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Eigenvalues of Matrices with Special Patterns Using Symmetry of Graphs

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In this paper, a simple and efficient method is developed for evaluating the eigenvalues of matrices having special patterns. This is achieved by decomposing the matrices into specific forms. The application is extended to calculate the eigenvalues of the Laplacian of graphs having special connectivity properties.

INTRODUCTION

Many engineering problems require the calculation of eigenvalues and eigenvectors of matrices. General methods are available in the literature for such calculations [1-3]. However, for matrices with special patterns, it is beneficial to make use of their extra properties.

In this paper, three simple forms are introduced for decomposing matrices. These forms cover many cases involved in structural engineering. The eigenvalues of the complete matrix are obtained by evaluating the eigenvalues of the submatrices formed after performing the decomposition.

One of the most popular matrices associated with graphs is the Laplacian matrix [4-6]. The eigenvalues and eigenvectors of this matrix can be applied to nodal numbering for bandwidth, profile and frontwidth reduction, graph partitioning and domain decomposition [7-9]. The application of the present method is extended to evaluate the eigenvalues of the Laplacian of graphs having special connectivity properties. Computer programs are developed for the construction of the patterns required for the suggested forms and examples are also studied.

DECOMPOSITION OF MATRICES TO SPECIAL FORMS AND THEIR EIGENVALUES

In this section, an $N \times N$ symmetric matrix $[\mathbf{M}]$ is considered with all entries being real. For three special forms, the eigenvalues of $[\mathbf{M}]$ are obtained using the properties of its submatrices.

Form I

In this case, $[\mathbf{M}]$ has the following pattern:

$$[\mathbf{M}] = \begin{bmatrix} [\mathbf{A}]_{n\times n} & [0]_{n\times n} \\ [0]_{n\times n} & [\mathbf{A}]_{n\times n} \end{bmatrix}_{N\times N}, \tag{1}$$

with $N = 2n$.

Considering the set of eigenvalues of the submatrix $[\mathbf{A}]$ as $\{\lambda\mathbf{A}\}$, the set of eigenvalues of $[\mathbf{M}]$ can be obtained as:

$$\{\lambda\mathbf{M}\} = \{\lambda\mathbf{A}\} \cup \{\lambda\mathbf{A}\}. \tag{2}$$

Since $\det \mathbf{M} = \det \mathbf{A} \times \det \mathbf{A}$, the above relation can easily be proved.

Form I can be generalized to a decomposed form with diagonal submatrices $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \cdots, \mathbf{A}_p$ and the eigenvalues can be calculated as:

$$\{\lambda\mathbf{M}\} = \{\lambda\mathbf{A}_1\} \cup \{\lambda\mathbf{A}_2\} \cup \{\lambda\mathbf{A}_3\} \cup \cdots \cup \{\lambda\mathbf{A}_p\}, \tag{3}$$

and the proof follows from the fact that $\det \mathbf{M} = \det \mathbf{A}_1 \times \det \mathbf{A}_2 \times \det \mathbf{A}_3 \times \cdots \times \det \mathbf{A}_p$.

As an example, consider the matrix $[\mathbf{M}]$ as:

$$[\mathbf{M}] = \left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ \hline 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{array}\right], \quad \text{with } [\mathbf{A}] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Since $\{\lambda\mathbf{A}\} = \{-0.3723, 5.3723\}$, therefore, $\{\lambda\mathbf{M}\} = \{-0.3723, 5.3723, -0.3723, 5.3723\}$.

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Form II

For this case, matrix [**M**] can be decomposed into the following form:

$$[\mathbf{M}] = \begin{bmatrix} [\mathbf{A}]_{n\times n} & [\mathbf{B}]_{n\times n} \\ [\mathbf{B}]_{n\times n} & [\mathbf{A}]_{n\times n} \end{bmatrix}_{N\times N}. \tag{4}$$

The eigenvalues of [**M**] can be calculated as:

$$\{\lambda \mathbf{M}\} = \{\lambda \mathbf{C}\} \cup \{\lambda \mathbf{D}\}, \tag{5}$$

where:

$$[\mathbf{C}] = [\mathbf{A}] + [\mathbf{B}] \text{ and } [\mathbf{D}] = [\mathbf{A}] - [\mathbf{B}], \tag{6}$$

[**C**] and [**D**] are called "condensed submatrices" of [**M**].

