Scientia Iranica, Vol. 13, No. 2, pp 206–216

SCIENTIA
IRANICA

Research Note

Free Vibration Analysis of Gantry Type Coordinate Measuring Machines

M.T. Ahmadian*, G.R. Vossoughi[1] and S. Ramezani[1]

Coordinate Measuring Machines (CMMs) are designed for precision inspection of complex industrial products. Even though CMMs have high accuracy in electrical systems, mechanical structures prevent very high accuracy in these machines. Mechanical accuracy of CMMs depends on both static and dynamic sources of error. In automated CMMs, one of the dynamic error sources is the vibration of probes, due to inertia forces resulting from parts deceleration. Modeling of a gantry type CMM, based on the Timoshenko beam theory, is developed and the natural frequencies of the CMM system, at different positions of the probe, are calculated. Findings from the analytical and finite elements method indicate high accuracy and good agreement between the results.

INTRODUCTION

Coordinate measuring machines are, nowadays, widely used for a large range of measuring tasks. These tasks are expected to be carried out with ever increasing accuracy, speed, flexibility and the ability to operate under shop floor conditions. Research is necessary to meet these demands. CMMs are prone to many error sources. Based on functional components of a CMM, an overview has been given by Weekers [1] of the most important error sources affecting the accuracy of a CMM:

- Geometric errors: Limited accuracy in manufacturing, assembling and adjustment of components, like guide ways and measurement systems;
- Drive system: For CNC operated CMMs, the axes are equipped with drives, transmission and a servo-control unit causing errors such as mechanical load and structural vibration;
- Measurement system: The actual coordinates of measuring points are derived from the values indicated by the linear scales of the CMM. The min. errors introduced by the scales are: Inaccuracy of scale pitch, misalignment and adjustment of reading device and interpolation errors;
- Errors due to mechanical loads: These are errors related to static or slowly varying forces on CMM components in combination with the compliance of components and are mainly caused by the weight of moving parts;
- Thermally induced errors: The difference between the temperature of the measuring scales of CMM and work piece and the temperature gradient in the machine components is a source of error;
- Dynamic errors: These are errors mainly caused by deceleration of moving parts before stopping. These errors depend on the CMMs structural properties, like mass distribution, component stiffness and damping characteristics, as well as on the control system and disturbing forces.

In this paper, the authors are interested in the dynamic errors of CMMs caused by the deceleration of moving parts. Some studies on the error analysis of CMMs have been done. Weekers and Schellekenes [2] proposed a method for compensation of the dynamic errors of CMMs using inductive position sensors for online measurement of the major dynamic errors. Barakat et al. [3] presented a kinematical and geometrical error compensation for CMMs based on experiment. Nijs et al. [4] presented a very simple model of CMM for obtaining natural

*. *Corresponding Author, Department of Mechanical Engineering, Sharif University of Technology, Tehran, I.R. Iran.*

1. *Department of Mechanical Engineering, Sharif University of Technology, Tehran, I.R. Iran.*

frequencies of a CMM. Vermeulen [5] generated high-accuracy 3-D coordinate machines using a new configuration with fewer dynamic errors. Several researchers have applied software compensation successfully on CMMs [6,7].

Most of these researchers have considered a very simple model for their analysis while most of the studies are based on experiment. In the present study, a full CMM modeling is analyzed. All columns and guide ways are modeled as a Timoshenko beam [8,9] with flexibility in all directions. All bearings are modeled as torsional springs and torsional deformation of the column and guide ways is considered. Dynamic equations of motion are derived using Hamilton's principle [10]. The derived equations are treated using a state space variables method. Natural frequencies of the CMM system are calculated at different positions of the probe in the system. Dynamic error analysis of CMM and optimization using a Genetic Algorithm, based on maximization of a fitness function containing the effect of natural frequency, dynamic error and weight of the CMM structure, is presented by Ramezani [11].

