on the propagation direction of information in the flowfield. This direction is related to the sign of the eigenvalues of Jacobian matrices. In these methods, solution of the Reimann problem (one-dimensional shock tube problem) is implemented to determine the fluxes of the Euler equations. These methods capture the shock waves properly and do not need dissipative terms, but they are more time-consuming. Gudonov [3], Osher [4] and Roe [5] methods belong to this category. The solution of 3-D Navier-Stokes equations for compressible flow is very time-consuming and problems involving these equations are usually solved using supercomputers [6,7]. Fortunately, it is possible to solve some reduced forms of the Navier-Stokes equations in many situations. In most of these cases, viscous derivatives in the streamwise direction are negligible and can be left out of Navier-Stokes equations. After omitting these terms, the steady state forms of the resulted equations called Parabolized Navier-Stokes (PNS) are obtained. Using PNS, the run-time and necessary memory are reduced but the application of these equations is limited to supersonic flow with small streamwise pressure gradient and without streamwise separation [8,9]. Another set of reduced forms of the Navier-Stokes equations are Thin Layer Navier-Stokes (TLNS) equations. In this case, time derivative terms are not dropped out in comparison with the PNS equations.

In this investigation, a 3-D TLNS code is developed. The Roe method is used to discretize the inviscid terms [10] and central differencing for the

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* Corresponding Author, Department of Mechanical Engineering, Shiraz University, Shiraz, I.R. Iran.
1. Department of Mechanical Engineering, Shiraz University, Shiraz, I.R. Iran.
viscous terms. Time derivative terms are discretized with the Explicit technique. The Baldwin-Lomax model [11] and Degani-Schief [12] modifications are used for turbulence modeling. The algorithm is based on a finite volume approach. The code is validated for two test cases. The first is a turbulent supersonic flow with Mach number 3 on a tangent-ogive cylinder at an incidence angle of 6° and the second is the same flow on a secant-ogive with an incidence angle of 10°. The addition of an implicit technique to the code and matching it with a PNS code [9] are currently being carried out and the results will be presented in the near future.

GOVERNING EQUATIONS

Navier-Stokes equations in Cartesian coordinates, in the conservative form are:

\[ Q_t + (E - E_v)_x + (F - F_v)_y + (G - G_v)_z = 0 \]  \hspace{1cm} (1)

\[
Q = \begin{bmatrix}
\rho e \\
\rho u \\
\rho v \\
\rho w
\end{bmatrix}, \quad E = \begin{bmatrix}
(e + p)u \\
\rho u^2 + p \\
\rho uv \\
\rho uw
\end{bmatrix},
\]

\[
F = \begin{bmatrix}
(e + p)v \\
\rho v^2 + p \\
\rho uv \\
\rho vw
\end{bmatrix}, \quad G = \begin{bmatrix}
(e + p)w \\
\rho w^2 + p \\
\rho uw \\
\rho vw
\end{bmatrix}.
\]  \hspace{1cm} (2)

\[
E_v = \begin{bmatrix}
\tau_{xx} + u\tau_{xy} + w\tau_{xz} - q_x \\
\tau_{xy} \\
\tau_{xz} \\
0
\end{bmatrix}, \quad F_v = \begin{bmatrix}
\tau_{yz} + u\tau_{yy} + w\tau_{yz} - q_y \\
\tau_{yy} \\
\tau_{yz} \\
0
\end{bmatrix}, \quad G_v = \begin{bmatrix}
\tau_{xz} + u\tau_{xy} + w\tau_{xz} - q_z \\
\tau_{xx} \\
\tau_{xy} \\
\tau_{xz}
\end{bmatrix}
\]  \hspace{1cm} (3)

in which:

\[
\tau_{xx} = \frac{2}{3} \mu \left(\frac{2\partial u}{\partial x} - \frac{\partial u}{\partial y} - \frac{\partial w}{\partial z}\right),
\]

\[
\tau_{yy} = \frac{2}{3} \mu \left(\frac{2\partial v}{\partial y} - \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right),
\]

\[
\tau_{zz} = \frac{2}{3} \mu \left(\frac{2\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right),
\]

\[
\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),
\]

\[
\tau_{xz} = \tau_{zx} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),
\]

\[
q_x = -K_x \frac{\partial T}{\partial x}, \quad q_y = -K_y \frac{\partial T}{\partial y}, \quad q_z = -K_z \frac{\partial T}{\partial z},
\]

\[
\mu = \mu_L + \mu_t, \quad K = K_L + K_t.
\]  \hspace{1cm} (4)