Let a 4×4 symmetric matrix be considered as:

$$\begin{bmatrix} L_1 & a & b & c \\ & L_2 & d & e \\ & & L_1 & f \\ Sym. & & & L_2 \end{bmatrix}. \tag{7}$$

In order to have a matrix in Form II, the following should hold:

$$c = d \text{ and } f = a,$$

i.e., [**M**] should be as:

$$[\mathbf{M}] = \left[\begin{array}{cc|cc} L_1 & a & b & d \\ & L_2 & d & e \\ \hline & & L_1 & a \\ Sym. & & & L_2 \end{array}\right]. \tag{8}$$

This matrix is symmetric and its transposed upper triangular submatrix is the same as its lower triangular submatrix and vice versa. The number of entries in its main diagonal is even.

Proof

Expanding the determinant of [**M**] in Equation 8, with respect to the first row and performing the necessary operations, leads to:

$$\det \mathbf{M} = [(L_1 + b)(L_2 + e) - (a + d)(a + d)] \times [(L_1 - b)(L_2 - e) - (a - d)(a - d)].$$

The first bracket is the determinant of [**C**] and the second is that of [**D**]. Therefore:

$$\det \mathbf{M} = \det \mathbf{C} \times \det \mathbf{D}. \tag{9}$$

In order to find the eigenvalues of [**M**], λ should be subtracted from the diagonal entries of [**M**]. In fact, L_1 and L_2 should be replaced by $L_1 - \lambda$ and $L_2 - \lambda$, respectively. Therefore:

$$\{\lambda \mathbf{M}\} = \{\lambda \mathbf{C}\} \cup \{\lambda \mathbf{D}\}. \tag{10}$$

This proof holds for any $2N \times 2N$ matrix with Form II pattern.

As an example, consider the matrix [**M**] as follows:

$$[\mathbf{M}] = \left[\begin{array}{cc|cc} 10 & 15 & 8 & 2 \\ 16 & 20 & 4 & -3 \\ \hline 8 & 2 & 10 & 15 \\ 4 & -3 & 16 & 20 \end{array}\right].$$

This matrix has the pattern of Form II and is decomposed according to Equation 4, leading to:

$$[\mathbf{A}] = \begin{bmatrix} 10 & 15 \\ 16 & 20 \end{bmatrix} \text{ and } [\mathbf{B}] = \begin{bmatrix} 8 & 2 \\ 4 & -3 \end{bmatrix}.$$

Matrices [**C**] and [**D**] are formed using Equation 5 as:

$$[\mathbf{C}] = [\mathbf{A}] + [\mathbf{B}] = \begin{bmatrix} 18 & 17 \\ 20 & 17 \end{bmatrix},$$

and:

$$[\mathbf{D}] = [\mathbf{A}] - [\mathbf{B}] = \begin{bmatrix} 2 & 13 \\ 12 & 23 \end{bmatrix}.$$

For these matrices, the set of eigenvalues are:

$$\{\lambda \mathbf{C}\} = \{35.9459, -0.9459\},$$

$$\{\lambda \mathbf{D}\} = \{-3.8172, 28.8172\},$$

hence:

$$\{\lambda \mathbf{M}\} = \{-0.9459, -3.8172, 28.8172, 35.9459\}.$$

A computer program is developed to transform a given matrix into Form II by row and column interchange operations. Naturally, the original matrix should have all the properties necessary to be transformed into Form II, otherwise the program will be terminated with a message indicating that no such form exists.

The computational time required for such a transformation is not much, e.g. a few seconds for a 100×100 matrix, on a 500MHz Pentium(r) III. However, as will be discussed in the next section, in this paper the symmetry of the graph is used for constructing this form.

