Hilbert Transform of Schwartz Distributions

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In this paper, an intrinsic description of the space \( H(D) \) and its topology is presented.

INTRODUCTION AND PRELIMINARIES

Let \( D(R) \) be the Schwartz space of \( C^\infty \) functions with compact support on \( R \) and let \( H(D) \) be the space of all \( C^\infty \) functions defined on \( R \) for which every element is the Hilbert transform of an element in \( D(R) \) that is:

\[
H(D) = \{ \psi : \psi(x) = p.v. \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x} \, dt \quad \text{where the integral is defined in the Cauchy principal-value sense.} \}
\]

where \( \phi \in D(R) \),

\[
\gamma_m(\phi) = \left( \int_{-\infty}^{\infty} |\phi^{(m)}(x)|^p \, dx \right)^{\frac{1}{p}}, \quad m = 0, 1, 2, \ldots .
\]

Since \( \gamma_0 \) is a norm, the sequence of semi-norms, \( \{\gamma_m\}_{m=0}^{\infty} \), is separating [3, p 8]. The space \( D_{L_p}(R) \) is a complete countably multi-normed space and \( D(R) \) is dense in it [4, p 199]. It is proved in [1] that the Hilbert transform \( H : D_{L_p}(R) \rightarrow D_{L_p}(R) \) defined by:

\[
\phi \xrightarrow{H} p.v. \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x} \, dt,
\]

is a linear homomorphism with its inverse given by:

\[
H^{-1} \phi = (-\frac{1}{\pi^2})H\phi \quad \text{for all} \quad \phi \in D_{L_p}(R).
\]

However, he did not describe the space \( H(D) \) and its topology in an intrinsic way. In this paper, an intrinsic description of the space \( H(D) \) and its topology is given, thereby providing a solution to an open problem posed by Pandey [2, p 90]. From the definition \( \psi(x) \) in Equation 1, it is shown in [1] that:

\[
\psi^{(k)}(x) = p.v. \int_{-\infty}^{\infty} \frac{\phi^{(k)}(t)}{t-x} \, dt = p.v. \int_{-a}^{a} \frac{\phi^{(k)}(t)}{t-x} \, dt,
\]

where, the support of \( \phi \) is contained in \([-a, a]\).

An infinitly differentiable functions \( \phi(x) (-\infty < x < \infty) \) is said to belong to the testing functions space

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This is due to the singularity of its kernel, but it can be extended as a holomorphic function, \( \psi(z) \), which is analytic outside the support of \( \phi \).

Before the main theorem is proven, some lemmas will be given, which will be used in the sequel.

**Lemma 1**

Let \( \{ \phi_j \}_{j=1}^\infty \) be a sequence of functions tending to zero in \( D_{L_p}(R) \) as \( j \to \infty \), that is:

\[
\gamma_k(\phi_j) \to 0 \quad \text{as} \quad j \to \infty \quad \text{for each} \quad k = 0, 1, 2, \ldots,
\]

then, for each \( k = 0, 1, 2, \ldots \):

\[
\phi_j^{(k)}(x) \to 0 \quad \text{as} \quad j \to \infty \quad \text{uniformly for all} \quad x \in R.
\]

**Proof**

This result is proven in [4,6]. A very simple proof can be given as follows. For \( \delta \in (D_{L_p}(R))' \),

\[
\phi^{(k)}(x) = \langle \delta(t), \phi^{(k)}(x - t) \rangle \quad \text{for all} \quad \phi \in D_{L_p}(R),
\]

(8)

Now, there exists a constant, \( c > 0 \) and a non-negative integer, \( r \), satisfying [7, p 8-19]:

\[
|\langle \delta(t), \phi^{(k)}(x - t) \rangle| \leq c\gamma'_r(\phi^{(k)}(t)),
\]

(9)

where:

\[
\gamma'_r(\phi) = \max(\gamma_0(\phi), \gamma_1(\phi), \ldots, \gamma_r(\phi)),
\]

and:

\[
\gamma'_r(\phi) = \gamma_r(\phi).
\]

