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Corotational Constitutive Modeling of Isotropic and Kinematic Hardening Materials

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In this paper, a corotational constitutive model for rigid plastic isotropic and kinematic hardening materials obeying von Mises yield criterion is introduced. This constitutive model relates the corotational rate of the back stress tensor to the corotational rate of the logarithmic strain tensor. It is illustrated that use of Jaumann and Zaremba corotational rates in the constitutive model yields the same results for a given problem. Applying the proposed constitutive model to the simple shear problem, the effects of nonlinear isotropic and kinematic hardenings on the stress components are studied.

INTRODUCTION

In the constitutive modeling of solids at large deformations, which generally, involves relations between stress and strain rates, it is necessary to account for the effect of material rotation [1]. In order to describe the material response independent of rigid rotations, corotational rates should be used. Thus, the use of corotational constitutive modeling is of great importance in solids and, particularly, in the plastic deformation of materials.

In finite deformation plasticity, it is a general practice to use incremental or rate constitutive models. There are different corotational rates in solid mechanics [2]. Although each one of these rates can be used for this purpose, it is worth noting that, even for purely elastic deformations, the choice of the corotational rate affects the solution of the given problem [1].

Nagtegaal and de Jong [3] have numerically analyzed the simple shear problem for large plastic deformations using a von Mises type, Prager-Ziegler kinematic hardening constitutive model. They used the corotational Jaumann rate, associated with the material spin, $\mathbf{W}$, for stress and back stress and obtained the oscillatory stress for monotonically increasing shear strain. They attributed this result to the kinematic hardening model.

In another attempt, Lee et al. [4] solved the same problem. They found out that for a simple shear problem, the corotational Jaumann rate gives an oscillatory solution for stress which is physically not acceptable. They attributed the stress oscillation to the use of the Jaumann rate rather than the Prager-Ziegler kinematic hardening. Based on the physical aspects of the kinematics of the simple shear problem, they proposed a "modified Jaumann rate" which eliminated the spurious stress oscillation.

Dafalias [5] considered a rigid plastic von Mises hardening material at large plastic deformations. Utilizing the corotational rate, associated with the body spin, $\mathbf{\Omega}$, he introduced a constitutive equation to suppress the shear oscillation. Dafalias' constitutive equation related the objective rate of Cauchy deviatoric shift stress (back stress) tensor to strain rate tensor, $\mathbf{D}$. Using his constitutive equation, Dafalias obtained a monotonically increasing solution for the corotational rate associated with $\mathbf{\Omega}$ and an oscillatory solution for the Jaumann corotational rate in the simple shear problem.

Metzger and Dubey [6] illustrated an approach in the special case of principal axes using different corotational rates and solved the simple shear problem. They found out that the solution is independent of the choice of the corotational stress rate used in that approach. Reinhardt and Dubey [7] introduced a corotational rate called $\mathbf{D}$-rate and showed that the strain rate tensor is $\mathbf{D}$-rate of the logarithmic strain tensor. Using this corotational rate, they proposed

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constitutive equations for hypoelastic and rigid plastic materials.

According to the principle of material objectivity or frame indifference, the material behavior is independent of any reference system [8]. Each of the spins mentioned above can be used to construct associated corotational tensor rates which are objective as observed from the reference system rotating with the corresponding spin. Accordingly, a particular corotational tensor rate is associated with a specific spinning reference system. Thus, the constitutive model, including stress and strain rates, should describe the material response, independent of the choice of the corotational rates used in the model.

In this paper, a corotational constitutive model for rigid plastic kinematic and isotropic hardening materials, obeying von Mises yield criterion, is introduced. This constitutive model relates the corotational rate of the back stress tensor to the same corotational rate of the logarithmic strain tensor in a basis-free form. Using this constitutive model for a Prager-Ziegler hardening material, it is shown that all corotational rates give the same result for the simple shear problem.

COROTATIONAL CONSTITUTIVE MODEL

Considering a rigid plastic von Mises hardening material at large plastic deformations, Dafalias [5] introduced the constitutive equation:

$$\overset{\circ}{\alpha} = \frac{2}{3}\bar{h}_\alpha \mathbf{D}, \tag{1}$$

where α is the back stress, $\bar{h}_\alpha$ is the kinematic hardening coefficient and the superscript ($^\circ$) represents any corotational rate. He showed that using the corotational rate of α, associated with the body spin, $\mathbf{\Omega}$, yields a monotonically increasing solution for stress. Using the Jaumann corotational rate of α, associated with the material spin, $\mathbf{W}$, in the constitutive Equation 1, yields an oscillatory solution for stress.

The back stress, α, is embedded in the material as residual stresses, generated due to the heterogeneous structure of anisotropic crystallities forming the polycrystalline material [4]. In particular, the principal component of α, having the largest absolute magnitude, produces the major influence on the yield surface and, hence, on the stress field and is carried in the lines of material elements oriented in the corresponding eigenvector direction. Thus, rotation of these lines of material may be considered to incorporate the major rotational influence of the back stress generated by the previous plastic flow.