MODELING OF CMM STRUCTURE

Structural components of a gantry type CMM are shown in Figure 1. Because of the very small deformation of each component, the x-, y- and z-axes are assumed to remain in the same direction as in the undeformed state. The origin of the coordinate system is along the axis of the left column and at a height which is equal to the axis of the x-guideway. The most important problem in this modeling is that the motions of the y-guide way, x-guide way and z-pinole are relative motions. They are the absolute deformation of each beam, but, they are not the absolute motion of each beam. So, the strain energy of each beam is only a function of its deformations, however, the kinetic energy of each component is influenced by the motion of other components. The system is assumed to be fixed at each position of the probe. The modeling of each component is presented in the following sections.

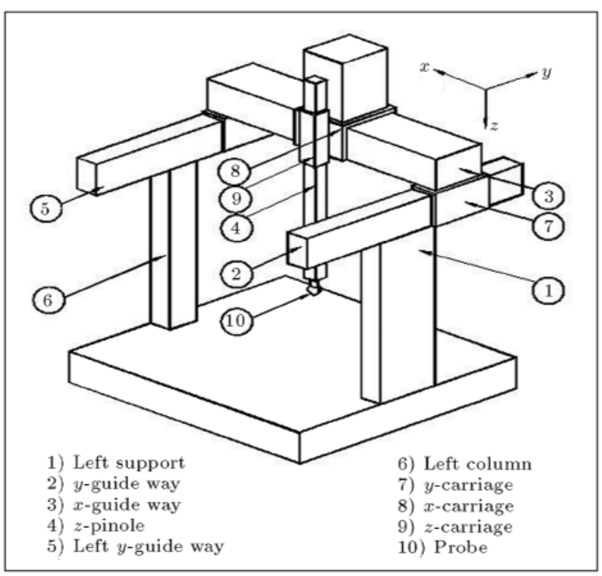

Figure 1. Schematic view of a gantry type CMM.

MODELING OF THE RIGHT COLUMN

For the right column, bending, torsional and longitudinal vibration is considered. The kinetic and strain energy of the column can be written as:

$$T_c = \frac{\rho_1}{2}\left\{\int_0^h \Big[A_1(\dot{A}^2(\xi,t) + \dot{B}^2(\xi,t) + \dot{C}^2(\xi,t)) + I_{1z}\dot{\alpha}^2(\xi,t) + I_{1y}\dot{\beta}^2(\xi,t) + I_{1z}\dot{\theta}_1^2(\xi,t)\Big]d\xi\right\}, \quad (1)$$

$$U_c = \frac{1}{2}\left\{\int_0^h \Big\{E_1 I_{1x}{\alpha'}^2(\xi,t) + E_1 I_{1y}{\beta'}^2(\xi,t) + kG_1A_1 \times + \left(\left[\alpha(\xi,t) \quad \frac{\partial A(\xi,t)}{\partial \xi}\right]^2 + \left[\beta(\xi,t) \quad \frac{\partial B(\xi,t)}{\partial \xi}\right]^2\right) + G_1 I_{1z}{\theta'}_1^2(\xi,t) + E_1A_1{C'}^2(\xi,t)\Big\}d\xi\right\}, \quad (2)$$

where:

- $A(\xi,t)$: Bending of column in xz-plane,
- $\alpha(\xi,t)$: Rotation of column around y-axis,
- $B(\xi,t)$: Bending of column in yz-plane,
- $\beta(\xi,t)$: Rotation of column around x-axis,
- $\theta_1(\xi,t)$: Torsion of column around z-axis,
- $C(\xi,t)$: Longitudinal vibration in z-direction,
- A_1, I_{1x}, I_{1y}, I_{1z}, E_1, G_1, ρ_1, h and k: Cross sectional area, actual and equivalent (for thin walled column) moments of inertia, Young modulus, shear modulus, density, length of column and Timoshenko shear modification constant, respectively.

Note that the symbols $(\dot{\,})$ and $(')$ denote derivation with respect to time and coordinates, respectively.