Therefore:

\[
|\phi_j^{(k)}(x)| \leq c\gamma'_r(\phi_j^{(k)}(t)) \to 0 \quad \text{as} \quad j \to \infty,
\]

that is, for each \( k = 0, 1, 2, \ldots \):

\[
\phi_j^{(k)}(x) \to 0 \quad \text{as} \quad j \to \infty \quad \text{independently of} \quad x.
\]

**Lemma 2** [1]

Let \( \phi(t) \in D \). Then as \( y \to 0^+ \):

i) \( \int_{-\infty}^{\infty} \frac{\phi(t)(t - z)}{(t - z)^2 + y^2} dt \to p.v. \int_{-\infty}^{\infty} \frac{\phi(t)}{t - z} dt \),

ii) \( \frac{1}{2} \int_{-\infty}^{\infty} \frac{\phi(t)(y - z)}{(t - z)^2 + y^2} dt \to \phi(x) \) in \( D_{L_p}(R) \), \( p > 1 \).

**Lemma 3**

Let \( \phi \in D \). Then, as \( y \to 0^+ \):

i) \( \int_{-\infty}^{\infty} \frac{\phi(t)}{t - z} dt \to p.v. \int_{-\infty}^{\infty} \frac{\phi(t)}{t - z} dt + i\pi\phi(x) \), uniformly for all \( x \in R \) and as \( y \to 0^+ \),

ii) \( \int_{-\infty}^{\infty} \frac{\phi(t)}{t - z} dt \to p.v. \int_{-\infty}^{\infty} \frac{\phi(t)}{t - z} dt - i\pi\phi(x) \), uniformly for all \( x \in R \).

**Proof**

One has:

\[
\int_{-\infty}^{\infty} \frac{\phi(t)}{t - z} dt = \int_{-\infty}^{\infty} \frac{\phi(t)(t - x)}{(t - z)^2 + y^2} dt + i \int_{-\infty}^{\infty} \frac{y\phi(t)}{(t - x)^2 + y^2} dt.
\]

(10)

Now the results (i) and (ii) follow using Lemmas 1 and 2.

**INTRINSIC DEFINITION OF THE SPACE \( H(D) \) AND ITS TOPOLOGY**

**Definition 1**

A function \( \psi(z) \) defined on the complex plane belongs to the space \( \Psi \) iff the following four properties hold:

(\( P_1 \)): \( \psi(z) \) is analytic outside some closed interval \([a, b] \), depending upon \( \psi \);

(\( P_2 \)): \( \psi^{(k)}(z) = o\frac{1}{|z|^k} \), \( |z| \to \infty \), for each fixed \( k = 0, 1, 2, \ldots \);

(\( P_3 \)): For each fixed \( k = 0, 1, 2, \ldots \), \( \psi^{(k)}(x + iy) \) converges uniformly for all \( x \in R \) as \( y \to 0^+ \);

(\( P_4 \)): \( \psi(x) = \psi_+(x) + \psi_-(x) \) where \( \psi_+(x) = \lim_{y \to 0^+} \psi(x + iy) \) and \( \psi_-(x) = \lim_{y \to 0} -\psi(x + iy) \).

**Theorem 1**

A necessary and sufficient condition that a function \( \psi(z) \) defined on the complex plane belongs to the space \( \Psi \), is that there exists a function \( \phi(t) \in D \) satisfying:

\[
\psi(z) = \int_{-\infty}^{\infty} \frac{\phi(t)}{t - z} dt, \quad Imz \neq 0
\]

\[
= p.v. \int_{-\infty}^{\infty} \frac{\phi(t)}{t - z} dt, \quad Imz = 0.
\]

**Proof (Necessity)**

If \( \psi(z) \in \Psi \), then in view of Conditions (\( P_1 \)) and (\( P_2 \)), \( \psi(x + iy) \) as a function of \( x \) belongs to \( D_{L_p}(R) \) for a fixed \( y \neq 0 \). In view of Conditions (\( P_1 \)) and (\( P_2 \))
it follows that if \( \{y_n\}_{n=1}^\infty \) is an arbitrary sequence of positive real numbers tending to zero then:

\[
\|\psi^{(k)}(x + iy_n) - \psi^{(k)}(x + iy_m)\|_p \to 0
\]