Large plastic deformation constitutive models have usually been formulated in terms of the strain rate tensor, $\mathbf{D}$, and a corotational rate of the stress or the back stress tensor [4,5,9]. Change of the corotational rate used in these models, yields different results for material behavior that is not acceptable. In order to satisfy the principle of material frame indifference, the constitutive model should yield the same result for a given problem using different corotational rates.

The logarithmic strain introduced by Hencky is a favored measure of strain in plasticity problems. In Hill's work [10], the logarithmic strain measures are believed to have inherent advantages in certain constitutive relations in solid mechanics. He has obtained the time rate of change of the logarithmic strain tensor in the principal axes of stretch. An explicit closed form relation for the material time derivative of the logarithmic strain tensor has been introduced by Naghdabadi et al. [11].

As the experimental results show, the kinematic hardening coefficient, $\bar{h}_\alpha$, depends on the history of the deformation. Thus, it is not a constant during the plastic deformation [12]. In order to account for the general case of this dependency, it is assumed that the back stress tensor is a function of $\bar{h}_\alpha$ and $\dot{\bar{h}}_\alpha$. Therefore, using the corotational rates of the logarithmic strain tensor, a corotational constitutive model for rigid plastic isotropic and kinematic hardening materials can be introduced in general form as:

$$\overset{\circ}{\alpha} = f(\bar{h}_\alpha, \dot{\bar{h}}_\alpha, (\ln \mathbf{V})^\circ). \tag{2}$$

Although $\bar{h}_\alpha$ is a fourth order tensor for anisotropic materials [4,12], it is a scalar for an isotropic material.

A simple form of the proposed model (Equation 2) can be written in the following form:

$$\overset{\circ}{\alpha} = h_\alpha (\ln \mathbf{V})^\circ + \dot{h}_\alpha \ln \mathbf{V}, \tag{3}$$

where $h_\alpha = \frac{2}{3}\bar{h}_\alpha$ for consistency with Equation 1. This form of constitutive equation can be used for isotropic materials with a variable kinematic hardening coefficient.

According to the assumption that the difference between the Cauchy deviatoric stress tensor and the back stress tensor is coaxial with the strain rate tensor [4,5], the flow rule is defined as:

$$\mathbf{D} = \dot{\phi}\,(\mathbf{S} - \alpha), \tag{4}$$

where $\dot{\phi}$ is a proportionality factor and $\mathbf{S}$ represents the deviatoric Cauchy stress tensor.

Using the von Mises yield criterion stated in the following form:

$$f = \frac{3}{2}(\mathbf{S} - \alpha) : (\mathbf{S} - \alpha) - k^2 = 0, \tag{5}$$

and Equation 4, the proportionality factor, $\dot{\phi}$, is obtained as:

$$\dot{\phi} = \frac{3\mathbf{D}_e}{2k}, \tag{6}$$

where k is the "size" of the yield surface, the symbol (:) stands for tensor inner product and $\mathbf{D}_e$ is the effective strain rate defined as:

$$\mathbf{D}_e = \sqrt{\frac{2}{3}\mathbf{D}:\mathbf{D}}. \tag{7}$$

Substituting ϕ from Equation 6 into Equation 4 , the flow rule can be written as:

$$\mathbf{S} - \alpha = \frac{2k}{3\mathbf{D}_e}\mathbf{D}. \tag{8}$$

Also, using a nonlinear isotropic hardening rule, k is defined as [5]:

$$k = 1 + (k_s - 1)[1 - \exp(-c\bar{\epsilon})], \tag{9}$$

where k_s and c are the isotropic hardening coefficients and $\bar{\epsilon}$ is the effective strain and can be determined by:

$$\bar{\epsilon} = \int \mathbf{D}_e dt. \tag{10}$$

In the constitutive Equation 3, $\overset{o}{\alpha}$ represents the corotational rate of the back stress tensor, defined as:

$$\overset{o}{\alpha} = \dot{\alpha} - \mathbf{\Lambda}\alpha + \alpha\mathbf{\Lambda}, \tag{11}$$

where $\mathbf{\Lambda}$ is a spin tensor which may be substituted with the material spin, $\mathbf{W}$, the body spin, $\mathbf{\Omega}$, or any other spin tensor of interest. Substituting $\mathbf{\Lambda}$ with $\mathbf{W}$ spin, it is obtained that:

$$\overset{J}{\alpha} = \dot{\alpha} - \mathbf{W}\alpha + \alpha\mathbf{W}, \tag{12}$$

where $\overset{J}{\alpha}$ is called Jaumann rate (J-rate) of tensor α. $\dot{\alpha}$ is the material time derivative of tensor α, which expresses the time rate of change of the concerned quantity as seen by an observer from a rigidly translating frame attached to the material particle. From this view point, $\overset{J}{\alpha}$ is, in fact, the time rate of change measured by an observer on the J-frame, which is assumed to have $\mathbf{W}$ spin. Thus, $\dot{\alpha}$ represents the time rate of change recorded in a fixed background, whereas $\overset{J}{\alpha}$ is the rate observed from a $\mathbf{W}$-spinning frame.