The laminar coefficients of viscosity and thermal conductivity, \( \mu_L \) and \( K_L \), are related to the thermodynamic variables using the kinetic theory. Sutherland's formula for viscosity is given by:

\[
\mu = \frac{C_1}{T^{3/2}} \frac{T + C_2}{T + C_2}
\]  \hspace{1cm} (5)

where \( C_1 \) and \( C_2 \) are constants for a given gas. For air at moderate temperature, \( C_1 = 1.458 \times 10^{-3} \text{kg/ms}^{1/2} \) and \( C_2 = 110.4 \text{ k} \). The Prandtl number \( Pr = \frac{C_1}{C_2} \) is often used to determine the coefficient of thermal conductivity. \( \mu_t \) and \( K_t \) will be defined later in this paper.

Equation 1 can be expressed in terms of generalized orthogonal curvilinear coordinates, as:

\[
\tilde{Q}_t + (\tilde{E} - \tilde{E}_v)_\xi + (\tilde{F} - \tilde{F}_v)_\eta + (\tilde{G} - \tilde{G}_v)_\zeta = 0,
\]  \hspace{1cm} (6)

\[
\tilde{Q} = \frac{Q}{J},
\]

\[
\tilde{E} = \frac{\xi_x}{J} E + \frac{\xi_y}{J} F + \frac{\xi_z}{J} G,
\]

\[
\tilde{E}_v = \frac{\xi_x}{J} E_v + \frac{\xi_y}{J} F_v + \frac{\xi_z}{J} G_v,
\]

\[
\tilde{F} = \frac{\eta_x}{J} E + \frac{\eta_y}{J} F + \frac{\eta_z}{J} G,
\]

\[
\tilde{F}_v = \frac{\eta_x}{J} E_v + \frac{\eta_y}{J} F_v + \frac{\eta_z}{J} G_v,
\]

\[
\tilde{G} = \frac{\zeta_x}{J} E + \frac{\zeta_y}{J} F + \frac{\zeta_z}{J} G,
\]

\[
\tilde{G}_v = \frac{\zeta_x}{J} E_v + \frac{\zeta_y}{J} F_v + \frac{\zeta_z}{J} G_v,
\]  \hspace{1cm} (7)

in which \( J \) is the Jacobian of the transformation.

The thin layer approximation is now applied to the transformed Navier-Stokes equations. This
where the volume of tetrahedrons, having vertices \(a, b, c, d\), are evaluated from:

\[
V_{\text{tet}}(a, b, c, d) = \frac{1}{6} \left[ x_a [y_b(z_c - z_d) - y_c(z_b - z_d) + y_d(z_b - z_c)] - x_b [y_c(z_a - z_d) - y_d(z_a - z_b) + y_d(z_a - z_c)] + x_c [y_d(z_b - z_d) - y_b(z_a - z_b) + y_b(z_a - z_c)] - x_d [y_a(z_b - z_c) - y_b(z_a - z_c) + y_c(z_a - z_b)] \right].
\]

(11)

**COMPUTATION OF CELL-FACE METRICS (NORMALS)**

The cell-face normals, having vertices \(a, b, c, d\), are evaluated as:

\[
n_{x}(a, b, c, d) = \frac{(dy_{cb}dz_{cb} - dy_{cb}dz_{ba})}{2} + \frac{(dy_{cd}dz_{ad} - dy_{ad}dz_{dc})}{2},
\]

\[
n_{y}(a, b, c, d) = \frac{(dz_{cb}dx_{cb} - dx_{cb}dy_{cb})}{2} + \frac{(dz_{cd}dx_{ad} - dx_{ad}dy_{dc})}{2},
\]

\[
n_{z}(a, b, c, d) = \frac{(dx_{cb}dy_{cb} - dy_{cb}dx_{cb})}{2} + \frac{(dx_{cd}dy_{ad} - dy_{ad}dx_{cd})}{2},
\]

(12)

in which \(ds_{12} = s_1 - s_2\).

