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Research Note

Hilbert Transform of Schwartz Distributions

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In this paper, an intrinsic description of the space $H(D)$ and its topology is presented.

INTRODUCTION AND PRELIMINARIES

Let $D(R)$ be the Schwartz space of C^∞ functions with compact support on R and let $H(D)$ be the space of all C^∞ functions defined on R for which every element is the Hilbert transform of an element in $D(R)$ that is:

$$H(D) = \{\psi : \psi(x) = p.v. \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x} dt = H[\phi](x); \phi \in D(R)\}, \tag{1}$$

where the integral is defined in the Cauchy principle-value sense. Introducing an appropriate topology in $H(D)$, Pandey [1] defined the Hilbert transform Hf of $f \in (D(R))'$ as an element of $(H(D))'$ by the following relation:

$$< Hf, \phi >=< f, -H\phi > \quad \text{for all} \quad \phi \in H(D), \tag{2}$$

and then, with an appropriate interpolation he proved that:

$$(-\frac{1}{\pi^2})H^2 f = f \quad \text{for all} \quad f \in (D(R))'. \tag{3}$$

However, he did not describe the space $H(D)$ and its topology in an intrinsic way. In this paper, an intrinsic description of the space $H(D)$ and its topology is given, thereby providing a solution to an open problem posed by Pandey [2, p 90]. From the definition $\psi(x)$ in Equation 1, it is shown in [1] that:

$$\psi^{(k)}(x) = p.v. \int_{-\infty}^{\infty} \frac{\phi^{(k)}(t)}{t-x} dt = p.v. \int_{-a}^{a} \frac{\phi^{(k)}(t)}{t-x} dt, \tag{4}$$

where, the support of ϕ is contained in $[-a, a]$.

An infinitly differentiable functions $\phi(x)$ $(-\infty < x < \infty)$ is said to belong to the testing functions space $D_{L_p}(R)$ iff:

$$\gamma_m(\phi) = \left(\int_{-\infty}^{\infty} |\phi^{(m)}(x)|^p dx \right)^{\frac{1}{p}} < \infty, \quad m = 0, 1, 2, \dots . \tag{5}$$

Since γ_0 is a norm, the sequence of semi-norms, $\{\gamma_m\}_{m=0}^{\infty}$, is separating [3, p 8]. The space $D_{L_p}(R)$ is a complete countably multi-normed space and $D(R)$ is dense in it [4, p 199]. It is proved in [1] that the Hilbert transform $H : D_{L_p}(R) \longrightarrow D_{L_p}(R)$ defined by:

$$\phi \xrightarrow{H} p.v. \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x} dt, \tag{6}$$

is a linear homomorphism with its inverse given by:

$$H^{-1}\phi = (-\frac{1}{\pi^2})H\phi \quad \text{for all} \quad \phi \in D_{L_p}(R). \tag{7}$$

Since $D(R)$ is dense in $D_{L_p}(R)$, it follows that the space $H(D)$, with the subspace topology on it, is dense in $D_{L_p}(R)$. The space $H(D)$ and its topology have not been yet described in an intrinsic way.

In [1] Pandey and Chaudhry developed the theory of the Hilbert transform of Schwartz distribution space, $(D_{L_p})'$, $p > 1$, which coincides with the corresponding theory for the Hilbert transform developed by Schwartz [4] by using the technique of convolution. However, the technique used by Pandey and Chaudhry in [1] is much simpler and can be easily used by applied scientists. In [2] Pandey extended the Hilbert transform to Schwartz distribution space, D', but he did not describe the space $H(D)$ and its topology in an intrinsic way. The object of this paper is to describe the space $H(D)$ and its topology in an intrinsic way by a method analogous to that used by Ehrenpreis [5] for the extension of the Fourier transform to the Schwartz distribution space, D'. It may however be noted that the inverse Fourier transform of $\phi \in D$ can be extended as an entire function, whereas the Hilbert transform of $\phi \in D$ cannot be extended as an entire function.

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This is due to the singularity of its kernel, but it can be extended as a holomorphic function, $\psi(z)$, which is analytic outside the support of ϕ.

Before the main theorem is proven, some lemmas will be given, which will be used in the sequel.