Obviously, one can use other frames which have different spins. Consider, for instance, the frame with $\mathbf{\Omega}$ spin, in which the time rate of change of the back stress tensor, α, is recorded as follows:

$$\overset{Z}{\alpha} = \dot{\alpha} - \mathbf{\Omega}\alpha + \alpha\mathbf{\Omega}, \tag{13}$$

where $\overset{Z}{\alpha}$ is defined as Z-rate of tensor α [7]. The superscript "Z" stands for Zaremba, who, for the first time, used a corotational time derivative of the stress tensor [13]. $\overset{Z}{\alpha}$ is also called Green-Naghdi [14] or Green-McInnis rate [15].

In the proposed constitutive model (Equation 3), $\overset{o}{(\ln \mathbf{V})}$ stands for the corresponding corotational rate of the logarithmic strain tensor defined by:

$$\overset{o}{(\ln \mathbf{V})} = \dot{(\ln \mathbf{V})} - \mathbf{\Lambda}(\ln \mathbf{V}) + (\ln \mathbf{V})\mathbf{\Lambda}. \tag{14}$$

Substituting $\mathbf{\Lambda}$ with the material or body spin tensor, $\mathbf{W}$ or $\mathbf{\Omega}$, respectively, the corotational J- or Z-rate of the logarithmic strain tensor is obtained. It is noted that the same corotational rate should be used for the back stress tensor, α, as well as the logarithmic strain tensor, $(\ln \mathbf{V})$, in the constitutive model (Equation 3).

KINEMATICS OF THE SIMPLE SHEAR PROBLEM

As an application of the proposed constitutive model (Equation 3), consider the simple shear deformation of a rectangle, shown in Figure 1. The deformation may be prescribed in terms of motion of a particle initially at X_i, which is currently occupying a position at x_i, such that:

$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \tag{15}$$

where γ is the shear displacement.

Let $\mathbf{F}$ denote the deformation gradient at a point in the deforming body with components:

$$F_{ij} = \frac{\partial x_i}{\partial X_j}. \tag{16}$$

Since det $\mathbf{F} > 0$, the polar decomposition theorem states that:

$$F_{ij} = V_{ik} R_{kj} = R_{ik} U_{kj}, \tag{17}$$

where U_{ij} and V_{ij} are the components of the right and left stretch tensors $\mathbf{U}$ and $\mathbf{V}$, respectively. Also, R_{ij} are

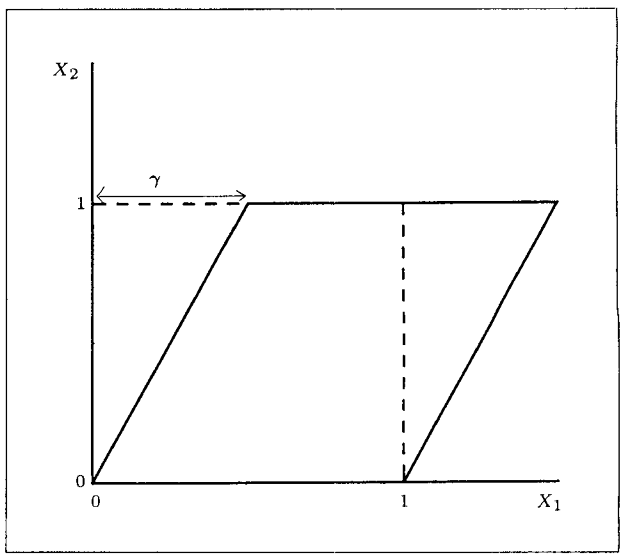

Figure 1. Simple shear problem.

components of the proper orthogonal rotation tensor $\mathbf{R}$. $\mathbf{U}$ and $\mathbf{V}$ are symmetric positive-definite tensors with the same eigenvalues, λ_i, which are the principal stretches. Let the set $\{\mathbf{N}_i\}$ be the principal directions of $\mathbf{U}$. Then, the sets $\{\mathbf{n}_i\}$ denoted by:

$$\mathbf{n}_i = \mathbf{R}\,\mathbf{N}_i, \tag{18}$$

are the principal directions of $\mathbf{V}$. Accordingly, tensor $\mathbf{V}$ has the following spectral representation:

$$\mathbf{V} = \sum_i \lambda_i\, \mathbf{n}_i \otimes \mathbf{n}_i. \tag{19}$$

For the given motion (Equation 15), the deformation gradient has the following components:

