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*Research Note*

# A Note on Fuzzy Process Capability Indices

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The notion of fuzzy process capability indices is studied by Parchami et al. [1], where the specification limits are triangular fuzzy numbers. In this note, their results are revised for the general case, where the specification limits are $L \quad R$ fuzzy intervals.

## INTRODUCTION

Process Capability Indices (PCIs) are used to measure the capability of a process to reproduce items within the specified tolerance preset by the product designers or customers. There are several PCIs, such as $C_p$, $C_{pk}$, $C_{pm}$ and $C_{pmk}$, which are used to estimate the capability of a manufacturing process, where, in most cases, the normal distribution and a large sample size are assumed for population of data [2-6]. These indices essentially compare the specification tolerance range with the actual production tolerance one. In some cases, Specification Limits (SLs) are not precise numbers and they are expressed in fuzzy terms, so that the classical capability indices could not be applied. For such cases, Yongting [7] introduced a process capability index, $C_p$, as a real number, which was used by Sadeghpour-Gildeh [8]. Lee investigated a process capability index, $C_{pk}$, as a fuzzy set [9]. Parchami et al. introduced fuzzy PCIs as fuzzy numbers, where upper and lower SLs are triangular fuzzy numbers and discussed the relations operating between them when SLs are fuzzy rather than crisp [1,10]. Some researchers also obtained fuzzy confidence intervals for these new process capability indices [11]. In this paper, the result of [1] is revised for cases where SLs are $L \quad R$ fuzzy intervals, which gives more flexibility to the product designers.

The organization of this paper is as follows. In the following section, some preliminaries are discussed. Then, traditional process capability indices are reviewed, the fuzzy process is considered and the fuzzy process capability indices introduced by Parchami et al. [1] are revised for the general case, where SLs are $L \quad R$ fuzzy. After that, the relation between fuzzy process capabilities indices are studied. Finally, there is the conclusion.

## PRELIMINARIES

In this section, some preliminaries are given, which will be needed throughout the paper. For more details, see [1,12]. Let $R$ be the set of real numbers. Let:

$$F(R)=\{A|A:R\rightarrow[0,1],\ A \text{ is a continuous function}\}.$$

**Definition 1**

Let $A \in F(R)$, then

a) $A$ is called normal, if, and only if, there exists $x \in R$, such that $A(x)=1$;

b) $A$ is called convex, if, and only if;

$$A(\lambda x+(1 \quad \lambda)y)\geq(A(x)\wedge A(y)),\ \forall x,\ y\in R,\ \forall\lambda\in[0,1],$$

where the symbol, $\wedge$, denotes the minimum operator.

**Definition 2**

1. A fuzzy number is a normal and convex fuzzy set of the real line, $R$, whose membership function is piecewise continuous;
2. A fuzzy number, $M$, is called positive (negative), denoted by $M>0(M<0)$, if its membership function satisfies $M(x)=0$, $\forall x<0(\forall x>0)$.

**Definition 3**

A fuzzy number is a fuzzy interval if its membership function, $M$, satisfies the following conditions:

1. $M$ is continuous mapping from $R$ to the closed interval $[0,1]$,

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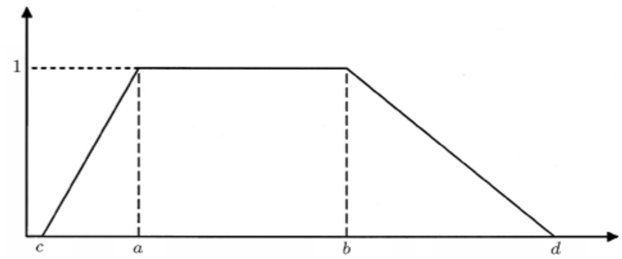


**Figure 1.** Fuzzy interval.

2. $M(x) = 0$ for all $x \in (-\infty, c]$,
3. $M$ is strictly increasing and continuous on $[c, a]$,
4. $M(x) = 1$ for all $x \in [a, b]$,
5. $M$ is strictly decreasing and continuous on $[b, d]$,
6. $M(x) = 0$ for all $x \in (d, +\infty]$,

where $a$, $b$, $c$, $d$ are real numbers and $(a + b)/2$ is the mean value of $M$ (see Figure 1).