as \( m, n \to \infty \),

independently of each other. Therefore, \( \{\psi(x + iy_n)\}_{n=1}^\infty \) is a Cauchy sequence in \( L_p(R) \), \( p > 1 \). Since \( D_L_p(R) \) is complete it follows that there exist a function \( \psi_+(x) \in D_L_p(R) \) such that \( \lim_{n \to \infty} \psi(x + iy_n) = \psi_+(x) \) in \( D_L_p(R) \), \( p > 1 \). Since \( \{y_n\} \) is an arbitrary sequence of positive numbers tending to zero, it follows that there exists a function \( \psi_+(x) \in D_L_p(R) \) such that:

\[
\lim_{y \to 0^+} \psi(x + iy) = \psi_+(x) \quad \text{in} \quad D_L_p. \tag{11}
\]

Similarly, it can be shown that there exists a function \( \psi_-(x) \in D_L_p(R) \) such that:

\[
\lim_{y \to 0^-} \psi(x + iy) = \psi_-(x) \quad \text{in} \quad D_L_p. \tag{12}
\]

Now using Condition \((P_4)\) and Equations 11 and 12, it follows that:

\[
\psi(x) = \frac{\psi_+(x) + \psi_-(x)}{2} \in D_L_p, \quad p > 1.
\]

From Lemma 1 and Equations 11 and 12, it follows that:

\[
\lim_{y \to 0^*} \psi(x \pm i0) = \psi_\pm(x) \quad \text{uniformly for all} \quad x \in R,
\]

and:

\[
\lim_{y \to 0^*} \psi(x + iy) = \psi_\pm(x) \quad \text{uniformly for all} \quad x \in R.
\]

Since \( \psi(z) \) is analytic outside a closed interval \([a, b]\), it follows that \( \psi_+(x) = \psi_-(x) = 0 \) outside \([a, b]\) and therefore belongs to \( D \). Using Cauchy's integral theorem and the technique used in [1], it can be shown that for \( \varepsilon > 0 \):

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi(t + i\varepsilon)}{t - z} dt = \psi(z + i\varepsilon), \quad Imz > 0,
\]

\[
= 0, \quad Imz < 0. \tag{13}
\]

Let \( \varepsilon \to 0^+ \) in Equation 13, it is deduced that:

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi_+(t)}{t - z} dt = \psi(z), \quad Imz > 0,
\]

\[
= 0, \quad Imz < 0. \tag{14}
\]

Similarly, it can be shown that:

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi_-(t)}{t - z} dt = -\psi(z), \quad Imz < 0,
\]

\[
= 0, \quad Imz > 0. \tag{15}
\]

Combining Equations 14 and 15, the following is obtained:

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi_+(t) - \psi_-(t)}{t - z} dt = \psi(z), \quad Imz \neq 0. \tag{16}
\]

Let \( \phi(t) = \frac{\psi_+(t) - \psi_-(t)}{2\pi i} \). Clearly \( \phi(t) \in D \) and thus:

\[
\psi(z) = \int_{-\infty}^{\infty} \frac{\phi(t)}{t - z} dt, \quad Imz \neq 0. \tag{17}
\]

In view of Lemmas 2 and 3 and Condition \((P_4)\), it follows that:

\[
\psi(x) = p.v. \int_{-\infty}^{\infty} \frac{\phi(t)}{t - x} dt,
\]

that is:

\[
\psi(z) = p.v. \int_{-\infty}^{\infty} \frac{\phi(t)}{t - z} dt, \quad Imz = 0. \tag{18}
\]

The proof for necessity follows from Equations 17 and 18.

**Proof (Sufficiency)**

If there exists a function \( \phi \in D \), which vanishes outside a closed interval satisfying:

\[
\psi(z) = \int_{-\infty}^{\infty} \frac{\phi(t)}{t - z} dt, \quad Imz \neq 0
\]

\[
= p.v. \int_{-\infty}^{\infty} \frac{\phi(t)}{t - x} dt, \quad Imz = 0,
\]

then, as proved in [1],

\[
\psi^{(k)}(z) = \int_{-\infty}^{\infty} \frac{\phi^{(k)}(t)}{t - z} dt, \quad Imz \neq 0,
\]

\[
= p.v. \int_{-\infty}^{\infty} \frac{\phi^{(k)}(t)}{t - x} dt, \quad Imz = 0.
\]

Clearly \( \psi \) satisfies Conditions \((P_1)\) and \((P_2)\). Conditions \((P_3)\) and \((P_4)\) are also satisfied in view of Lemma 3. This completes the proof of Theorem 1.