MODELING OF THE RIGHT Y-GUIDE WAY

For the right y-guide way, kinetic and strain energy have been considered to be as follows:

$$T_y = \frac{\rho_2}{2}\Bigg\{\int_0^{y_s}\Big\{A_2([\dot{E}(y,t) + y\dot{\alpha}(h,t) \quad \dot{C}(h,t)]^2$$

$$+ \dot{A}^2(h,t) + [\dot{D}(y,t) + \dot{B}(h,t) + y\dot{\theta}_1(h,t)]^2)$$

$$+ I_{2z}\dot{\gamma}_1^2(y,t) + I_{2y}[\dot{\theta}_3(y,t) + \dot{\beta}(h,t)]^2$$

$$+ I_{2x}\dot{\gamma}_2^2(y,t)\Big\}dy + \int_{y_s}^{L_y}\Big\{A_2([\dot{E}_1(y,t) + y\dot{\alpha}(h,t)$$

$$\dot{C}(h,t)]^2 + \dot{A}^2(h,t) + [\dot{D}_1(y,t) + \dot{B}(h,t)$$

$$+ y\dot{\theta}_1(h,t)]^2) + I_{2x}\dot{\gamma}_{22}^2(y,t) + I_{2y}[\dot{\theta}_3(y,t) + \dot{\beta}(h,t)]^2$$

$$+ I_{2z}\dot{\gamma}_{11}^2(y,t) + I_{2y}[\dot{\theta}_3(y,t) + \dot{\beta}(h,t)]^2$$

$$+ I_{2x}\dot{\gamma}_2^2(y,t)\Big\}dy\Bigg\}, \qquad (3)$$

$$U_y = \frac{1}{2}\Bigg\{\int_0^{y_s}\Big\{E_2 I_{2z}{\gamma'}_1^2(y,t) + E_2 I_{2x}{\gamma'}_2^2(y,t)$$

$$+ kG_2A_2\Big(\Big[\gamma_2(y,t) \quad \frac{\partial D(y,t)}{\partial y}\Big]^2 + \Big[\gamma_1(y,t)$$

$$\frac{\partial E(y,t)}{\partial y}\Big]^2\Big) + G_2 I_{2y}{\theta'}_3^2(y,t)\Big\}dy$$

$$+ \int_{y_s}^{L_y}\Big\{E_2 I_{2z}{\gamma'}_{11}^2(y,t) + \Big[\Big(\gamma_{22}(y,t) \quad \frac{\partial D_1(y,t)}{\partial y}\Big)\Big]^2$$

$$+ kG_2A_2\Big(\Big[\gamma_{11}(y,t) \quad \frac{\partial E_1(y,t)}{\partial y}\Big]^2\Big) + E_2 I_{2x}$$

$$\times {\gamma'}_{22}^2(y,t) + G_2 I_{2y}{\theta'}_{33}^2(y,t)\Big\}dy\Bigg\}, \qquad (4)$$

where:

- y_s: Location of y-carriage on the y-guide way,
- $D(y,t)$: Bending in xy-plane for $0 < y < y_s$,
- $D_1(y,t)$: Bending in xy-plane for $y_s < y < L_y$,
- $\gamma_1(y,t)$: Rotation around z-axis $0 < y < y_s$,
- $\gamma_{11}(y,t)$: Rotation around z-axis $y_s < y < L_y$,
- $E(y,t)$: Bending in yz-plane $0 < y < y_s$,
- $E_1(y,t)$: Bending in yz-plane $y_s < y < L_y$,
- $\gamma_2(y,t)$: Rotation around x-axis $0 < y < y_s$,
- $\gamma_{22}(y,t)$: Rotation around x-axis $y_s < y < L_y$,
- $\theta_3(y,t)$: Torsion around y-axis $0 < y < y_s$,
- $\theta_{33}(y,t)$: Torsion around y-axis $y_s < y < L_y$,
- A_2, I_{2x}, I_{2z}, I_{2y}, E_2, G_2, ρ_2 and L_y: cross sectional area, actual and equivalent moments of inertia, Young modulus, shear modulus and density, length of y-guide way, respectively.