Lemma 1

Let $\{\phi_j\}_{j=1}^{\infty}$ be a sequence of functions tending to zero in $D_{L_p}(R)$ as $j \to \infty$, that is:

$$\gamma_k(\phi_j) \longrightarrow 0 \text{ as } j \to \infty \quad \text{for each} \quad k = 0, 1, 2, \ldots,$$

then, for each $k = 0, 1, 2, \ldots$:

$$\phi_j^{(k)}(x) \longrightarrow 0 \text{ as } j \to \infty \quad \text{uniformly for all} \quad x \in R.$$

Proof

This result is proven in [4,6]. A very simple proof can be given as follows. For $\delta \in (D_{L_p}(R))'$,

$$\phi^{(k)}(x) = <\delta(t), \phi^{(k)}(x-t)> \quad \text{for all} \quad \phi \in D_{L_p}(R). \tag{8}$$

Now, there exists a constant, $c > 0$ and a non-negative integer, r, satisfying [7, p 8-19]:

$$|<\delta(t), \phi^{(k)}(x-t)>| \leq c\gamma_r'\left(\phi^{(k)}(x-t)\right)$$

or:

$$|\phi^{(k)}(x)| \leq c\gamma_r'(\phi^{(k)}(t)), \tag{9}$$

where:

$$\gamma_r'(\phi) = \max(\gamma_0(\phi), \gamma_1(\phi), \ldots, \gamma_r(\phi)),$$

and:

$$\gamma_0'(\phi) = \gamma_0(\phi).$$

Therefore:

$$|\phi_j^{(k)}(x)| \leq c\gamma_r'(\phi_j^{(k)}) \longrightarrow 0 \quad \text{as} \quad j \longrightarrow \infty,$$

that is, for each $k = 0, 1, 2, \ldots$:

$$\phi_j^{(k)}(x) \longrightarrow 0 \quad \text{as} \quad j \longrightarrow \infty \quad \text{independently of } x.$$

Lemma 2 [1]

Let $\phi(t) \in D$. Then as $y \longrightarrow 0^+$:

i) $\int_{-\infty}^{\infty} \frac{\phi(t)(t-x)}{(t-x)^2+y^2}dt \longrightarrow p.v. \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x}dt$,

ii) $\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\phi(t)(y-x)}{(t-x)^2+y^2}dt \longrightarrow \phi(x)$ in $D_{L_p}(R)$, $p > 1$.

Lemma 3

Let $\phi \in D$. Then, as $y \longrightarrow 0^+$:

i) $\int_{-\infty}^{\infty} \frac{\phi(t)}{t-z}dt \longrightarrow p.v. \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x}dt + i\pi\phi(x)$ uniformly for all $x \in R$ and as $y \longrightarrow 0^-$,

ii) $\int_{-\infty}^{\infty} \frac{\phi(t)}{t-z}dt \longrightarrow p.v. \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x}dt - i\pi\phi(x)$ uniformly for all $x \in R$.

Proof

One has:

$$\int_{-\infty}^{\infty} \frac{\phi(t)}{t-z}dt = \int_{-\infty}^{\infty} \frac{\phi(t)(t-x)}{(t-z)^2+y^2}dt + i\int_{-\infty}^{\infty} \frac{y\phi(t)}{(t-x)^2+y^2}dt. \tag{10}$$

Now the results (i) and (ii) follow using Lemmas 1 and 2.

INTRINSIC DEFINITION OF THE SPACE $H(D)$ AND ITS TOPOLOGY

Definition 1

A function $\psi(z)$ defined on the complex plane belongs to the space Ψ iff the following four properties hold:

(P_1): $\psi(z)$ is analytic outside some closed interval $[a, b]$, depending upon ψ;

(P_2): $\psi^{(k)}(z) = o(\frac{1}{|z|})$, $|z| \longrightarrow \infty$, for each fixed $k = 0, 1, 2, \ldots$;

(P_3): For each fixed $k = 0, 1, 2, \ldots$, $\psi^{(k)}(x+iy)$ converges uniformly for all $x \in R$ as $y \longrightarrow 0^{\pm}$;

(P_4): $\psi(x) = \frac{\psi_+(x)+\psi_-(x)}{2}$ where $\psi_+(x) = \lim_{y\to 0} \psi(x+iy)$ and $\psi_-(x) = \lim_{y \to 0} -\psi(x+iy)$.