Form III

This form has a Form II submatrix, augmented by K rows and columns, as shown in the following:

$$[\mathbf{M}] = \left[\begin{array}{c|c|ccc} [\mathbf{A}] & [\mathbf{B}] & L_1 & \cdots & L_K \\ & & L_1 & \cdots & L_K \\ \hline & & \vdots & \cdots & \vdots \\ [\mathbf{B}] & [\mathbf{A}] & L_1 & \cdots & L_K \\ \hline C(2n+1,1) & C(2n+1,2n) & C(2n+1,2n+1) & \cdots & C(2n+1,2n+K) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ Z(2n+K,1) & Z(2n+K,2n) & Z(2n+K,2n+1) & \cdots & Z(2n+K,2n+K) \end{array}\right], \tag{11}$$

where [**M**] is a $(2n + K) \times (2n + K)$ matrix, with a $2n \times 2n$ submatrix with the pattern of Form II and K augmented columns and rows. The entries of the augmented columns are the same in each column and all the entries of [**M**] are real numbers.

Now [**D**] is obtained as $[\mathbf{D}] = [\mathbf{A}] - [\mathbf{B}]$ and [**E**] is constructed as the following:

$$[\mathbf{E}] = \left[\begin{array}{ccc|ccc} \multicolumn{3}{c|}{} & L_1 & \cdots & L_K \\ \multicolumn{3}{c|}{[\mathbf{A}+\mathbf{B}]} & L_1 & \cdots & L_K \\ \multicolumn{3}{c|}{} & \cdot & \cdots & \cdot \\ \hline & & & L_1 & \cdots & L_K \\ C(2n+1,1)+C(2n+1,n+1) & \cdots & C(2n+1,2n+1) & & \cdots & C(2n+1,2n+K) \\ \cdot & \cdots & \cdot & & \cdots & \cdot \\ Z(2n+K,1)+Z(2n+K,n+1) & \cdots & Z(2n+K,2n+1) & & \cdots & Z(2n+K,2n+K) \end{array}\right]. \tag{12}$$

The set of eigenvalues for [**M**] is obtained as:

$$\{\lambda\mathbf{M}\} = \{\lambda\mathbf{D}\} \cup \{\lambda\mathbf{E}\}, \tag{13}$$

and:

$$\det \mathbf{M} = \det \mathbf{D} \times \det \mathbf{E}.$$

A special form, with only one augmented row and column, is as follows:

$$[\mathbf{M}] = \left[\begin{array}{ccc|c} \mathbf{A} & \mathbf{B} & & L_1 \\ \mathbf{B} & \mathbf{A} & & L_1 \\ & & & \cdot \\ & & & L_1 \\ \hline C(1,1) & \cdots & C(n+1,n+1) & C(2n+1,2n+1) \end{array}\right]_{(2n+1)(2n+1)}, \tag{14}$$

with:

$$[\mathbf{D}] = [\mathbf{A}] - [\mathbf{B}],$$

and:

$$[\mathbf{E}] = \begin{bmatrix} [\mathbf{A}+\mathbf{B}] & L_1 \\ & L_1 \\ C(1,1)+C(n+1,n+1) & C(2n+1,2n+1) \end{bmatrix}. \tag{15}$$

As an example, consider [**M**] as follows:

$$[\mathbf{M}] = \left[\begin{array}{rr|rr|r} -1 & 0.5 & -0.7 & -0.7 & -10.3 \\ 3 & 4 & 0.8 & 0.9 & -10.3 \\ \hline -0.7 & -0.7 & -1 & 0.5 & -10.3 \\ 0.8 & 0.9 & 3 & 4 & -10.3 \\ \hline -11.3 & -12.3 & -13.3 & 1.3 & -5.7 \end{array}\right].$$

Condensed submatrices are calculated using Equation 15 as:

$$[\mathbf{D}] = \begin{bmatrix} -1 & 0.5 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} -0.7 & -0.7 \\ 0.8 & 0.9 \end{bmatrix} = \begin{bmatrix} -0.3 & 1.2 \\ 2.2 & 3.1 \end{bmatrix},$$

and:

$$[\mathbf{E}] = \begin{bmatrix} -1-0.7 & 0.5-0.7 & -10.3 \\ 3+0.8 & 4+0.9 & -10.3 \\ -13.3-11.3 & -12.3+1.3 & -5.7 \end{bmatrix}$$

$$= \begin{bmatrix} -1.7 & -0.2 & -10.3 \\ 3.8 & 4.9 & -10.3 \\ -24.6 & -11 & -5.7 \end{bmatrix}.$$