**Definition 4**

A fuzzy interval, $M$, is said to be an $L-R$ fuzzy interval, if

$$M(x) = \begin{cases} L\left(\frac{m_1-x}{\alpha}\right) & x < m_1 \quad \alpha > 0, \\ 1 & m_1 \leq x \leq m_2, \\ R\left(\frac{x-m_2}{\beta}\right) & x > m_2 \quad \beta > 0, \end{cases}$$

where $\alpha$ and $\beta$ are left and right spreads, respectively, and the function, $L(.)$, is a left shape function satisfying:

1. $L(x) = L(-x)$,
2. $L(0) = 1$,
3. $L(x)$ is non-increasing on $[0, \infty)$.

Naturally, a right shape function, $R(.)$, is similarly defined as $L(.)$. $\frac{m_1+m_2}{2}$ is called the mean value of $M$.

Using its mean value, left and right spreads and shape function, such an $L-R$ fuzzy interval, $M$, is written as:

$$M = (m_1, m_2, \alpha, \beta)_{LR}.$$

By an $L-R$ fuzzy number, one means a fuzzy interval, $M = (m_1, m_2, \alpha, \beta)_{LR}$, where $m_1 = m_2$. It is written as: $M = (m_1, \alpha, \beta)_{LR}$. Also, when $\alpha = \beta = 0$ and $m_1 = m_2$, then, $M$ reduces to $M = (m_1, m_1, 0, 0)_{LR}$, which is the real number, $m_1$.

**Definition 5**

Let $M = (m_1, m_2, \alpha, \beta)_{LR}$ and $N = (n_1, n_2, \gamma, \delta)_{RL}$ be fuzzy intervals. Then:

1. One calls $M = (m_1, m_2, \alpha, \beta)_{LR}$ lower bounded, if there exists $x_U = m_1 - \alpha L^{-1}(0)$, such that $L\left(\frac{m_1-x_U}{\alpha}\right) = 0$ for all $x \leq x_U$;

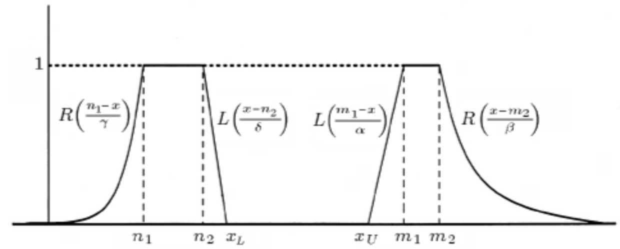


**Figure 2.** Upper and lower bounded fuzzy intervals.

2. One calls $N = (n_1, n_2, \gamma, \delta)_{RL}$ upper bounded, if there exists $x_L = n_2 + \delta L^{-1}(0)$, such that $L\left(\frac{x_L-n_2}{\delta}\right) = 0$ for all $x \geq x_L$ (see Figure 2);
3. Let:

$$F_{LR} = \{M = (m_1, m_2, \alpha, \beta)_{LR} |$$
$$M \text{ is a lower bounded fuzzy interval}\},$$

and:

$$F_{RL}^{+} = \{N = (n_1, n_2, \gamma, \delta)_{RL} |$$
$$N \text{ is an upper bounded fuzzy interval}\}.$$

**Definition 6**

For an $L-R$ fuzzy interval, $M = (m_1, m_2, \alpha, \beta)_{LR} \in F_{LR}$, and an $L-R$ fuzzy interval, $N = (n_1, n_2, \gamma, \delta)_{RL} \in F_{RL}^{+}$, the following formulas for the extended opposite and subtraction hold, respectively:

1. Opposite:

$$-M = (-m_2, -m_1, \beta, \alpha)_{RL},$$

2. Subtraction:

$$M \Theta N = (m_1 - n_2, m_2 - n_1, \alpha + \delta, \beta + \gamma)_{LR}. \quad (1)$$

As a special case, observe that scalar multiplication for an $L-R$ fuzzy interval, $M = (m_1, m_2, \alpha, \beta)_{LR}$, can be given by the following formula, depending on the sign of $\lambda$.