Theorem 1 shows that there is a one to one correspondence between the space \( \Psi \) and the space \( H(D) \). Therefore, it can be defined that the space \( H(D) \) is a genuinely intrinsic way as follows.

A \( C^\infty \) function \( \psi(z) \) belongs to \( H(D) \) iff there exists a holomorphic function \( \psi(z) \) satisfying Conditions \((P_1)\), \((P_2)\), \((P_3)\) and \((P_4)\). In other words \( \psi(x) \in H(D) \) iff it is the average of the upper and lower limit of a holomorphic function satisfying Conditions \((P_1)\), \((P_2)\) and \((P_3)\). That is, \( \psi(x) \) can be extended uniquely as a holomorphic function satisfying Conditions \((P_1)\), \((P_2)\), \((P_3)\) and \((P_4)\).

The convergence of a sequence \( \{\psi_n(x)\}_{n=1}^{\infty} \) to zero in \( H(D) \) can be defined in an intrinsic way as follows.

A sequence \( \{\psi_n\}_{n=1}^{\infty} \) in \( H(D) \) converges to zero in \( H(D) \) iff
\( A_1 \) The associated functions \( \psi(x) \) in accordance with Theorem 1 are analytic outside a closed interval \([a, b]\) or else \( \psi(x) \) is analytic outside a fixed closed interval \([a, b]\).

\( A_2 \) \( \psi(x) \to 0 \) in \( D_{L_p} \) as \( \mu \to \infty \).

Clearly if \( \{\phi(x)\}_{\mu=1}^{\infty} \) is a sequence in \( D \) tending to zero in \( D \) as \( \mu \to \infty \) and:

\[
\psi(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x} \, dt, \quad (19)
\]

and:

\[
\psi(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x} \, dt, \quad \text{Im} z \neq 0, \quad (20)
\]

then \( \psi(x) \) is analytic outside the closed interval \([a, b]\) (see [8]) and:

\[
\|\psi^{(k)}\|_{p} \leq C \|\phi^{(k)}\|_{p} \to 0 \quad \text{as} \quad \mu \to \infty.
\]

Therefore, Conditions \( A_1 \) and \( A_2 \) are satisfied. If Conditions \( A_1 \) and \( A_2 \) are assumed then, there exists a closed interval \([a, b]\) containing the supports of all \( \phi(x) \). From Equations 19 and 20 and the fact that \(-\frac{1}{\pi^2}H^2 = I\), it follows that:

\[
\phi(x) = -\frac{1}{\pi^2} \text{p.v.} \int_{-\infty}^{\infty} \psi(x) \, dt.
\]

Therefore:

\[
\|\phi^{(k)}(x)\|_{p} \leq \frac{1}{\pi^2}C \|\psi^{(k)}\|_{p} \to 0 \quad \text{as} \quad \mu \to \infty,
\]

that is, \( \phi(x) \to 0 \) in \( D_{L_p} \) as \( \mu \to \infty \). Thus, by Lemma 1, \( \phi(x) \to 0 \) uniformly for all \( x \in \mathbb{R} \) as \( \mu \to \infty \). By Condition \( A_2 \), all \( \phi(x) \) have supports contained in a fixed interval \([a, b]\).

Therefore, if \( \{\psi(x)\}_{\mu=1}^{\infty} \to 0 \) in \( H(D) \) as \( \mu \to \infty \), then, it has been proven that:

\[
\phi(x) \to 0 \quad \text{in} \quad H(D) \quad \text{as} \quad \mu \to \infty.
\]

Thus, Conditions \( A_1 \) and \( A_2 \) together describe intrinsically the convergence of a sequence \( \{\psi(x)\}_{\mu=1}^{\infty} \) to zero in \( H(D) \) as \( \mu \to \infty \).

REFERENCES