Eigenvalues for [**D**] and [**E**] are calculated as:

$$\{\lambda\mathbf{D}\} = \{-0.9516, 3.7516\},$$

$$\{\lambda\mathbf{E}\} = \{1.6224, 17.6885, -21.8109\}.$$

Therefore, the eigenvalues of [**M**] are obtained:

$$\{\lambda\mathbf{M}\} = \{-0.9516, 3.7516, 1.6224, 17.6885, -21.8109\}.$$

THREE FORMS FOR LAPLACIAN OF GRAPHS AND THEIR EIGENVALUES

The Laplacian $\mathbf{L}(G) = [l_{ij}]_{N\times N}$ of a graph G is an $N \times N$ matrix defined as follows:

$$\mathbf{L}(G) = \mathbf{D}(G) - \mathbf{A}(G), \tag{16}$$

where:

$$l_{ij} = \begin{cases} -1 & \text{if } n_i \text{ is adjacent to } n_j \\ \text{degree } n_j & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

$\mathbf{D}(G)$ and $\mathbf{A}(G)$ are the degree matrix and adjacency matrix of G, respectively [4-7].

In this section, [**M**] is taken as the Laplacian of G and, for different forms of [**M**], the corresponding graphs are introduced. This correspondence provides efficient means for calculating the eigenvalues of the Laplacian of graphs.

Form I

This form for the Laplacian, in graph theoretical terms corresponds to disjoint graph G. Obviously, the eigenvalues of G will be the union of the eigenvalues for the components of G.

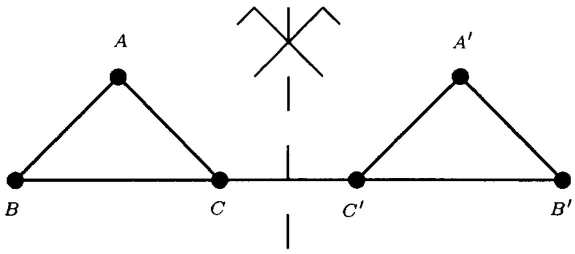

Figure 1. A symmetric graph G.

Form II

Consider the symmetric graph shown in Figure 1.

The nodes A, B and C in the first half have corresponding nodes A', B' and C' in the second half. Two types of nodes can be recognized, namely, 'linked' and 'unlinked' nodes. A 'link' is an edge (member) connecting two halves, e.g. CC'. The nodes A and B are unlinked and C is a linked node.

If A, B, C, A', B' and C' are numbered as 1 to 6, then, the Laplacian of G in Figure 1 can be written as:

$$[\mathbf{L}] = \left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{S} & \mathbf{LI} \\ \hline \mathbf{LI} & \mathbf{S} \end{array}\right]_{N \times N}.$$

The entry -1 in $\mathbf{LI}$ corresponds to the link CC', i.e. $\mathbf{LI}(i,j) = \mathbf{LI}(j,i) = -1$ with $j = i + N/2$.

The condensed matrices $[\mathbf{C}]$ and $[\mathbf{D}]$ in this form are obtained as:

$$[\mathbf{C}] = [\mathbf{S}] + [\mathbf{LI}] \quad \text{and} \quad [\mathbf{D}] = [\mathbf{S}] - [\mathbf{LI}]. \tag{18}$$

Matrix $[\mathbf{C}]$ is the same as $[\mathbf{S}]$ with -1 added to its linked node $\mathbf{C}$ and $[\mathbf{D}]$ is the same as $[\mathbf{S}]$ with $-(-1)$ added to its linked node C'. Therefore, $[\mathbf{C}]$ and $[\mathbf{D}]$ can be viewed as the Laplacian matrices of two subgraphs, C and D, as shown in Figure 2, with one loop being added to D. A decomposition of G is obtained where modifications are made to include the effect of link member CC'.

Therefore, in place of finding the eigenvalues of the Laplacian of G, those of C and D can be calculated and:

$$\{\lambda\mathbf{L}(G)\} = \{\lambda\mathbf{C}(G)\} \cup \{\lambda\mathbf{D}(G)\}. \tag{19}$$

If the subgraphs obtained in this way have symmetry, then further decomposition can be performed.