**NUMERICAL INVISCID FLUXES (ROE APPROXIMATE RIEMANN SOLVER)**

The Roe method is used for the calculation of the fluxes. So Equation 7 may be condensed to:

\[
\tilde{F}, \tilde{G} = f(Q, n_x, n_y, n_z) = f(Q, N).
\]

(13)

The Jacobian matrix of the flux, \(f\), with respect to \(Q\), can be denoted by \(\frac{Df}{DQ}\). The eigenvalues of the Jacobian matrix are shown by \(\lambda_i\) and the corresponding left and right eigenvectors by \(\ell^i\) and \(r^i\) [13].

The left eigenvectors matrix is formed by left eigenvectors as rows and the right eigenvectors matrix is organized by right eigenvectors as a column.

In the Roe approach, the cell interface, values of density, velocity and enthalpy \(\{h = \frac{\gamma p}{[\gamma - 1] \rho] + (\frac{\omega^2 + \omega' + \omega''}{2})\) are computed using a special averaging procedure:

\[
\rho_{m+1/2} = \sqrt{\frac{\rho_m \rho_{m+1}}{2}};
\]

\[
(u, v, w)_{m+1/2} = (u, v, w)_{m+1} \sqrt{\rho_{m+1}} + (u, v, w)_{m} \sqrt{\rho_{m}} / \sqrt{\rho_{m+1} + \rho_{m}};
\]

\[
h_{m+1/2} = \frac{h_{m+1} \sqrt{\rho_{m+1}} + h_{m} \sqrt{\rho_{m}}}{\sqrt{\rho_{m+1} + \rho_{m}}},
\]

(14)
where \( m = j \) or \( k \) or \( l \) and speed of sound may be computed by:

\[
C_{m+1/2} = \sqrt{(\gamma m_{m+1/2} - (u^2 + v^2 + w^2))/\gamma - 1}.
\]  

(15)

The formula for eigenvalues and eigenvector matrices are now presented.

Defining the contravariant velocity by \( \vec{U} = n_z u + n_y v + n_x w \), eigenvalues are determined as:

\[
\lambda^1 = \vec{U} - c\sqrt{n_x^2 + n_y^2 + n_z^2}, \\
\lambda^{2,3,4} = \vec{U}, \\
\lambda^5 = \vec{U} + c\sqrt{n_x^2 + n_y^2 + n_z^2}.
\]

(16)

At each cell face, the positive and negative projections of the eigenvalues may be defined by:

\[
\lambda^\pm = (\lambda_{m+1/2}^i \pm |\lambda_{m+1/2}^i|)/2 \\
i = 1, \cdots, 5.
\]

(17)

To avoid expansion shock, only at sonic rarefaction's \( [\lambda^i(U_m) < 0 < \lambda^i(U_{m+1})] \) equations are modified as:

\[
\lambda^\pm_{m+1/2} = \lambda^\pm_{m+1/2} \pm |\lambda_{m+1/2}^i - \lambda_{m}^i(U)|/4.
\]

(18)

Defining \( \hat{n}_{x,y,z} = (n_x, n_y, n_z)/\sqrt{n_x^2 + n_y^2 + n_z^2} \) and \( \theta = (u^2 + v^2 + w^2)/2 \), the left and right eigenvectors are presented in Table 1.