3. Scalar multiplication:

$$\lambda \otimes M = \begin{cases} (\lambda m_1, \lambda m_2, \lambda\alpha, \lambda\beta)_{LR} & \lambda \geq 0, \\ (\lambda m_2, \lambda m_1, -\lambda\beta, -\lambda\alpha)_{RL} & \lambda < 0. \end{cases} \quad (2)$$

## TRADITIONAL PROCESS CAPABILITY INDICES

In this section, some notions of traditional PCIs will be reviewed. A process capability index is a real number, as a summary that compares the behavior of a product or process characteristic to engineering specifications. This measure is also called the performance index. Several PCIs are introduced in the literature, such as $C_p$,

$C_{pk}$, $C_{pm}$ and so on [3-5]. For convenience, the upper and lower specification limits will be denoted by $U$ and $L$, respectively, rather than the more customary USL and LSL notations. Where univariate measurements are concerned, the corresponding random variate will be denoted by $X$. The expected value and standard deviation of $X$ will be denoted by $\mu$ and $\sigma$, respectively. The situation will be limited to where $\mu$ is in the specification interval, i.e. $L \leq \mu \leq U$ and it will be assumed that the measured characteristic should have a normal distribution (at least, approximately), although it is difficult to see why a good industrial process must result in a normal distribution for every measured characteristic.

The commonly recognized PCIs are:

$$C_p = \frac{U - L}{6\sigma} = \frac{w}{6\sigma}, \tag{3}$$

where $w = U - L$. This $C_p$ is used when $\mu = M$ and where $M = (U + L)/2$.

$$C_{pk} = \frac{w - 2|\mu - M|}{6\sigma} = \frac{\min\{U - \mu, \mu - L\}}{3\sigma}, \tag{4}$$

and:

$$C_{pm} = \frac{w}{6\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{w}{6\sqrt{E[(X - T)^2]}}, \tag{5}$$

where $T$ is target value and $E[.]$ denotes the expected value.

There is also the hybrid index:

$$C_{pmk} = \frac{w - 2|\mu - M|}{6\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{w - 2|\mu - M|}{6\sqrt{E[(X - T)^2]}}. \tag{6}$$

Usually, $T = M$. If $T \neq M$, the situation is sometimes described as "asymmetric tolerances" (see [13-15]). Introduction of $C_p$ is ascribed to Juran [16]; that of $C_{pk}$ to Kane [2]; that of $C_{pm}$, for the most part to Hsiang and Taguchi [17], and $C_{pmk}$ to Pearn et al. [5].

Clearly $C_p \geq C_{pk} \geq C_{pmk}$ and $C_p \geq C_{pm} \geq C_{pmk}$. Some more relations between PCIs could be realized. From Equations 3 and 4, one has:

$$C_{pk} = C_p - \frac{1}{3}\left|\frac{\mu - M}{\sigma}\right|, \tag{7}$$

and from Equations 3 and 5, one has:

$$C_{pm} = \frac{C_p}{\sqrt{1 + \left(\frac{\mu - T}{\sigma}\right)^2}}. \tag{8}$$

From Equations 4 and 6, one has:

$$C_{pmk} = \frac{C_{pk}}{\sqrt{1 + \left(\frac{\mu - T}{\sigma}\right)^2}}. \tag{9}$$

A further interesting relation is:

$$C_{pmk} = \frac{C_{pm} C_{pk}}{C_p}. \tag{10}$$

An enlightening view of relations between the PCIs can be obtained from studies of the "superstructure PCIs", introduced by Vännman [18], as follows:

$$C_p(u, v) = \frac{w - 2u|\mu - M|}{6\sqrt{\sigma^2 + v(\mu - T)^2}} \quad (u, v \geq 0). \tag{11}$$

The four PCIs introduced in Equations 3 to 6 are special cases of $C_p(u, v)$. Indeed;

$$C_p = C_p(0, 0), \qquad C_{pk} = C_p(1, 0),$$

$$C_{pm} = C_p(0, 1), \qquad C_{pmk} = C_p(1, 1).$$

See more details concerning this section in Kotz [3].