MODELING OF THE Y-CARRIAGE AND BEARING

For the drive system effect, only the rotation of a y-carriage bearing around the y-axis is considered. The kinetic and strain energy in this member can be written as:

$$T_{bc3} = \frac{1}{2}M_{bc3}\Big\{[\dot{A}(h,t)]^2 + [y_s\dot{\theta}_1(h,t) + \dot{D}(y_s,t)$$

$$+ \dot{B}(h,t)]^2 + [y_s\dot{\alpha}(h,t) + \dot{E}(y_s,t) \quad \dot{C}(h,t)]^2\Big\}$$

$$+ J_{bc3z} \times \dot{\theta}_2(t)^2 + J_{bc3y}[\dot{\theta}_3(y_s,t) + \dot{\beta}(y_s,t)]^2, \qquad (5)$$

$$U_{bc3} = \frac{1}{2}K_{bc3}\theta_2(t)^2, \qquad (6)$$

where:

- $\theta_2(t)$: Torsion around y-axis
- K_{bc3}, M_{bc3}, J_{bc3y} and J_{bc3z}: Torsional stiffness of bearing and drive system, mass and mass moments of inertia of y-carriage and its bearing, respectively.

MODELING OF THE X-GUIDE WAY

For the x-guide way, bending and torsion kinetic and strain energy can be written as:

$$T_x = \frac{\rho_3}{2}\Bigg\{\int_0^{x_s}\Big\{A_3([\dot{V}(x,t) + \dot{A}(h,t) \quad x\dot{\theta}_2(t)]^2 + [y_s$$

$$\times\dot{\theta}_1(h,t) + \dot{D}(y_s,t) + \dot{B}(h,t)]^2 + [\dot{W}(x,t) \quad \dot{C}(h,t)$$

$$+ \dot{E}(y_s,t) + y_s\dot{\alpha}(h,t) + x\dot{\theta}_3(x,t)]^2) + I_{3y}\dot{\psi}^2(x,t)$$

$$+ I_{3z}\dot{\varphi}^2(x,t) + I_{3x}\dot{\eta}^2(x,t)\Big\}dx + \int_{x_s}^{L_x}\Big\{A_3([\dot{D}(y_s,t)$$

$$+ \dot{B}(h,t) + y_s\dot{\theta}_1(h,t)]^2 + [\dot{W}_1(x,t) + y_s \times \dot{\alpha}(h,t)$$

$$+ \dot{E}(y_s,t) \quad \dot{C}(h,t) + x\dot{\theta}_3(x,t)]^2) + I_{3y}\dot{\psi}_1^2(x,t)$$

$$+ I_{3z}\dot{\varphi}_1^2(x,t) + I_{3x}\dot{\eta}_x^2(x,t)\Big\} \, dx \Bigg\}, \tag{7}$$

$$U_x = \frac{1}{2}\Bigg\{ \int_0^{x_s} \Big\{ E_3[I_{3y}{\psi'}^2(x,t) + I_{3z}{\varphi'}^2(x,t)]$$

$$+ G_3 I_{3x}{\eta'}^2(x,t) + kG_3A_3\Big(\Big[\psi(x,t) \quad \frac{\partial W(x,t)}{\partial x}\Big]^2$$

$$+ \Big[\varphi(x,t) \quad \frac{\partial V(x,t)}{\partial x}\Big]^2\Big)\Big\}dx + \int_{x_s}^{L_x}\Big\{E_3 I_{3y}[{\psi'}_1^2(x,t)$$

$$+ I_{3z}{\varphi'}_1^2(x,t)] + G_3 I_{3x}{\eta'}_x^2(x,t) + kG_3A_3\Big(\Big[\varphi_1(x,t)$$

$$\frac{\partial V_1(x,t)}{\partial x}\Big]^2 + \quad \frac{\partial V_1(x,t)}{\partial x}\Big]^2$$

$$+ \Big[\psi_1(x,t) \quad \frac{\partial W_1(x,t)}{\partial x}\Big]^2\Big)\Big\}dx\Bigg\}, \tag{8}$$

where:

- x_s: Location of y-carriage on the y-guide way,
- $W(x,t)$: Bending in xz-plane for $0 < x < x_s$,
- $W_1(x,t)$: Bending in xz-plane for $x_s < x < L_x$,
- $\psi(x,t)$: Rotation around y-axis $0 < x < x_s$,
- $\psi_1(x,t)$: Rotation around y-axis $x_s < x < L_x$,
- $V(y,t)$: Bending in xy-plane $0 < x < x_s$,
- $V_1(y,t)$: Bending in xy-plane $x_s < x < L_x$,
- $\varphi(x,t)$: Rotation around z-axis $0 < x < x_s$,
- $\varphi_1(x,t)$: Rotation around z-axis $x_s < x < L_x$,
- $\eta(x,t)$: Torsion around x-axis $0 < x < x_s$,
- $\eta_x(x,t)$: Torsion around x-axis $x_s < x < L_x$,
- A_3, I_{3x}, I_{3y}, I_{3z}, E_3, G_3, ρ_3 and L_x are area, actual and equivalent moments of inertia, Young modulus, shear modulus, density and length of x-guide way, respectively.