Now, the equation for calculating the first order upwind flux is defined as:

\[
f_{m+1/2} = 1/2[f(Q_{m+1}, N_{m+1/2}) + f(Q_m, N_{m+1/2})]
\]

\[
- 1/2 \left( \sum_i \lambda^{i+}_{m+1/2} - \lambda^{i-}_{m+1/2} \right) r^i_{m+1/2}
\]

\[
= f(Q_m, N_{m+1/2}) = \sum_i \lambda^{i-}_{m+1/2} \alpha^{i-}_{m+1/2}
\]

\[
= f + df^{-}
\]

\[
= f(Q_{m+1}, N_{m+1/2}) = \sum_i \lambda^{i+}_{m+1/2} \alpha^{i+}_{m+1/2}
\]

\[
= f - df^{+},
\]

(19)

in which:

\[
\alpha^{i+}_{m+1/2} = \epsilon_{m+1/2}^i (Q_{m+1} - Q_m),
\]

and:

\[
df^{i\pm}_{m+1/2} = \lambda^{i\pm}_{m+1/2} \alpha^{i}_{m+1/2} r^i_{m+1/2}.
\]

**VISCOSOUS TERMS**

In this section, the discretization of viscous terms in the TLNS equations are explained.

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**Table 1.** The left and right eigenvectors of Jacobian matrix.

<table>
<thead>
<tr>
<th>( \xi^1 )</th>
<th>( \xi^2 )</th>
<th>( \xi^3 )</th>
<th>( \xi^4 )</th>
<th>( \xi^5 )</th>
</tr>
</thead>
</table>
| \[
\left[ \frac{2 \xi - 1}{c} \right] / \sqrt{2}, \left[ \frac{2 \xi - 1}{c} \theta + \hat{n}_{y} u + \hat{n}_{x} v + \hat{n}_{z} w \right] / \sqrt{2}, \left[ \frac{2 \xi - 1}{c} u - \hat{n}_{z} \right] / \sqrt{2}
\] | \[
\left[ \frac{2 \xi - 1}{c} \right] / \sqrt{2}, \left[ \frac{2 \xi - 1}{c} u + \hat{n}_{z} \right] / \sqrt{2}, \left[ \frac{2 \xi - 1}{c} \right] / \sqrt{2}
\] | \[
\left[ \frac{2 \xi - 1}{c} \right] / \sqrt{2}, \left[ \frac{2 \xi - 1}{c} u - \hat{n}_{z} \right] / \sqrt{2}, \left[ \frac{2 \xi - 1}{c} \right] / \sqrt{2}
\] | \[
\left[ \frac{2 \xi - 1}{c} \right] / \sqrt{2}, \left[ \frac{2 \xi - 1}{c} u + \hat{n}_{z} \right] / \sqrt{2}, \left[ \frac{2 \xi - 1}{c} \right] / \sqrt{2}
\] | \[
\left[ \frac{2 \xi - 1}{c} \right] / \sqrt{2}, \left[ \frac{2 \xi - 1}{c} u - \hat{n}_{z} \right] / \sqrt{2}, \left[ \frac{2 \xi - 1}{c} \right] / \sqrt{2}
\] |
| \[
\left[ \frac{2 \xi - 1}{c} \right] / \sqrt{2}, \left[ \frac{2 \xi - 1}{c} \theta + \hat{n}_{y} u + \hat{n}_{x} v + \hat{n}_{z} w \right] / \sqrt{2}, \left[ \frac{2 \xi - 1}{c} u - \hat{n}_{z} \right] / \sqrt{2}
\] | | | | |
Viscous terms in the TLNS equations are expressed as:

\[
\widetilde{F}_v = \frac{\eta_v}{J} E_v + \frac{\eta_v}{J} F_v + \frac{\eta_v}{J} G_v.
\]  

(20)

Considering Relations 3 and 4, it is obvious that the stress tensors should be computed. Computation of stresses using the finite volume method in a given cell is done by applying the divergence theorem. Let \( \Omega \) be an arbitrary volume and \( S \) be the closed boundary surface of the cell, therefore:

\[
\int_\Omega \nabla \cdot \mathbf{U} \, d\Omega = \oint_S \mathbf{U} \cdot d\mathbf{S}.
\]  

(21)