$$F_{11} = F_{22} = 1,$$

$$F_{12} = \gamma,$$

$$F_{21} = 0. \tag{20}$$

Using the polar decomposition theorem, components of the left stretch tensor are calculated as follows:

$$V_{11} = \frac{2+\gamma^2}{c_1},$$

$$V_{12} = V_{21} = \frac{\gamma}{c_1},$$

$$V_{22} = \frac{2}{c_1}, \tag{21}$$

where:

$$c_1 = \sqrt{4+\gamma^2}. \tag{22}$$

Also, the rotation angle associated with the components of the orthogonal rotation tensor, R_{ij}, is given by:

$$\theta = \arctan\,(-\gamma/2). \tag{23}$$

The principal stretches, λ_i, and the angle β of the orientation of the principal directions are obtained from Equation 21 in the following form:

$$\lambda_1, \lambda_2 = \frac{1}{2}(c_1 \pm \gamma), \tag{24}$$

$$\beta = \frac{1}{2}\arctan(2/\gamma). \tag{25}$$

Assuming that $\mathbf{F}$ is a continuously differentiable tensor of time, the velocity gradient $\mathbf{L}$ is expressed by:

$$\mathbf{L} = \dot{\mathbf{F}}\,\mathbf{F}^{-1} = \mathbf{D} + \mathbf{W}, \tag{26}$$

where the strain rate tensor, $\mathbf{D}$, and the material spin tensor, $\mathbf{W}$, are the symmetric and skew symmetric parts of $\mathbf{L}$, respectively. Also, let the body spin tensor, $\boldsymbol{\Omega}$, denote the relative spin between the sets $\{\mathbf{N}_i\}$ and $\{\mathbf{n}_i\}$, so that:

$$\dot{\mathbf{R}} = \boldsymbol{\Omega}\,\mathbf{R}, \tag{27}$$

where $\boldsymbol{\Omega}$ is also called the rate of body rotation. In the development of the principal axes method, Hill [10] has introduced two more spins, $\boldsymbol{\Omega}^{L}$ and $\boldsymbol{\Omega}^{E}$, which are called the Lagrangian and Eulerian spins. They are assumed to represent the spins of the sets $\{\mathbf{N}_i\}$ and $\{\mathbf{n}_i\}$, respectively.

$$\dot{\mathbf{N}}_i = \boldsymbol{\Omega}^{L}\mathbf{N}_i,$$

$$\dot{\mathbf{n}}_i = \boldsymbol{\Omega}^{E}\mathbf{n}_i. \tag{28}$$

Using the chain rule for partial derivatives of the deformation gradient components (Equation 20, together with Equation 26), the non-zero components of the strain rate tensor are obtained as:

$$D_{12} = D_{21} = \dot{\gamma}/2. \tag{29}$$

Also, the non-zero components of the material spin tensor, W_{12} and W_{21}, are calculated as follows:

$$W_{12} = -W_{21} = \dot{\gamma}/2. \tag{30}$$

Using Equations 23 and 27, the non-zero components of the body spin tensor, Ω_{12} and Ω_{21}, are obtained in the form:

$$\Omega_{12} = -\Omega_{21} = \frac{2\dot{\gamma}}{4+\gamma^2}. \tag{31}$$

Differentiating the components of the rotation associated with Equation 25, the non-zero components of the Eulerian spin tensor are calculated as:

$$\Omega^E_{12} = -\Omega^E_{21} = \frac{\dot{\gamma}}{4+\gamma^2}. \tag{32}$$

The tensor logarithm maps symmetric, positive-definite tensors into symmetric tensors [16]. Let the logarithmic strain tensor, $\ln\mathbf{V}$, be defined by:

$$\ln\mathbf{V} = \sum_i \ln\lambda_i \mathbf{n}_i \otimes \mathbf{n}_i, \tag{33}$$

where λ_i are the principal stretches. Using the definition of the logarithmic strain tensor, together with the principal values of $\mathbf{V}$, Equation 24, the principal components of the logarithmic strain tensor are as follows:

$$(\ln V)^*_{ij} = \begin{bmatrix} \ln\lambda_1 & 0 \\ 0 & \ln\lambda_2 \end{bmatrix}. \tag{34}$$

Transforming the principal components of the logarithmic strain tensor, $(\ln V)^*_{ij}$, to the fixed axes, one can obtain:

$$(\ln V)_{ij} = \frac{1}{c_1}\begin{bmatrix} \lambda_1 \ln \lambda_1 + \lambda_2 \ln \lambda_2 & \ln \lambda_1 - \ln \lambda_2 \\ \ln \lambda_1 - \ln \lambda_2 & \lambda_2 \ln \lambda_1 + \lambda_1 \ln \lambda_2 \end{bmatrix}, \tag{35}$$

where λ_1 and λ_2 are the principal values of the left stretch tensor introduced by Equation 24.