MODELING OF THE BEARINGS AT Z-PINOLE

The kinetic and strain energy resulting from the motion of the z-pinole and the torsion of bearings is as follows:

$$T_{bcz} = \frac{1}{2}\Big\{M_{bcz}\Big\{[y_s\dot{\theta}_1(h,t) + \dot{D}(y_s,t) + \dot{B}(h,t)]^2$$

$$+ [y_s\dot{\alpha}(h,t) + \dot{E}(y_s,t) \quad \dot{C}(h,t)]^2 + [\dot{A}(h,t)$$

$$+ \dot{V}(x_s,t) \quad x_s\dot{\theta}_2(t)]^2\Big\} + J_{bc4x}[\dot{\theta}_4(t) + \dot{\eta}(x_s,t)]^2$$

$$+ J_{bc5x}[\dot{\theta}_4(t) + \dot{\theta}_5(t) + \dot{\eta}(x_s,t)]^2 + J_{bc6y}\dot{\theta}_6^2(t)\Big\}, \tag{9}$$

$$U_{bcz} = \frac{1}{2}[K_{bc4}\theta_4(t)^2 + K_{bc5}\theta_5(t)^2 + K_{bc6}\theta_6(t)^2], \tag{10}$$

where:

- $\theta_4(t)$: Torsion of x-carriage bearing around x-axis,
- $\theta_5(t)$: Torsion of z-axis bearing around x-axis,
- $\theta_6(t)$: Torsion of z-axis bearing around y-axis,
- K_{bc4}, K_{bc5}, K_{bc6}, M_{bcz}, J_{bc4x}, J_{bc5x} and J_{bc6y}: Torsional stiffness of bearings and drive system, mass and mass moments of inertia of z-pinole assembly, respectively.

MODELING OF THE Z-PINOLE

Bending of the z-pinole in the xz- and yz-planes is considered. Kinetic and strain energy can be written as follows:

$$T_z = \frac{\rho_4}{2}A_4\int_0^{L_z}\Bigg\{[\dot{S}(z,t) + \dot{D}(y_s,t) + y_s\dot{\theta}_1(h,t)$$

$$+ \dot{B}(h,t) + (z+z_0)\dot{\theta}_6(t)]^2 + [\dot{R}(z,t) + \dot{A}(h,t)$$

$$+ (z+z_0)(\dot{\theta}_4(t) + \dot{\theta}_5(t) + \dot{\eta}(x_s,t)) + \dot{V}(x_s,t)$$

$$x_s\dot{\theta}_2(t)]^2 + [\dot{W}(x_s,t) + \dot{E}(y_s,t) + y_s\dot{\alpha}(h,t)$$

$$\dot{C}(h,t) + x_s\dot{\theta}_3(x,t) + l(\dot{\theta}_4(t) + \dot{\theta}_5(t)$$

$$+ \dot{\eta}(x_s,t))]^2 + I_{4x}\dot{\eta}_1^2(z,t) + I_{4y}\dot{\eta}_2^2(z,t)\Bigg\}dz, \tag{11}$$

$$U_x = \frac{1}{2}\int_0^{L_z}\Big\{E_4 I_{4x}\eta'^2_1(z,t) + E_4 I_{4x}\eta'^2_2(z,t)$$
$$+ kG_4A_4\Big[(\eta'_1(z,t) \quad \frac{\partial R(z,t)}{\partial z}\Big]^2$$
$$+ \Big[(\eta_2(z,t) \quad \frac{\partial S(z,t)}{\partial z}\Big]^2\Big\}dz, \tag{12}$$

where:

- $R(z,t)$: Bending in yz-plane,
- $\eta_1(z,t)$: Rotation around x-axis,
- $S(z,t)$: Bending in xz-plane,
- $\eta_2(z,t)$: Rotation around y-axis
- A_4, I_{4x}, I_{4y}, I_{4z}, E_4, G_4, ρ_4 and L_z: Cross sectional area, moments of inertia, Young modulus, shear modulus, density and length of z-axis, respectively.