## FUZZY PROCESS CAPABILITY INDICES

In this section, the recent results on fuzzy process capability indices obtained by Parchami et al. [1] will be revised. The $C_p$ index based on fuzzy SLs was introduced as a real number by Yongting [7] and was also used by other authors [8]. But, it would be more realistic to have a $C_p$ which is also fuzzy, since a fuzzy process capability index would be much more appropriate than a precise number, if SLs are fuzzy.

### Definition 7

A process with fuzzy specification limits, called a fuzzy process for short, is one which approximately satisfies the normal distribution condition and its specification limits are fuzzy [1].

In the following Definition 4.1, 4.2 of [1] is revisited, which gives a $\widetilde{C}_p$ as having more flexibility for the product designers by the shape of its SLs.

### Definition 8

Suppose one has a fuzzy process with fixed $\sigma$, for which the specification limits are the fuzzy sets $U_{LR} = (m_1, m_2, \alpha, \beta)_{LR} \in F_{LR}$, $L_{RL} = (n_1, n_2, \gamma, \delta)_{RL} \in F^+_{RL}$ and $x_U \geq x_L$, where $x_L = n_2 + \delta L^{-1}(0)$ and $x_U = m_1 - \alpha L^{-1}(0)$. Then:

a) The width between fuzzy process specification limits is a $L - R$ fuzzy number, $\widetilde{w}_{LR} \in F_{LR}$, defined by;

$$\begin{aligned} \widetilde{w}_{LR} &= U_{LR} \Theta L_{RL} \\ &= (m_1, m_2, \alpha, \beta)_{LR} \Theta (n_1, n_2, \gamma, \delta)_{RL} \\ &= (m_1 - n_2, m_2 - n_1, \alpha + \delta, \beta + \gamma)_{LR}. \end{aligned} \tag{12}$$

b) The fuzzy process capability index is a $L$ $R$ fuzzy number, $\widetilde{C}_p \in F(R)$, defined by;

$$\widetilde{C}_p = \frac{1}{6\sigma} \otimes \widetilde{w}_{LR}. \tag{13}$$

By Equations 1 and 2, one can obtain $\widetilde{C}_p$, as follows:

$$\widetilde{C}_p = \left(\frac{m_1 \quad n_2}{6\sigma}, \frac{m_2 \quad n_1}{6\sigma}, \frac{\alpha+\delta}{6\sigma}, \frac{\beta+\gamma}{6\sigma}\right)_{LR}. \tag{14}$$

Note that $\widetilde{C}_p$ is useful when $\mu = m$, where:

$$m = \frac{m_1 + m_2 + n_1 + n_2}{4}.$$

In the following, an example is given to clear the idea of $\widetilde{C}_p$.

**Example 1**

For a special product, suppose that the specification limits are considered to be $U_{LR} = (5, 5.5, 0.5, 0.5)_{LR} \in F_{LR}$ and $L_{RL} = (2.5, 3, 0.5, 1)_{RL} \in F^+_{RL}$, respectively, where $L(x) = 1 \quad x^2$ and $R(x) = e^{\ x^2}$, as in Figure 3. Assume that the process mean $\mu$ is 6 and the estimated process standard deviation is 1/2. By Definition 8, one can compute the width between process SLs as $\widetilde{w}_{LR} = (2, 3, 1.5, 1)_{LR}$. Therefore, $\hat{\widetilde{C}}_p = (2/3, 1, 1/2, 1/3)_{LR}$ is the estimate of $\widetilde{C}_p$, which is depicted in Figure 4.