An explicit relation for the material time rate of change of the logarithmic strain tensor, $(\ln \mathbf{V})^{\cdot}$, has been derived in the following form [11]:

$$(\ln \mathbf{V})^{\cdot} = [\mathbf{D} + \mathbf{W} - \mathbf{\Omega}^{E} - \mathbf{V}(\mathbf{\Omega} - \mathbf{\Omega}^{E})\mathbf{V}^{-1}] + [\mathbf{\Omega}^{E}(\ln \mathbf{V}) - (\ln \mathbf{V})\mathbf{\Omega}^{E}]. \tag{36}$$

Using Equation 36, the components of the material time derivative of the logarithmic strain tensor are calculated as follows:

$$(\ln V)^{\cdot}_{ij} = \frac{\dot\gamma}{c_1^3}\begin{bmatrix} \gamma c_1 + 4e_1 & 2c_1 - 2\gamma e_1 \\ 2c_1 - 2\gamma e_1 & -\gamma c_1 - 4e_1 \end{bmatrix}, \tag{37}$$

where:

$$e_1 = \ln[\frac{1}{2}(c_1 + \gamma)]. \tag{38}$$

In order to study the effect of the choice of the corotational rates associated with the material and body spins, J-rate and Z-rate, respectively, the proposed constitutive model is applied to the simple shear problem.

APPLICATION OF JAUMANN COROTATIONAL RATE

Considering the proposed constitutive model (Equation 3) with the Jaumann corotational rate, one can obtain:

$$\overset{J}{\alpha} = h_\alpha \overset{J}{(\ln \mathbf{V})} + \dot h_\alpha \ln \mathbf{V}, \tag{39}$$

where $\overset{J}{\alpha}$ is the J-rate of the back stress tensor. Substitution of $\overset{J}{\alpha}$ from Equation 12 in Equation 39, yields:

$$\dot\alpha = h_\alpha \overset{J}{(\ln \mathbf{V})} + \dot h_\alpha \ln \mathbf{V} + \mathbf{W}\alpha - \alpha\mathbf{W}. \tag{40}$$

In order to solve the shear deformation (Equation 15) with the constitutive Equation 39, the Jaumann corotational rate of the logarithmic strain tensor, $\overset{J}{(\ln \mathbf{V})}$, should be calculated. With the help of Equation 36 and substitution of the spin tensor $\mathbf{\Lambda}$ with $\mathbf{W}$ in Equation 14, one can obtain:

$$\overset{J}{(\ln \mathbf{V})} = [\mathbf{D} + \mathbf{W} - \mathbf{\Omega}^{E} - \mathbf{V}(\mathbf{\Omega} - \mathbf{\Omega}^{E})\mathbf{V}^{-1}] + [(\mathbf{\Omega}^{E} - \mathbf{W})(\ln \mathbf{V}) - (\ln \mathbf{V})(\mathbf{\Omega}^{E} - \mathbf{W})]. \tag{41}$$

Using Equation 41, together with the required tensor components associated with the shear deformation, it is found that:

$$\overset{J}{(\ln V)}_{ij} = \frac{\dot\gamma}{c_1^3}\begin{bmatrix} \gamma c_1 - 2e_1(2+\gamma^2) & 2c_1 + \gamma e_1(2+\gamma^2) \\ 2c_1 + \gamma e_1(2+\gamma^2) & -\gamma c_1 + 2e_1(2+\gamma^2) \end{bmatrix}. \tag{42}$$

Substitution of Equations 30, 35 and 42 in Equation 40, yield the following set of rate equations:

$$\dot\alpha_{11} = \frac{h_\alpha \dot\gamma}{c_1^3}[\gamma c_1 - 2e_1(2+\gamma^2)] + \frac{\dot h_\alpha \gamma e_1}{c_1} + \dot\gamma \alpha_{12} = -\dot\alpha_{22} \tag{43}$$

$$\dot\alpha_{12} = \frac{h_\alpha \dot\gamma}{c_1^3}[2c_1 + \gamma e_1(2+\gamma^2)] + 2\frac{\dot h_\alpha e_1}{c_1} + \dot\gamma \frac{\alpha_{22} - \alpha_{11}}{2}. \tag{44}$$