The average velocity gradients are defined as:

\[
\left( \frac{\partial u}{\partial x} \right)_\Omega = \frac{1}{\Omega} \int_\Omega \frac{\partial u}{\partial x} \, d\Omega = \frac{1}{\Omega} \oint_S \mathbf{U} \cdot d\mathbf{S}_z
\]

\[
= \frac{1}{\text{vol}} \sum U_{nb}(n_z)_{nb}
\]

\[
= \frac{1}{\text{vol}} \left\{ U_{j+1/2,k,i} \left( \frac{\epsilon_x}{J} \right)_{j+1/2,k,i} + U_{j-1/2,k,i} \left( \frac{\epsilon_x}{J} \right)_{j-1/2,k,i} + U_{j,k+1/2,i} \left( \frac{\eta_x}{J} \right)_{j,k+1/2,i} - U_{j,k-1/2,i} \left( \frac{\eta_x}{J} \right)_{j,k-1/2,i} + U_{j,i+1/2} \left( \frac{\epsilon_y}{J} \right)_{j,i+1/2} - U_{j,i-1/2} \left( \frac{\epsilon_y}{J} \right)_{j,i-1/2} \right\},
\]  

(22)

\[
\left( \frac{\partial u}{\partial y} \right)_\Omega = \frac{1}{\Omega} \int_\Omega \frac{\partial u}{\partial y} \, d\Omega = \frac{1}{\Omega} \oint_S \mathbf{U} \cdot d\mathbf{S}_y
\]

\[
= \frac{1}{\text{vol}} \sum U_{nb}(n_y)_{nb}
\]

\[
= \frac{1}{\text{vol}} \left\{ U_{j+1/2,k,i} \left( \frac{\epsilon_y}{J} \right)_{j+1/2,k,i} - U_{j-1/2,k,i} \left( \frac{\epsilon_y}{J} \right)_{j-1/2,k,i} + U_{j,k+1/2,i} \left( \frac{\eta_y}{J} \right)_{j,k+1/2,i} - U_{j,k-1/2,i} \left( \frac{\eta_y}{J} \right)_{j,k-1/2,i} \right\},
\]  

(23)

The other derivatives \((\partial u/\partial z, \ldots)\) are discretized in the same way.

**ALGEBRAIC MODEL OF TURBULENCE**

There are many turbulence models to simulate a turbulent boundary layer, but a useful and simple model, which is examined in many computer assessments and needs less amount of computer time, has been offered by Baldwin and Lomax [11]. This is a zeroth model, in which the effect of turbulence is simulated in terms of an eddy viscosity coefficient \((\mu_t)\). Thus, in stress terms of the laminar Navier-Stokes equations, the molecular coefficient of viscosity, \((\mu)\), is replaced by \((\mu + \mu_t)\). In heat flux terms \((K/Cp = \mu/Pr)\) is replaced by \((\mu/Pr + \mu_t/Pr_t)\).

The Baldwin-Lomax model is patterned after an algebraic model developed by Cebeci et al. [14]. This turbulence model is named as a two-layer zeroth equation model and shows that the turbulent flow is divided into an inner and an outer region. A different set of equations is used in each region to determine the turbulent eddy viscosity, \(\mu_t\). The final value of \(\mu_t\) is defined as:

\[
\mu_t = \min \left[ (\mu_t)_{\text{inner}}, (\mu_t)_{\text{outer}} \right].
\]  

(24)

For the inner layer, the Prandtl-Van Driest formula is used to determine \(\mu_t\) (the eddy viscosity coefficient), which is defined by:

\[
(\mu_t)_{\text{inner}} = \rho l^2 |\Omega|,
\]  

(25)

where \(l\), the mixing length, is given as:

\[
l = ky \left[ 1 - \text{exp}(-y^+ A^+) \right],
\]  

(26)

where \(k\) and \(A^+\) are constant and equal to 0.41 and 26, respectively, \(|\Omega|\) is the magnitude of the local vorticity vector and is defined as:

\[
|\Omega| = \sqrt{(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y})^2 + (\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z})^2},
\]  

(27)

and:

\[
y^+ = \frac{\sqrt{\rho_w \tau_w}}{\mu_w} y,
\]  