MODELING OF LEFT COLUMN AND Y-GUIDE WAY

For the left-side support column and y-guide way, the same motions $C(\xi,t)$, $E(y,t)$ and $\gamma_2(y,t)$ have been considered in a similar fashion to the right side column and y-guide way. The kinetic and strain energy can be written as follows:

$$T_L = \frac{1}{2}\Big\{\rho_5 A_5\int_0^h \dot{C}^2(\xi,t)d\xi + \rho_6 A_6\int_0^{L_y}[\dot{E}(y,t)$$
$$\dot{C}(h,t)]^2 dy\Big\}, \tag{13}$$

$$U_L = \frac{1}{2}\Big\{E_5 A_5\int_0^h C'^2(\xi,t)d\xi + \int_0^{y_s}\Big\{E_6 I_{6x}\gamma'^2_2(y,t)$$
$$+ kG_6A_6\Big[\Big(\gamma_2(y,t) \quad \frac{\partial D(y,t)}{\partial y}\Big]^2\Big\}dy\Big\}, \tag{14}$$

where A_5, A_6, I_{6x}, E_5, E_6, G_5, G_6, ρ_5 and ρ_6 are cross sectional area, moments of inertia, Young modulus, shear modulus and density of left column and left y-guide way, respectively.

Note that because of the effect of the drive system, some motions, such as the rotation of the x-carriage around the z-axis and the longitudinal vibration of the z-axis, the horizontal and vertical motion of the y-carriage have been neglected. Furthermore, the gravitational strain energy is neglected.

EQUATIONS OF MOTION

Using Hamilton's principle, equations of motion can be found.

$$\int_{t_1}^{t_2}(\delta T \quad \delta U + \delta W)dt = 0, \tag{15}$$

where δ denotes the variation operator and:

$$T = T_c + T_x + T_y + T_z + T_{bcz} + T_L, \tag{16}$$

$$U = U_c + U_x + U_y + U_z + U_{bcz} + U_L. \tag{17}$$

In the case of a fixed system, neglecting gravitational effects, it can be written that:

$$W = 0. \tag{18}$$

It is interesting to find natural frequencies using a state space variables approach. State variables are defined as follows:

$$\begin{array}{llll}
A = x_1, & A' = x_2, & \alpha = x_3, & \alpha' = x_4,\\
B = x_5, & B' = x_6, & \beta = x_7, & \beta' = x_8,\\
C = x_9, & C' = x_{10}, & \theta_1 = x_{11}, & \theta'_1 = x_{12},\\
D = x_{13}, & D' = x_{14}, & \gamma_1 = x_{15}, & \gamma'_1 = x_{16},\\
E = x_{17}, & E' = x_{18} & \gamma_2 = x_{19}, & \gamma'_2 = x_{20},\\
\theta_3 = x_{21}, & \theta'_3 = x_{22}, & D_1 = x_{23}, & D'_1 = x_{24},\\
\gamma_{11} = x_{25}, & \gamma'_1 = x_{26}, & E_1 = x_{27}, & E'_1 = x_{28},\\
\gamma_{22} = x_{29}, & \gamma'_{22} = x_{30}, & \theta_{33} = x_{31}, & \theta'_{33} = x_{32},\\
W = x_{33}, & W' = x_{34}, & \psi = x_{35}, & \psi' = x_{36},\\
V = x_{37}, & V' = x_{38}, & \varphi = x_{39}, & \varphi' = x_{40},\\
\eta = x_{41}, & \eta' = x_{42}, & W_1 = x_{43}, & W'_1 = x_{44},\\
\psi_1 = x_{45}, & \psi'_1 = x_{46}, & V = x_{47}, & V' = x_{48},\\
\varphi_1 = x_{49}, & \varphi'_1 = x_{50}, & \eta_x = x_{51}, & \eta'_x = x_{52},\\
R = x_{53}, & R' = x_{54}, & \eta_1 = x_{55}, & \eta'_1 = x_{56},\\
S = x_{57}, & S' = x_{58}, & \eta_2 = x_{59}, & \eta'_2 = x_{60}.
\end{array} \tag{19}$$