**Example 2**

In a manufacturing process, 75 samples extracted from the vane-manufacturing process, which is under statistical control, are shown in Table 1 (as on page 781 of [19]).

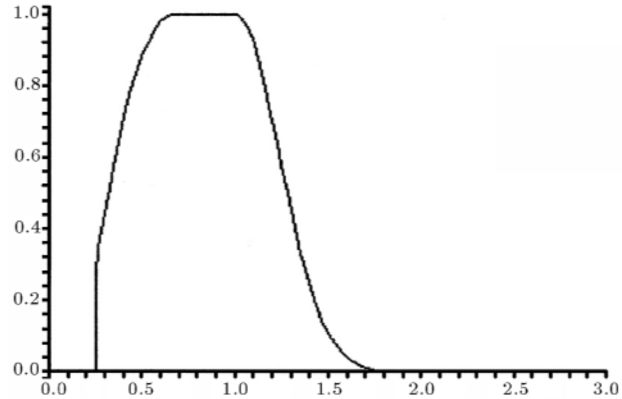


**Figure 4.** The membership function of fuzzy process capability index in Example 1.

From the data, $s = 2.15$ is estimated. Assume $U_{LR} = (39.5, 40.5, 0.7, 0.5)_{LR}$ and $L_{RL} = (19.5, 20.5, 0.4, 0.9)_{RL}$ are specification limits, where $L(x) = R(x) = 1 \quad x^2$. Then, by Equation 14, $\hat{\widetilde{C}}_p = (1.47, 1.62, 0.12, 0.07)_{LR}$.

**Remark 1**

Let, in a fuzzy process, $U_{LR} = (5, 5.5, 0.5, 0.5)_{LR}$, $L_{RL} = (2.5, 3, 2.5, 1)_{RL}$, $L(x) = 1 \quad x^2$ and $R(x) = e^{\ x^2}$. For several values of standard deviation, the pictorial representation of $\hat{C}_p$ is depicted in Figure 5 by Maple software. The bold curve is $\hat{C}_p$ for $s = 1.4$, 3.1 and 5.4, respectively. Note that, as $s$ increases, the $\hat{C}_p$ tends to be a sharper and smaller $L$ $R$ fuzzy interval.

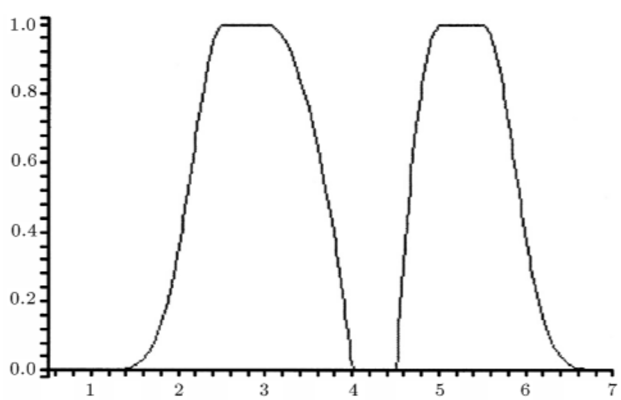


**Figure 3.** The membership function of fuzzy process specification limits in Example 1.

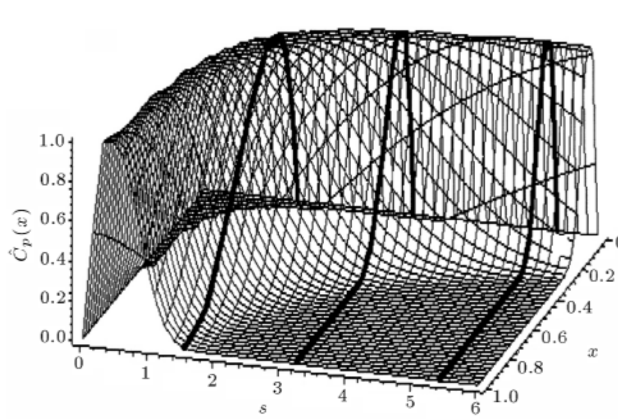


**Figure 5.** Three-dimensional representation of fuzzy PCI related to standard deviation.

**Table 1.** Data vane-manufacturing process for Example 2.

| **Observation** | 27 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 39 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Frequency** | 2 | 1 | 7 | 8 | 8 | 16 | 10 | 13 | 4 | 4 | 2 |