It is noted that Equations 43 and 44 are coupled in $\dot\alpha_{ij}$, which are the components of the time rate of change of the back stress tensor. If the body is initially unstressed, Equations 43 and 44 can be written as follows:

$$\alpha'_{11} = -\alpha'_{22} = \alpha_{12} + \frac{h_\alpha}{c_1^3}[\gamma c_1 - 2e_1(2+\gamma^2)] + \frac{h'_\alpha \gamma e_1}{c_1}, \tag{45}$$

$$\alpha'_{12} = \alpha_{11} + \frac{h_\alpha}{c_1^3}[2c_1 + \gamma e_1(2+\gamma^2)] + 2\frac{h'_\alpha e_1}{c_1}, \tag{46}$$

where:

$$\alpha'_{11} = \frac{d\alpha_{11}}{d\gamma} = \frac{1}{\dot\gamma}\dot\alpha_{11}, \quad \alpha'_{12} = \frac{d\alpha_{12}}{d\gamma} = \frac{1}{\dot\gamma}\dot\alpha_{12}, \quad h'_\alpha = \frac{dh_\alpha}{d\gamma} = \frac{1}{\dot\gamma}\dot h_\alpha. \tag{47}$$

In general, h_α is a path dependent parameter. Having h_α as a function of γ, it can be substituted into Equations 45 and 46 and the resulting system of differential equations can be solved.

In order to compare the results of this paper with other works, a linear kinematic hardening material with a constant h_α was assumed. For such a material, solving Equations 45 and 46 for the initial conditions of an unstressed body, the components of the back stress tensor, α_{11}, α_{22} and α_{12}, are obtained for the Jaumann corotational rate. By substitution of these components in Equation 8, the normal and shear components of the Cauchy stress are calculated.

APPLICATION OF ZAREMBA COROTATIONAL RATE

Considering the proposed constitutive model (Equation 3) with the Zaremba corotational rate, the following rate constitutive equation may be obtained:

$$\alpha^{Z} = h_\alpha (\ln \mathbf{V})^{Z} + \dot{h}_\alpha \ln \mathbf{V}, \tag{48}$$

where α^{Z} is the Z-rate of the back stress tensor. Substitution of α^{Z} from Equation 13 in Equation 48, yields:

$$\dot{\alpha} = h_\alpha (\ln \mathbf{V})^{Z} + \dot{h}_\alpha (\ln \mathbf{V}) + \mathbf{\Omega}\alpha - \alpha\mathbf{\Omega}. \tag{49}$$

In order to solve, analytically, the shear deformation (Equation 15) with the constitutive Equation 48, the Z-rate of the logarithmic strain tensor, $(\ln \mathbf{V})^{Z}$, should be calculated. With the help of Equation 36 and substitution of the spin tensor $\mathbf{\Lambda}$ with $\mathbf{\Omega}$ in Equation 14, it is obtained that:

$$\begin{aligned}(\ln \mathbf{V})^{Z} &= [\mathbf{D} + \mathbf{W} - \mathbf{\Omega}^{E} - \mathbf{V}(\mathbf{\Omega} - \mathbf{\Omega}^{E})\mathbf{V}^{-1}] \\ &\quad + [(\mathbf{\Omega}^{E} - \mathbf{\Omega})(\ln \mathbf{V}) - (\ln \mathbf{V})(\mathbf{\Omega}^{E} - \mathbf{\Omega})].\end{aligned} \tag{50}$$

Using Equation 50, together with the required tensor components associated with the shear deformation, it is found that:

$$(\ln V)^{Z}_{ij} = \frac{\dot{\gamma}}{c_1^3}\begin{bmatrix} \gamma c_1 - 4e_1 & 2c_1 + 2\gamma e_1 \\ 2c_1 + 2\gamma e_1 & -\gamma c_1 + 4e_1 \end{bmatrix}. \tag{51}$$

Substitution of Equations 31, 35 and 51 in Equation 49, yields the following set of rate equations:

$$\begin{aligned}\dot{\alpha}_{11} &= \frac{h_\alpha \dot{\gamma}}{c_1^3}(\gamma c_1 - 4e_1) + \frac{\dot{h}_\alpha \gamma e_1}{c_1} + \frac{4\dot{\gamma}}{c_1^2}\alpha_{12} \\ &= -\dot{\alpha}_{22},\end{aligned} \tag{52}$$

$$\begin{aligned}\dot{\alpha}_{12} &= \frac{h_\alpha \dot{\gamma}}{c_1^3} 2c_1 + 2\gamma e_1 (2+\gamma^2) + 2\frac{\dot{h}_\alpha e_1}{c_1} \\ &\quad + 2\dot{\gamma}\frac{\alpha_{22} - \alpha_{11}}{c_1^2}.\end{aligned} \tag{53}$$

It is noted that Equations 52 and 53 are coupled in $\dot{\alpha}_{ij}$. If the body is initially unstressed, these equations can be written as follows:

$$\alpha'_{11} = -\alpha'_{22} = \frac{4}{c_1^2}\alpha_{12} + \frac{h_\alpha}{c_1^3}(\gamma c_1 - 4e_1) + \frac{h'_\alpha \gamma e_1}{c_1}, \tag{54}$$

$$\alpha'_{12} = -\frac{4}{c_1^2}\alpha_{11} + \frac{h_\alpha}{c_1^3}(2c_1 + 4e_1) + 2\frac{h'_\alpha e_1}{c_1}, \tag{55}$$

where α'_{11}, α'_{12} and h'_α are defined by Equations 47. According to the previous explanations, h_α are considered constant. Solving Equations 45 and 55, the components of the back stress tensor, α_{11}, α_{22} and α_{12}, are calculated for the Zaremba corotational rate. By substituting the back stress components into the flow rule (Equation 8), the components of the Cauchy stress tensor are determined.