(28)

where, \(\rho_w, \tau_w\) and \(\mu_w\) are the density, shear stress and viscosity coefficient at the wall, respectively, and \(y\)
is the normal distance from the wall. In the outer region, for attached boundary layers, the turbulent eddy viscosity, \( \mu_t \), is defined as:

\[
(\mu_t)_{\text{outer}} = KC_{cp} \rho F_{\text{wake}} F_{\text{Kleb}}(y),
\]

where \( K \) and \( C_{cp} \) are constant and equal to 0.0168 and 1.6, respectively and \( F_{\text{Kleb}}(y) \) is the Klebanoff intermittence factor:

\[
F_{\text{Kleb}} = \left[ 1 + 5.5 \left( \frac{C_{\text{Kleb}} y}{y_{\text{max}}} \right)^6 \right]^{-1},
\]

and:

\[
F_{\text{wake}} = \min \left\{ \frac{y_{\text{max}} F_{\text{max}}}{C_{\text{wk}} y_{\text{max}} U_{\text{Diff}}^3 / F_{\text{max}}} \right\},
\]

where \( C_{\text{Kleb}} \) is the Klebanoff constant and equal to 0.3, \( C_{\text{wk}} \) is the wake constant and equal to 0.25 and:

\[
U_{\text{Diff}} = \left( \sqrt{u^2 + v^2 + w^2} \right)_{\text{max}} - \left( \sqrt{u^2 + v^2 + w^2} \right)_{\text{min}}.
\]

\( F_{\text{max}} \) and \( y_{\text{max}} \) are determined from the following function:

\[
F(y) = y/\Omega \left[ 1 - \exp(-y^+/A^+) \right],
\]

such that the peak value of \( F(y) \) between the wall and the flow field is defined as \( F_{\text{max}} \) and the value of \( y \) at which this occurs is defined as \( y_{\text{max}} \).

**TURBULENT MODIFICATION FOR BALDWIN-LOMAX MODEL**

A problem with Baldwin-Lomax model is encountered when it is applied to treat flow about slender bodies at incidence [15]. In the separated flow region, it becomes difficult to determine the correct value of \( F_{\text{max}} \), which is necessary for evaluation of \( \mu_t_{\text{outer}} \). In an attached flow, there is only one maximum for \( F(y) \) in the radial direction and \( F_{\text{max}} \) is simply determined. When separated flow occurs, two maxima for \( F(y) \) are encountered. The first peak occurs in the boundary layer and a second layer peak exists, due to the presence of a vortex sheet. If Baldwin-Lomax model is used to obtain \( F_{\text{max}} \), the second maximum in \( F(y) \) is obtained. This results in values of \( \mu_t_{\text{outer}} \) that are too high, resulting in distortion or a washout of the features in the computed flow [15]. A modification to Baldwin-Lomax model has been proposed by Degani and Schiff and applied by Weinacht et al. [12]. For each axial station, a maximum value of the scaling length, \( y_{\text{max}} \), is defined as 1.8 times the value of \( y_{\text{max}} \) on the windward ray for non-spinning bodies. A peak in \( F(y) \) is defined if the value of \( F(y) \) drops below 90% of the local maximum. Where two separate distinct peaks in \( F(y) \) exist, the peak closer to the body is chosen. If the two peaks in \( F(y) \) merge into one abnormally large peak (or a peak cannot be found at all), the value of \( F_{\text{max}} \) is frozen at the value used for the previous roll angle.

**EXPLICIT SOLUTION PROCEDURE**

In this section, an explicit method is considered and discussed. The conservative form of the TLNS equations is discretized in the following form:

\[
\frac{\hat{Q}^{n+1} - \hat{Q}^n}{\Delta \tau} + \left( \hat{E}_{j+1/2,k,l} - \hat{E}_{j-1/2,k,l} \right)^n
+ \left( \hat{F}_{j,k+1/2,l} - \hat{F}_{j,k-1/2,l} \right)^n
+ \left( \hat{G}_{j,k,l+1/2} - \hat{G}_{j,k,l-1/2} \right)^n = 0.
\]

Therefore:

\[
\hat{Q}^{n+1} = \hat{Q}^n - \Delta \tau \left\{ \left( \hat{E}_{j+1/2,k,l} - \hat{E}_{j-1/2,k,l} \right)^n
+ \left( \hat{F}_{j,k+1/2,l} - \hat{F}_{j,k-1/2,l} \right)^n
+ \left( \hat{G}_{j,k,l+1/2} - \hat{G}_{j,k,l-1/2} \right)^n \right\}.
\]

The solution of the system of equations with the above procedure is straightforward and this procedure is repeated until the flow field is steady, that is, the residue \( R = \sum_{m=1}^{m} \rho_m^{n+1} - \rho_m^n \) decreases less than a desired number.