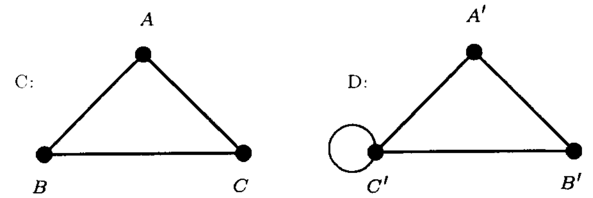

Figure 2. Decomposition of a symmetric graph G into two subgraphs C and D.

Table 1. Subgraphs of G and their eigenvalues.

Subgraphs	Eigenvalues
CC	0, 7, 4, 5
DC	1.4364, 9.8053, 6.3596
CD	1.4364, 9.8053, 4.3987, 6.3596
DD	2.1518, 5.6727, 7, 9.1755

Example

Consider graph G as shown in Figure 3a. The Laplacian of G is a 16×16 matrix, which is put in Form II with suitable ordering. The subgraphs, corresponding to the condensed submatrices $[\mathbf{C}]$ and $[\mathbf{D}]$, are obtained, as shown in Figure 3b. Further decompositions result in the subgraphs illustrated in Figure 3c. The eigenvalues are then calculated for the subgraphs, as provided in Table 1.

The eigenvalues for the Laplacian $[\mathbf{L}]$ of G are obtained as:

$$\{\lambda\mathbf{L}(G)\} = \{0, 7, 4, 5, 1.4364, 9.8053, 6.3596, 1.4364, 9.8053, 4.3987, 6.3596, 2.1518, 5.6727, 7, 9.1755\}.$$

Using the symmetry, the Laplacian matrix of G with a dimension of 16×16, having 256 entries, is reduced to four matrices of dimension 4×4, having counted together 64 entries.

Form III

Consider the Laplacian for Form II and augment it by a row and a column as:

$$[\mathbf{L}] = \left[\begin{array}{c|c|c} \mathbf{S} & \mathbf{LI} & \begin{matrix} a \\ b \\ c \end{matrix} \\ \hline \mathbf{LI} & \mathbf{S} & \begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix} \\ \hline a \quad b \quad c & \cdot \quad \cdot \quad \cdot & z \end{array}\right]. \tag{20}$$

Here, we have no column with equal entries and the only augmented row is the transpose of the augmented column. Similar to the general case, many augmented rows and columns may be included.

Consider the graph shown in Figure 4. The Laplacian is formed as:

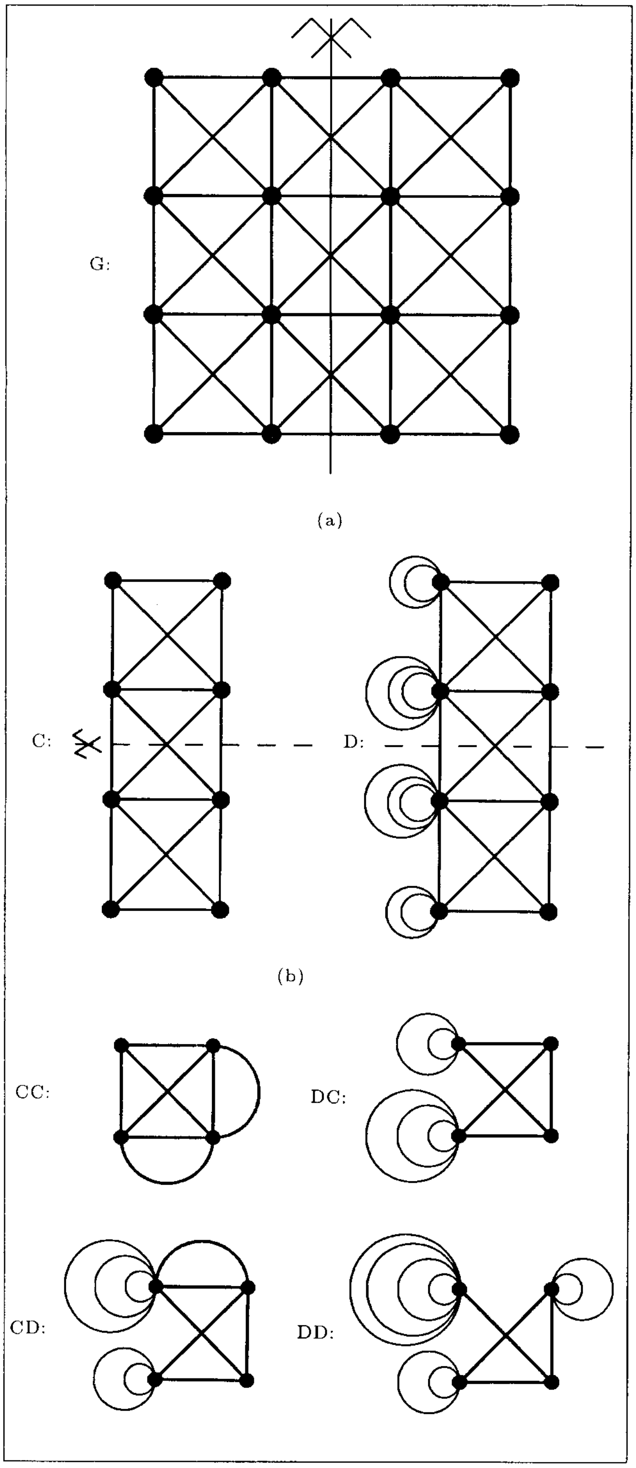

Figure 3. A graph G with symmetry and its decomposition.

$$
[\mathbf{L}] = \left[\begin{array}{cc|c|c} 2 & -1 & -1 & 0 \\ \hline -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ \hline 0 & 0 & -1 & 1 \end{array}\right],
$$

where $[\mathbf{S}] = [2]$ and $[\mathbf{LI}\} = [-1]$.

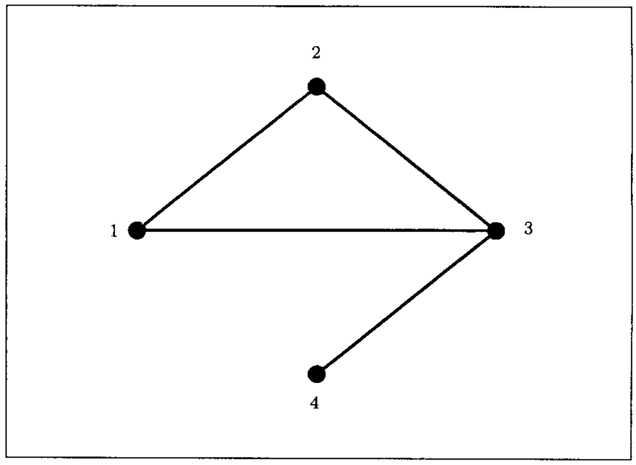

Figure 4. Graph G.

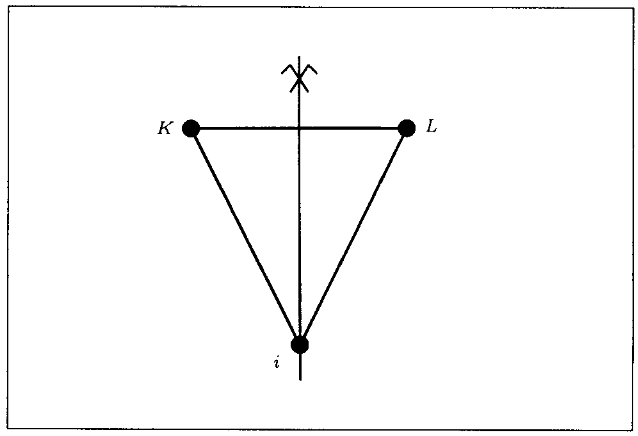

Figure 5. A graph with a symmetric core.