DISCUSSION AND CONCLUSIONS

The current practice in the constitutive modeling of rigid plastic materials is based on relating a corotational rate of the back stress tensor to the strain rate tensor, $\mathbf{D}$. The corotational model presented in the current study renders a different style for constitutive modeling of rigid plastic materials. The presented constitutive model relates the corotational rate of the back stress tensor to the same corotational rate of the logarithmic strain tensor.

Based on the proposed constitutive model, the simple shear problem is solved for an isotropic and kinematic hardening material. Figures 2 and 3 show the normal and shear components of the Cauchy stress for a kinematic hardening material using the corotational Jaumann and Zaremba rates in the simple shear problem. The material properties for kinematic hardening used in this example are $\bar{h}_\alpha = 0.1$ and $k = 1$. The solution with the same material properties are also shown, based on different models [3,5].

In order to study the effect of isotropic hardening, the same example has been solved for a nonlinear isotropic hardening material. Knowing that the isotropic hardening does not affect the normal stress

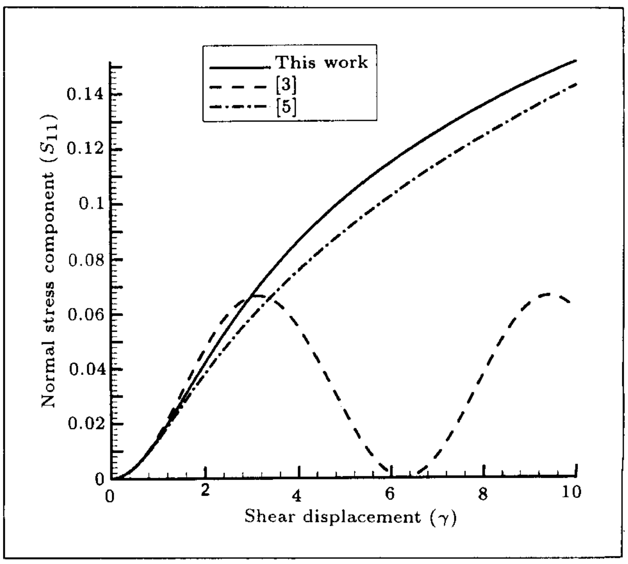

Figure 2. Normal stress component vs shear displacement for a kinematic hardening material with $\bar{h}_\alpha = 0.1, k = 1$.

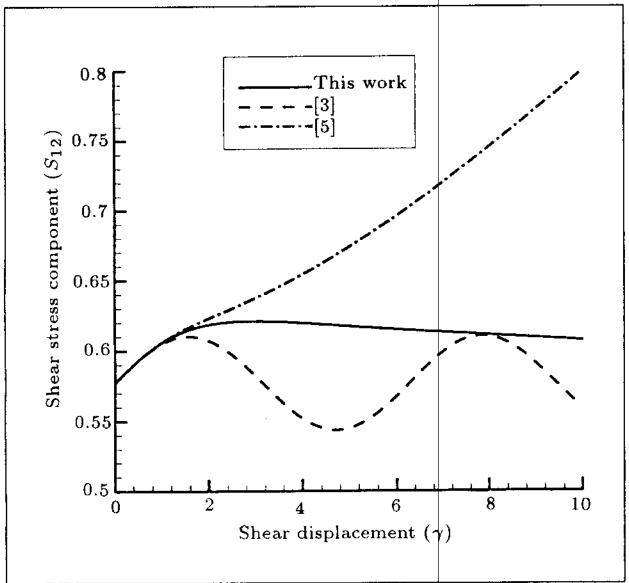

Figure 3. Shear stress component vs shear displacement for a kinematic hardening material with $\bar{h}_\alpha = 0.1, k = 1$.

components in the simple shear problem, Figure 4 shows the shear stress component vs. shear displacement for an isotropic and kinematic hardening material with properties $\bar{h}_\alpha = 0.1$, $k_s = 2.25$ and $c = 2.57$. Also, the solution with the same material properties is shown, based on different models [3,5].