Theorem 1

A necessary and sufficient condition that a function $\psi(z)$ defined on the complex plane belongs to the space Ψ, is that there exists a function $\phi(t) \in D$ satisfying:

$$\psi(z) = \int_{-\infty}^{\infty} \frac{\phi(t)}{t-z}dt, \quad Imz \neq 0$$

$$= p.v. \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x}dt, \quad Imz = 0.$$

Proof (Necessity)

If $\psi(z) \in \Psi$, then in view of Conditions (P_1) and (P_2), $\psi(x+iy)$ as a function of x belongs to $D_{L_p}(R)$ for a fixed $y \neq 0$. In view of Conditions (P_1) and (P_2)

it follows that if $\{y_n\}_{n=1}^{\infty}$ is an arbitrary sequence of positive real numbers tending to zero then:

$$\|\psi^{(k)}(x+iy_n) - \psi^{(k)}(x+iy_m)\|_p \longrightarrow 0$$

$$\text{as} \quad m, n \longrightarrow \infty,$$

independently of each other. Therefore, $\{\psi(x+iy_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $D_{L_p}(R)$, $p > 1$. Since $D_{L_p}(R)$ is complete it follows that there exist a function $\psi_+(x) \in D_{L_p}(R)$ such that $\lim_{n\to\infty} \psi(x+iy_n) = \psi_+(x)$ in $D_{L_p}(R)$, $p > 1$. Since $\{y_n\}$ is an arbitrary sequence of positive numbers tending to zero, it follows that there exists a function $\psi_+(x) \in D_{L_p}(R)$ such that:

$$\lim_{y\to 0^+} \psi(x+iy) = \psi_+(x) \quad \text{in} \quad D_{L_p}. \tag{11}$$

Similarly, it can be shown that there exists a function $\psi_-(x) \in D_{L_p}(R)$ such that:

$$\lim_{y\to 0^-} \psi(x+iy) = \psi_-(x) \quad \text{in} \quad D_{L_p}. \tag{12}$$

Now using Condition (P_4) and Equations 11 and 12, it follows that:

$$\psi(x) = \frac{\psi_+(x) + \psi_-(x)}{2} \in D_{L_p}, \quad p > 1.$$

From Lemma 1 and Equations 11 and 12, it follows that:

$$\lim_{y\to 0^+} \psi(x+it) = \psi_+(x) \quad \text{uniformly for all} \quad x \in R,$$

and:

$$\lim_{y\to 0^-} \psi(x+iy) = \psi_-(x) \quad \text{uniformly for all} \quad x \in R.$$

Since $\psi(z)$ is analytic outside a closed interval $[a,b]$, it follows that $\psi_+(x) = \psi_-(x) = 0$ outside $[a,b]$ and therefore belongs to D. Using Cauchy's integral theorem and the technique used in [1], it can be shown that for $\varepsilon > 0$:

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\psi(t+i\varepsilon)}{t-z}dt = \psi(z+i\varepsilon), \quad Imz > 0,$$

$$= 0, \quad Imz < 0. \tag{13}$$

Let $\varepsilon \longrightarrow 0^+$ in Equation 13, it is deduced that:

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\psi_+(t)}{t-z}dt = \psi(z), \quad Imz > 0,$$

$$= 0, \quad Imz < 0. \tag{14}$$

Similarly, it can be shown that:

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\psi_-(t)}{t-z}dt = -\psi(z), \quad Imz < 0,$$

$$= 0, \quad Imz > 0. \tag{15}$$

Combining Equations 14 and 15, the following is obtained:

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\psi_+(t) - \psi_-(t)}{t-z}dt = \psi(z), \quad Imz \neq 0. \tag{16}$$

Let $\phi(t) = \frac{\psi_+(t)-\psi_-(t)}{2\pi i}$. Clearly $\phi(t) \in D$ and, thus:

$$\psi(z) = \int_{-\infty}^{\infty} \frac{\phi(t)}{t-z}dt, \quad Imz \neq 0. \tag{17}$$

In view of Lemmas 2 and 3 and Condition (P_4), it follows that:

$$\psi(x) = p.v. \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x}dt,$$

that is:

$$\psi(z) = p.v. \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x}dt, \quad Imz = 0. \tag{18}$$

The proof for necessity follows from Equations 17 and 18.

Proof (Sufficiency)

If there exists a function $\phi \in D$, which vanishes outside a closed interval satisfying:

$$\psi(z) = \int_{-\infty}^{\infty} \frac{\phi(t)}{t-z}dt, \quad Imz \neq 0$$

$$= p.v. \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x}dt, \quad Imz = 0,$$

then, as proved in [1],

$$\psi^{(k)}(z) = \int_{-\infty}^{\infty} \frac{\phi^{(k)}(t)}{t-z}dt, \quad Imz \neq 0,$$

$$= p.v. \int_{-\infty}^{\infty} \frac{\phi^{(k)}(t)}{t-z}dt, \quad Imz = 0.$$

Clearly ψ satisfies Conditions (P_1) and (P_2). Conditions (P_3) and (P_4) are also satisfied in view of Lemma 3. This completes the proof of Theorem 1.