The condensed matrices $[\mathbf{D}]$ and $[\mathbf{E}]$ and their eigenvalues are obtained as:

$$[\mathbf{D}] = [2] - [-1] = [3] \text{ and } \{\lambda\mathbf{D}\} = 3,$$

and:

$$
[\mathbf{E}] = \begin{bmatrix} 2+(-1) & -1 & 0 \\ -1+(-1) & 3 & -1 \\ 0+0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix},
$$

$$\{\lambda\mathbf{E}\} = \{0, 1, 4\}.$$

Hence, $\{\lambda\mathbf{L}(G)\} = \{0, 1, 4, 3\}$.

In Form III, the matrix $[\mathbf{L}]$ contains a submatrix,

$$
[\mathbf{H}] = \begin{bmatrix} \mathbf{S} & \mathbf{LI} \\ \mathbf{LI} & \mathbf{S} \end{bmatrix},
$$

corresponding to the symmetric core of the graph. As an example, for the graph shown in Figure 5, the edge KL is the symmetric core.

Node i is linked to nodes K and L in a symmetric manner. K and L are called "in-core" nodes and i is known as the "out-of-core" node.

In order to construct Form III, the in-core and out-of-core nodes should be ordered. In-core nodes are numbered in a suitable manner for Form II, followed by an arbitrary ordering of the out-of-core nodes.

Example 1

Let G be a graph, as shown in Figure 6a and, then, decompose and order the nodes of the subgraphs as illustrated in Figure 6b.

The Laplacian of the graph G is formed as:

$$[\mathbf{L}] = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

The core in Form II is shown in the upper-left part of the matrix. Augmented rows and columns are illustrated by separating lines.

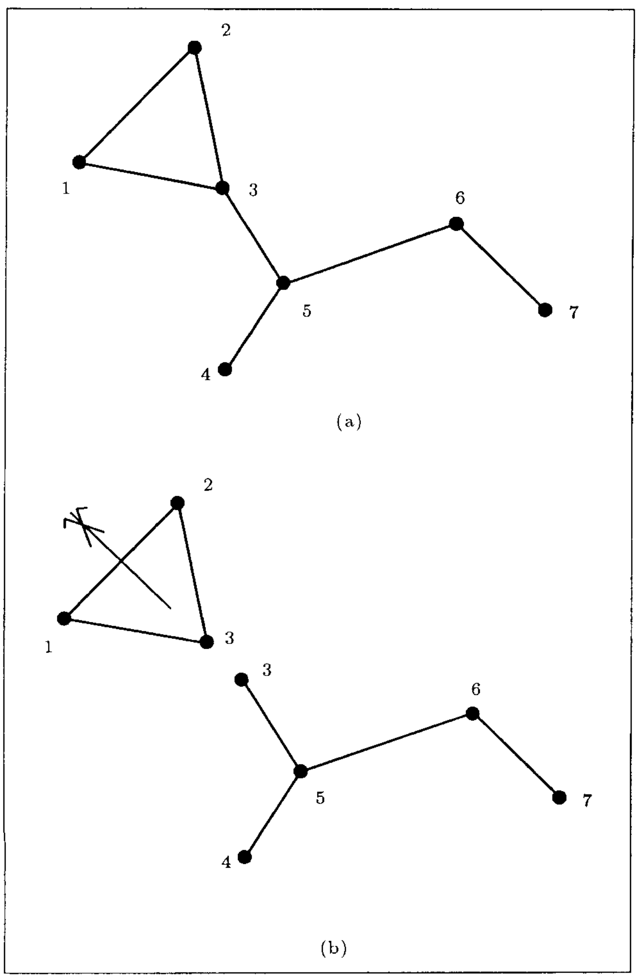

Figure 6. Graph G and its decomposition.