**Definition 9**

Let $U_{LR} = (m_1, m_2, \alpha, \beta)_{LR} \in F_{LR}$ and $L_{RL} = (n_1, n_2, \gamma, \delta)_{RL} \in F^+_{RL}$ be the engineering fuzzy specification limits and $x_U \geq x_L$, where $x_L = n_2 + \delta L^{-1}(0)$ and $x_U = m_1 - \alpha L^{-1}(0)$. The following revised fuzzy PCIs are introduced:

$$\tilde{C}_{pk} = \left(\frac{m_1 - n_2 - 2|\mu - m|}{6\sigma}, \frac{m_2 - n_1 - 2|\mu - m|}{6\sigma}, \frac{\alpha+\delta}{6\sigma}, \frac{\beta+\gamma}{6\sigma}\right)_{LR}, \tag{15}$$

$$\tilde{C}_{pm} = \left(\frac{m_1 - n_2}{6\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{m_2 - n_1}{6\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\alpha+\delta}{6\sqrt{\sigma^2+(\mu - T)^2}}, \frac{\beta+\gamma}{6\sqrt{\sigma^2+(\mu - T)^2}}\right)_{LR}, \tag{16}$$

$$\tilde{C}_{pmk} = \left(\frac{m_1 - n_2 - 2|\mu - m|}{6\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{m_2 - n_1 - 2|\mu - m|}{6\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\alpha+\delta}{6\sqrt{\sigma^2+(\mu - T)^2}}, \frac{\beta+\gamma}{6\sqrt{\sigma^2+(\mu - T)^2}}\right)_{LR}, \tag{17}$$

where $m = \frac{m_1+m_2+n_1+n_2}{4}$ and $T$ is target value.

The revised "superstructure fuzzy PCI" is defined as follows:

$$\tilde{C}_p(u,v) = \left(\frac{m_1 - n_2 - 2u|\mu - m|}{6\sqrt{\sigma^2 + v(\mu - T)^2}}, \frac{m_2 - n_1 - 2u|\mu - m|}{6\sqrt{\sigma^2 + v(\mu - T)^2}}, \frac{\alpha+\delta}{6\sqrt{\sigma^2+v(\mu - T)^2}}, \frac{\beta+\gamma}{6\sqrt{\sigma^2+v(\mu - T)^2}}\right)_{LR}$$

for $u, v \geq 0$. (18)

**Remark 2**

i) The process capability indices defined by Equations 3 to 6 could be expressed by the fuzzy process capability indices as follows:

$$C_p = \tilde{C}_p = \left(\frac{m_1 - n_1}{6\sigma}, 0, 0\right)_{LR},$$

$$C_{pk} = \tilde{C}_{pk} = \left(\frac{m_1 - n_1 - 2|\mu - m|}{6\sigma}, 0, 0\right)_{LR},$$

$$C_{pm} = \tilde{C}_{pm} = \left(\frac{m_1 - n_1}{6\sqrt{\sigma^2 + (\mu - T)^2}}, 0, 0\right)_{LR},$$

$$C_{pmk} = \tilde{C}_{pmk} = \left(\frac{m_1 - n_1 - 2|\mu - m|}{6\sqrt{\sigma^2 + (\mu - T)^2}}, 0, 0\right)_{LR}.$$

ii) When the process specification limits, $U_{LR} = (m_1, m_2, \alpha, \beta)_{LR} \in F_{LR}$ and $L_{RL} = (n_1, n_2, \gamma, \delta)_{RL} \in F^+_{RL}$, are precise numbers, i.e., $m_1 = m_2$, $n_1 = n_2$ and $\alpha = \beta = \gamma = \delta = 0$, then, all the introduced revised fuzzy PCIs are precise numbers and coincide with the traditional PCIs.