Theorem 1 shows that there is a one to one correspondence between the space Ψ and the space $H(D)$. Therefore, it can be defined that the space $H(D)$ is a genuinely intrinsic way as follows.

A C^∞ function $\psi(x)$ belongs to $H(D)$ iff there exists a holomorphic function $\psi(z)$ satisfying Conditions (P_1), (P_2), (P_3) and (P_4). In other words $\psi(x) \in H(D)$ iff it is the average of the upper and lower limit of a holomorphic function satisfying Conditions (P_1), (P_2) and (P_3). That is, $\psi(x)$ can be extended uniquely as a holomorphic function satisfying Conditions (P_1), (P_2), (P_3) and (P_4).

The convergence of a sequence $\{\psi_\mu(x)\}_{\mu=1}^{\infty}$ to zero in $H(D)$ can be defined in an intrinsic way as follows.

A sequence $\{\psi_\mu\}_{\mu=1}^{\infty}$ in $H(D)$ converges to zero in $H(D)$ iff:

A$_1$) The associated functions $\psi_\mu(z)$ in accordance with Theorem 1 are analytic outside a closed interval $[a,b]$ or else $\psi_\mu(x)$ is analytic outside a fixed closed interval $[a,b]$.

A$_2$) $\psi_\mu(x) \longrightarrow 0$ in D_{L_p} as $\mu \longrightarrow \infty$.

Clearly if $\{\phi_\mu(x)\}_{\mu=1}^{\infty}$ is a sequence in D tending to zero in D as $\mu \longrightarrow \infty$ and:

$$\psi_\mu(x) = p.v. \int_{-\infty}^{\infty} \frac{\phi_\mu(t)}{t-x} dt, \tag{19}$$

and:

$$\psi_\mu(z) = p.v. \int_{-\infty}^{\infty} \frac{\phi_\mu(t)}{t-z} dt, \quad Imz \neq 0, \tag{20}$$

then $\psi_\mu(z)$ is analytic outside the closed interval $[a,b]$ (see [8]) and:

$$\|\psi_\mu^{(k)}\|_p \leq C_p \|\phi_\mu^{(k)}\|_p \longrightarrow 0 \quad \text{as} \quad \mu \longrightarrow \infty.$$

Therefore, Conditions (A$_1$) and (A$_2$) are satisfied. If Conditions (A$_1$) and (A$_2$) are assumed then, there exists a closed interval $[a,b]$ containing the supports of all $\phi_\mu(x)$. From Equations 19 and 20 and the fact that $-\frac{1}{\pi^2}H^2 = I$, it follows that:

$$\phi_\mu(x) = -\frac{1}{\pi^2} p.v. \int_{-\infty}^{\infty} \frac{\psi_\mu(x)}{t-x} dt.$$

Therefore:

$$\|\phi_\mu^{(k)}(x)\|_p \leq \frac{1}{\pi^2} C_p \|\psi_\mu^{(k)}\|_p \longrightarrow 0 \quad \text{as} \quad \mu \longrightarrow \infty,$$

that is, $\phi_\mu(x) \longrightarrow 0$ in D_{L_p} as $\mu \longrightarrow \infty$. Thus, by Lemma 1, $\phi_\mu(x) \longrightarrow 0$ uniformly for all $x \in R$ as $\mu \longrightarrow \infty$. By Condition (A$_1$), all $\phi_\mu(x)$ have supports contained in a fixed interval $[a,b]$.

Therefore, if $\{\psi_\mu\}_{\mu=1}^{\infty} \longrightarrow 0$ in $H(D)$ as $\mu \longrightarrow \infty$, then, it has been proven that:

$$\phi_\mu \longrightarrow 0 \text{ in } D \text{ as } \mu \longrightarrow \infty \Longleftrightarrow \psi_\mu \longrightarrow 0 \text{ in } H(D) \text{ as } \mu \longrightarrow \infty.$$

Thus, Conditions (A$_1$) and (A$_2$) together describe intrinsically the convergence of a sequence $\{\psi_\mu\}_{\mu=1}^{\infty}$ to zero in$H(D)$ as $\mu \longrightarrow \infty$.

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