Note that for free vibration, each state variable can be separated into time and spatial coordinates, so:

$$x_i(\text{coordinates, time}) = X_i(\text{coordinates}).e^{i\omega t}. \tag{20}$$

Now, the equations and boundary conditions of the system are written. Note that the attractiveness of the energy method is that it gives both equations of motion and corresponding boundary conditions.

EQUATIONS OF THE RIGHT COLUMN

$$X'_1(\xi) = X_2(\xi), \tag{21a}$$

$$X'_2(\xi) = X_4(\xi) \quad (\rho_1\omega^2/kG1)X_1(\xi), \tag{21b}$$

$$X_3'(\xi) = X_4(\xi), \tag{21c}$$

$$X_4'(\xi) = \quad (\rho_1\omega^2/E1)X_3(\xi) + (kG_1A_1/E_1I_{1x}) \times [X_3(\xi) \quad X_2(\xi)], \tag{21d}$$

$$X_5'(\xi) = X_6(\xi), \tag{22a}$$

$$X_6'(\xi) = X_8(\xi) \quad (\rho_1\omega^2/kG1)X_5(\xi), \tag{22b}$$

$$X_7'(\xi) = X_8(\xi), \tag{22c}$$

$$X_8'(\xi) = \quad (\rho_1\omega^2/E1)X_7(\xi) + (kG_1A_1/E_1I_{1x}) \times [X_3(\xi) \quad X_2(\xi)], \tag{22d}$$

$$X_9'(\xi) = X_{10}(\xi), \tag{23a}$$

$$X_{10}'(\xi) = \omega^2(\rho_1A_1 + \rho_5A_5)[E_1A_1 + E_5A_5]^{\ \ 1} \times X_9(\xi), \tag{23b}$$

$$X_{11}'(\xi) = X_{12}(\xi), \tag{24a}$$

$$X_{12}'(\xi) = \omega^2\rho_1/G_1X_9(\xi), \tag{24b}$$

EQUATIONS OF RIGHT Y-GUIDE WAY

$$X_{13}'(y) = X_{14}(y), \tag{25a}$$

$$X_{14}'(y) = X_{16}(y) \quad \omega^2\rho_2A_2[k(G_6A_6 + G_2A_2)]^{\ \ 1} \times [X_{13}(y) + yX_{11}(h) + X_5(h)], \tag{25b}$$

$$X_{15}'(y) = X_{16}(y), \tag{25c}$$

$$X_{16}'(y) = [E_6I_{6z} + E_2I_{2z}]^{\ \ 1}\{\quad \omega^2\rho_2I_{2z}X_{15}(y) + k(G_6A_6 + G_2A_2)[X_{15}(y) \quad X_{14}(y)\}, \tag{25d}$$

$$X_{17}'(y) = X_{18}(y), \tag{26a}$$

$$X_{18}'(y) = X_{20}(y) \quad \omega^2[k(G_6A_6 + G_2A_2)]^{\ \ 1} \times \{(\rho_2A_2 + \rho_6A_6)[X_{17}(y) \quad X_9(h)] + \rho_2A_2yX_3(h)\}, \tag{26b}$$

$$X_{19}'(y) = X_{20}(y), \tag{26c}$$

$$X_{20}'(y) = [E_6I_{6x} + E_2I_{2x}]^{\ \ 1}\{\quad \omega^2[\rho_2I_{2x} + \rho_6I_{6x}] \times X_{19}(y) + k(G_6A_6 + G_2A_2)[X_{19}(y) \ \ X_{18}(y)]\}, \tag{26d}$$