**GRID GENERATION**

The grid generation is performed in two steps. In the first, a few 2-D grids of O type are generated at different cross-flow planes. In the second step, these 2-D grids are connected to make a 3-D grid. The outer radius of the grids is obtained using the Taylor-Makcoll solution around the nose of the ogive to include the outer shock. A circumferential plane and a cross-flow plane of a generated grid for a secant-ogive with an incidence angle of 10° are shown in Figure 2.

**RESULTS AND DISCUSSION**

Figures 3 to 6 show circumferential pressure distribution in different axial sections for a supersonic flow
with Mach number of 3 and $Re_\infty = 4.101 \times 10^6$/m past a tangent-ogive. The incidence angle is $6^\circ$. The numerical results are compared with experiments [16] and Euler solution results. Because of the cross flow separation at $x/D > 3$, the accuracy of results in the leeward region is decreased. As expected, TLNS results are more exact than Euler results.

Figures 7 to 10 show circumferential pressure distribution in different axial sections for a supersonic flow with Mach number of 3 and $Re_\infty = 4.101 \times 10^6$/m past a secant-ogive. The incidence angle is $10^\circ$. The numerical results are compared with experiments [16] and Euler solution results. Due to the higher incidence angle, the separation zone is bigger in comparison with the incidence angle of $6^\circ$. Also, a supersonic cross-flow occurred in this case and the interaction of this shock with the viscous layer creates a complex flow.

In Figure 11, the axial distribution of pressure at the windward and leeward angle is presented. An expansion is captured at $x/D = 3$, in which the ogive
Figure 6. The comparison of pressure, $\alpha=6$, Mach number = 3 and $x/D = 5.77$.

Figure 7. The comparison of pressure, $\alpha=10$, Mach number = 3 and $x/D = 1.56$.

Figure 8. The comparison of pressure, $\alpha=6$, Mach Number = 3 and $x/D = 3.13$.

Figure 9. The comparison of pressure, $\alpha=10$, Mach number = 3 and $x/D = 4.14$.

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NOMENCLATURE

$C$ speed of sound
$E$ total energy
$E, F, G$ inviscid fluxes
$E_v, F_v, G_v$ viscous fluxes
$\sim, \sim$ fluxes in general curvilinear coordinates
Figure 10. The comparison of pressure, $\alpha=10$, Mach number = 3 and $x/D = 5.77$.

Figure 11. Axial distribution of pressure at windward and leeward angle, $\alpha = 10$, Mach number = 3.

Figure 12. Cross-flow tangential velocity contours, $\alpha = 10$, Mach number = 3 and $x/D = 5.77$.

Figure 13. Cross-flow Mach number contours, $\alpha = 10$, Mach number = 3 and $x/D = 5.77$.

$F_{\text{Kleb}}$ Klebanoff intermittence factor
$F_{\text{Wake}}$ wake function
$J$ Jacobian of transformation
$H$ enthalpy
$K$ thermal conductivity
$n_x, n_y, n_z$ components of surface vector
$q_x, q_y, q_z$ heat conduction terms
$Q$ primitive variable matrix
$P$ pressure
$Pr$ Prandtl number
$r^1$ right eigen vector

Greek Symbols
$\Omega$ volume of the element
$\lambda$ eigenvalue
$\mu$ viscosity coefficient
$\mu_t$ turbulent viscosity coefficient

$T$ temperature
$u, v, w$ velocities
$V$ volume
$x, y, z$ coordinates of Cartesian.
\[ \rho \quad \text{specific mass} \]
\[ \tau_{ij} \quad \text{tensor of stresses} \]
\[ \xi, \eta, \zeta \quad \text{general curvilinear coordinates} \]
\[ \ell_i \quad \text{left eigenvector.} \]
\[ (L_x y z)^{j+1/2} (L_x y z)^{k+1/2} \]
\[ \text{metrics and components of surface vector.} \]

REFERENCES