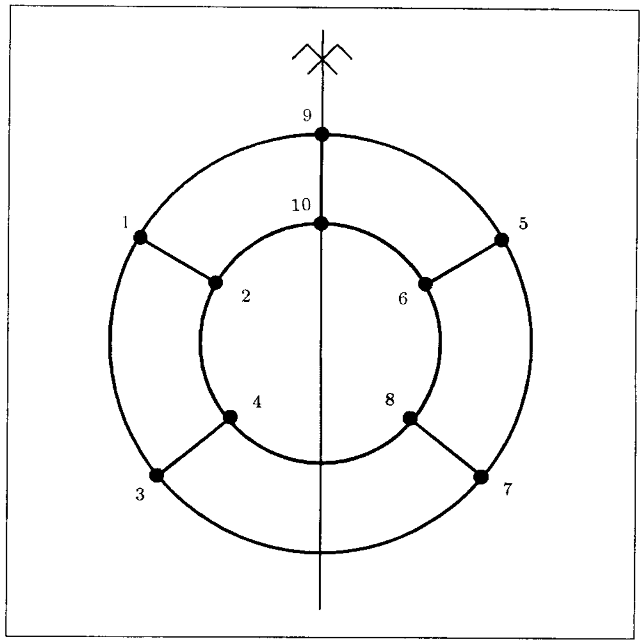

Figure 7. A graph with 10 nodes.

Example 2

Consider the graph shown in Figure 7 with 10 nodes. This graph has a core in Form II with nodes 1-8. Nodes 9 and 10 are connected to this core and the entire graph has Form III.

The 10×10 Laplacian matrix for the graph of Figure 7 is:

$$\mathbf{L} = \left[\begin{array}{cccc|cccc|cc} 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 3 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 3 & -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 3 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & 3 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 3 & -1 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 \end{array}\right]$$

This matrix has Form III and can be decomposed into its cores with the following submatrices:

$$\mathbf{D} = \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & 0 & -1 & 0 & -1 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 & 3 & -1 \\ 0 & -2 & 0 & 0 & -1 & 3 \end{bmatrix}.$$

Now, the eigenvalues can be calculated as:

$$\{\lambda \mathbf{D}\} = \{3.3820, 3.6180, 1.3820, 5.6180\},$$

$$\{\lambda \mathbf{E}\} = \{5.6180, 0, 2, 3.6180, 1.3820, 3.3820\},$$

$$\{\lambda \mathrm{L}\} = \{\lambda_{\mathbf{D}}\} \cup \{\lambda_{\mathbf{E}}\},$$

$$\{\lambda\} = \{5.6180, 0, 2, 3.6180, 1.3820, 3.3820, 3.3820, 3.6180, 1.3820, 5.6180\}.$$

Therefore, a graph with 10 nodes, having a 10×10 Laplacian matrix with 100 entries, is decomposed into two 4×4 and 6×6 matrices, having 52 entries.

CONCLUDING REMARKS

Civil engineering structures contain a large number of members and nodes with many symmetries. The present method simplifies the numerical operations required for calculating the eigenvalues of the corresponding matrices.

Applications of these forms can be extended to include different civil engineering problems, where eigenvalues and eigenvectors of matrices are involved. The present method can also be employed in other fields of engineering where eigenproblems are encountered.

For future developments, other useful forms should be constructed and their properties explored.

REFERENCES

1. Bathe, K.J. and Wilson, E.L., *Numerical Methods for Finite Element Analysis*, Englewood Cliffs, N.J., Prentice Hall (1976).
2. Jennings, A. and McKeown, J.J., *Matrix Computation*, John Wiley and Sons (1992).
3. Livesley, R.K., *Mathematical Methods for Engineers*, Ellis Horwood Ltd, Chichester, UK (1989).
4. Biggs, N.E., *Algebraic Graph Theory*, Cambridge University Press, 2nd Edition, Cambridge (1993).
5. Kaveh, A., *Optimal Structural Analysis*, Research Studies Press, John Wiley, UK (1997).
6. Paulino, G.H., Menezes, I.F.M., Gattas, M. and Mukherjee, S. "Node and element resequencing using Laplacian of finite element graph: Part I- general concepts and algorithms", *Int. J. Numer. Meths Engng*, **37**, pp 1511-1530 (1994).
7. Mohar, B. "Eigenvalues, diameter and mean distance in graphs", *Graphs and Combinatorics*, **7**, pp 53-64 (1991).
8. Mohar, B. "The Laplacian spectrum of graphs, in a book entitled", *Graph Theory, Combinatorics and Applications*, Y. Alavi, Ed., John Wiley and Sons, New York, pp 871-898 (1991).
9. Simon, H.D. "Partitioning of un-structured problems for parallel processing", *Computing Systems in Engineering*, **2**, pp 135-148 (1991).