**Remark 3**

Fuzzy intervals $M = (m_1, m_2, \alpha, \beta)_{LR} \in F_{LR}$ and $N = (n_1, n_2, \gamma, \delta)_{RL} \in F^+_{RL}$ are triangular fuzzy numbers if $m_1 = m_2$, $n_1 = n_2$ and $L(x) = R(x) = 1 - x$. Thus, as a special case, SLs, as $U_{LR} = (b_u, b_u, b_u - a_u, c_u - b_u)_{LR} \in F_{LR}$ and $L_{RL} = (b_l, b_l, b_l - a_l, c_l - b_l)_{RL} \in F^+_{RL}$, are triangular fuzzy numbers, $T(a_u, b_u, c_u)$ and $T(a_l, b_l, c_l)$, respectively, used by Parchami et al. [1], where:

$$T(a,b,c) = \begin{cases} (x-a)/(b-a) & \text{if } a \leq x \leq b \\ (c-x)/(c-b) & \text{if } b \leq x \leq c \\ 0 & \text{if elsewhere} \end{cases},$$

and $L(x) = R(x) = 1 - x$.

## RELATION BETWEEN FUZZY PROCESS CAPABILITY INDICES

In this section, some relations governing the revised fuzzy process capability indices are given, as introduced in the previous section.

**Theorem 1**

In a fuzzy process, assume $x_U \geq x_L$, where $x_L = n_2 + \delta L^{-1}(0)$ and $x_U = m_1 - \alpha L^{-1}(0)$. Then;

$$\tilde{C}_{pk}(x) = \tilde{C}_p\left(x + \frac{|\mu - m|}{3\sigma}\right),$$

where:

$$m = \frac{m_1 + m_2 + n_1 + n_2}{4}, \tag{19}$$

$$\tilde{C}_{pm}(x) = \tilde{C}_p\left(x\sqrt{1 + \left(\frac{\mu - T}{\sigma}\right)^2}\right), \tag{20}$$

$$\tilde{C}_{pmk}(x) = \tilde{C}_{pk}\left(x\sqrt{1 + \left(\frac{\mu - T}{\sigma}\right)^2}\right). \tag{21}$$

***Proof***

Let $y = \frac{|\mu - m|}{3\sigma}$. Then, by some calculations one has:

$$\widetilde{C}_p(x+y)$$

$$= \begin{cases} L\left(\frac{\frac{m_1 - n_2}{6\sigma} - x - y}{\frac{\alpha+\delta}{6\sigma}}\right) & \text{if } x+y < \frac{m_1 - n_2}{6\sigma} \\ 1 & \text{if } \frac{m_1 - n_2}{6\sigma} \leq x+y \leq \frac{m_2 - n_1}{6\sigma} \\ R\left(\frac{x+y - \frac{m_2 - n_1}{6\sigma}}{\frac{\beta+\gamma}{6\sigma}}\right) & \text{if } x+y > \frac{m_2 - n_1}{6\sigma} \end{cases},$$

$$\widetilde{C}_{pk}(x)$$

$$= \begin{cases} L\left(\frac{\frac{m_1 - n_2}{6\sigma} - y - x}{\frac{\alpha+\delta}{6\sigma}}\right) & \text{if } x < \frac{m_1 - n_2}{6\sigma} - y \\ 1 & \text{if } \frac{m_1 - n_2}{6\sigma} - y \leq x \leq \frac{m_2 - n_1}{6\sigma} - y. \\ R\left(\frac{x - \frac{m_2 - n_1}{6\sigma} + y}{\frac{\beta+\gamma}{6\sigma}}\right) & \text{if } x > \frac{m_2 - n_1}{6\sigma} - y \end{cases}$$

Therefore, Equation 19 is proved. Similarly, Equations 20 and 21 can be proven. □