It is noted that the model, based on the strain rate tensor, **D**, gives an oscillatory solution with period 2π for increasing shear displacement, using the Jaumann corotational rate of the back stress tensor. Such a solution is not physically acceptable, but the proposed constitutive model with the Jaumann corotational rate

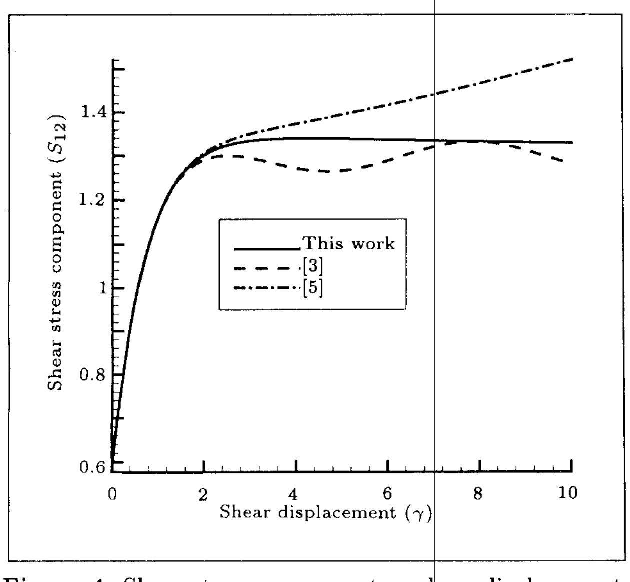

Figure 4. Shear stress component vs shear displacement for an isotropic and kinematic hardening material with $\bar{h}_\alpha = 0.1, k_s = 2.25, c = 2.57$.

does not give an oscillatory solution. Using the Zaremba corotational rate, the results are different from the results obtained by relating the Zaremba rate of the back stress tensor to **D** (Dafalias' model [5]), they are the same with the solutions obtained by Metzger and Dubey in the special case of principal axes [6].

It should be noted that the deviation of solutions presented in [3] and [5] with those predicted by the proposed constitutive model is merely due to the fact that $(\ln \mathbf{V})^{J} \neq \mathbf{D} \neq (\ln \mathbf{V})^{Z}$, in general.

The proposed constitutive model with the Zaremba corotational rate gives the same results as those obtained for the Jaumann rate. Thus, the presented constitutive model yields the same results independent of the choice of the corotational rates of the stress tensor.

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REFERENCES

1. Dienes, J.K. "On the analysis of rotation and stress rate in deforming bodies", *Acta Mechanica*, pp 217-232 (1979).
2. Dubey, R.N. "Choice of tensor rates- a methodology", *SM Archives*, **12**, pp 233-244 (1987).
3. Nagtegaal, J.C. and de Jong, J.E. "Some aspects of non-isotropic work-hardening in finite strain plasticity", *Proceeding of the Workshop on Plasticity of Metals at Finite Strain: Theory, Experimental and Computation*, pp 65-102, Stanford University (1981).
4. Lee, E.H., Mallett, R.L. and Wertheimer, T.B. "Stress analysis for anisotropic hardening in finite deformation plasticity", *ASME, J. Appl. Mech.*, **50**, pp 554-560 (1983).
5. Dafalias, Y.F. "Corotational rates for kinematic hardening at large plastic deformations", *ASME, J. Appl. Mech.*, pp 561-565 (1983).
6. Metzger, D.R. and Dubey, R.N. "Corotational rates in constitutive modeling of elastic-plastic deformation", *Int. J. Plasticity*, pp 341-368 (1987).
7. Reinhardt, W. and Dubey, R.N. "Application of objective rates in mechanical modeling of solids", *ASME, J. Appl. Mech.*, **118**, pp 692-698 (1996).
8. Truesdell, C. "The elements of continuum mechanics", Springer-Verlag, New York Inc, USA (1966).
9. Schieck, B. and Stumpf, H. "The appropriate corotational rate, exact formulation for the plastic spin and constitutive model for finite elasto-plasticity", *Int. J. Solids Structures*, pp 3643-3667 (1995).
10. Hill, R. "Aspects of invariance in solid mechanics", In *Advances in Applied Mechanics*, **18**, Academic Press, New York, USA (1978).

11. Naghdabadi, R., Dubey, R.N. and Heppler, G.R. "On material time derivative of logarithmic strain", *Proceedings of CANCAM'95*, pp 844-845 (1995).

12. Ishikawa, H. "Subsequent yield surface probed from its current center", *Int. J. Plasticity*, **13**, pp 533-549 (1997).

13. Truesdell, C. and Noll, W. "Non-linear field theories of mechanics", In *Handbuch der Physik*, **III/3**, Springer, Berlin, Germany (1965).

14. Green, A.E. and Naghdi, P.M. "A general theory of an elastic-plastic continuum", *Arch. Rat. Mech. Anal.*, **18**, pp 251-281 (1965).

15. Green, A.E. and McInnis, B.C. "Generalized hypoelasticity", *Proc. Roy. Soc.*, Edinburgh, Scotland, UK, A67, part III, pp 220-230 (1967).

16. Heiduschke, K. "The logarithmic strain space description", *Int. J. Solids Structures*, **32**, pp 1047-1062 (1995).