**Theorem 2**

The four revised fuzzy PCIs introduced in Equations 14 to 17 are special cases of $\widetilde{C}_p(u,v)$. Indeed:

$$\widetilde{C}_p = \widetilde{C}_p(0,0), \quad \widetilde{C}_{pk} = \widetilde{C}_p(1,0), \quad \widetilde{C}_{pm} = \widetilde{C}_p(0,1),$$

$$\widetilde{C}_{pmk} = \widetilde{C}_p(1,1).$$

***Proof***

It is obvious by using Equations 14 to 18. □

Further interesting relations are stated in the following theorem. First, an important definition is given.

**Definition 10**

For $L-R$ fuzzy intervals, $M = (m_1, m_2, \alpha, \beta)_{LR}$ and $N = (n_1, n_2, \gamma, \delta)_{LR}$, the following approximate formula for the extended multiplication holds, as follows:

1. Multiplication [20]:
   If $M > 0$ and $N > 0$, then,

   $$M \otimes N \cong (m_1 n_1, m_2 n_2, m_1\gamma + n_1\alpha, m_2\delta + n_2\beta)_{LR}.$$

2. For positive $L-R$ fuzzy intervals, $M, N, M'$ and $N'$, one can say $M \otimes N \approx M' \otimes N'$, if $M \otimes N \cong (m,n,\lambda,\nu)_{LR}$ and $M' \otimes N' \cong (m,n,\lambda,\nu)_{LR}$, for some fuzzy interval, $(m,n,\lambda,\nu)_{LR}$.

**Theorem 3**

Let in a fuzzy process, $x_U > x_L$, where $x_L = n_2 + \delta L^{-1}(0)$ and $x_U = m_1 - \alpha L^{-1}(0)$, and $\widetilde{C}_p, \widetilde{C}_{pmk}, \widetilde{C}_{pm}$ and $\widetilde{C}_{pk}$ can be defined by Equations 14 to 17, respectively. Then, $\widetilde{C}_p \otimes \widetilde{C}_{pmk} \approx \widetilde{C}_{pm} \otimes \widetilde{C}_{pk}$.

***Proof***

According to Definitions 5 and 8 to 10, one has the following:

$$\widetilde{C}_p \otimes \widetilde{C}_{pmk} \cong \left(\frac{(m_1 - n_2)(m_1 - n_2 - 2|\mu - m|)}{36\sigma\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{(m_2 - n_1)(m_2 - n_1 - 2|\mu - m|)}{36\sigma\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{(\alpha+\delta)(2m_1 - 2n_2 - 2|\mu - m|)}{36\sigma\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{(\beta+\gamma)(2m_2 - 2n_1 - 2|\mu - m|)}{36\sigma\sqrt{\sigma^2 + (\mu - T)^2}}\right)_{LR},$$

$$\widetilde{C}_{pm} \otimes \widetilde{C}_{pk} \cong \left(\frac{(m_1 - n_2)(m_1 - n_2 - 2|\mu - m|)}{36\sigma\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{(m_2 - n_1)(m_2 - n_1 - 2|\mu - m|)}{36\sigma\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{(\alpha+\delta)(2m_1 - 2n_2 - 2|\mu - m|)}{36\sigma\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{(\beta+\gamma)(2m_2 - 2n_1 - 2|\mu - m|)}{36\sigma\sqrt{\sigma^2 + (\mu - T)^2}}\right)_{LR}.$$

Therefore, the theorem is proved when $\widetilde{C}_p, \widetilde{C}_{pmk}, \widetilde{C}_{pm}$ and $\widetilde{C}_{pk}$ are $L-R$ fuzzy intervals and $x_U > x_L$, where $x_L = n_1 + \delta L^{-1}(0)$ and $x_U = m_1 - \alpha L^{-1}(0)$. □

## CONCLUSION

In this paper, the revised fuzzy process capability indices have been introduced, when the engineering specification limits are $L-R$ fuzzy intervals. Also, several relations between them have been revealed.

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