$$X_{21}'(y) = X_{22}(y), \tag{27a}$$

$$X_{22}'(y) = \quad \omega^2\rho_2/G_2[X_{21}(y) + X_7(h)]. \tag{27b}$$

EQUATION CORRESPONDING TO ROTATION θ_2

$$\omega^2[J_{bc3z} + M_{bcz}x_s^2 + A_3L_x^3\rho_3/3 + x_s^2A_4L_z\rho_4]K_{bc3}\theta_2 \quad \omega^2x_sA_4\rho_4[(L_z + z_0)^2 \quad z_0^2]/2(\theta_4 + \theta_5) \quad \omega^2[M_{bcz}x_s + A_3L_x^2\rho_3/2 + x_sA_4L_z\rho_4]X_1(h) \quad \omega^2x_s[M_{bcz} + A_4L_z\rho_4]X_{27}(y_s) \quad A_4\rho_4[(L_z + z_0)^2 \quad z_0^2]/2X_{31}(x_s) = 0. \tag{28}$$

EQUATIONS OF X-GUIDE WAY

$$X_{23}'(x) = X_{24}(x), \tag{29a}$$

$$X_{24}'(x) = X_{26}(x) \quad \omega^2\rho_3/kG_3\{X_{23}(x) + X_{17}(y_s) + y_sX_3(h) \quad X_9(h)]\}, \tag{29b}$$

$$X_{25}'(x) = X_{26}(x), \tag{29c}$$

$$X_{26}'(x) = 1/E_3I_{3y}\{\quad \omega^2\rho_3I_{3y}X_{25}(x) + kG_3A_3 \times [X_{25}(x) \quad X_{24}(x)]\}, \tag{29d}$$

$$X_{27}'(x) = X_{28}(x), \tag{30a}$$

$$X_{28}'(x) = X_{30}(x) \quad \omega^2\rho_3/kG_3[X_{27}(x) \quad x\theta_2 + X_1(h)], \tag{30b}$$

$$X_{29}'(x) = X_{30}(x), \tag{30c}$$

$$X_{30}'(x) = 1/E_3I_{3z}\{\quad \omega^2\rho_3I_{3z}X_{29}(x) + kG_3A_3[X_{29}(x) \quad X_{28}(x)]\}, \tag{30d}$$

$$X_{31}'(x) = X_{32}(x), \tag{31a}$$

$$X_{32}'(x) = \quad \omega^2\rho_3/G_3X_{31}(x), \tag{31b}$$

EQUATIONS CORRESPONDING TO THE ROTATIONS θ_4, θ_5 AND θ_6

$$\{K_{bc4} \quad \omega^2[J_{bc4x} + J_{bc5x} + A_4\rho_4(L_zl^2 + [(L_z + z_0)^3 \quad z_0^3]/3)]\}\theta_4 \quad \omega^2\{J_{bc5x} + A_4\rho_4(L_zl^2 + [(L_z + z_0)^3 \quad z_0^3]/3)\}\theta_6 + x_s(L_z + z_0)^2 \quad z_0^2]/2\omega^2\theta_2 = \omega^2[J_{bc4x} + J_{bc5x} + A_4\rho_4(L_zl^2 + [(L_z + z_0)^3 \quad z_0^3]/3)]X_{31}(x_s) + \omega^2A_4\rho_4\{[(L_z + z_0)^2$$

$$z_0^2]/2][X_1(h)+X_{27}(x_s)]+[L_z l^2 X_{27}(x_s)$$
$$+L_z l y_s X_3(h)+l x_s X_{21}(y_s)+L_z l X_{17}(y_s)$$
$$+L_z l X_9(h)\}, \tag{32}$$

$$A_4\rho_4 x_s[(L_z+z_0)^2 \quad z_0^2]/2]\omega^2\theta_2 \quad \{J_{bc5x}+A_4\rho_4(L_z l^2$$
$$+[(L_z+z_0)^3 \quad z_0^3]/3)]\}\omega^2(\theta_4+\theta_5)+K_{bc5}\theta_5$$
$$=\omega^2\{J_{bc5x}+A_4\rho_4(L_z l^2+[(L_z+z_0)^3 \quad z_0^3]/3)]\}$$
$$+\omega^2 A_4\rho_4\times\{[(L_z+z_0)^2 \quad z_0^2]/2][X_1(h)+X_{27}(x_s)]$$
$$+L_z l^2(y_s X_3(h)+x_s X_{21}(h) \quad X_9(h)+X_{23}(x_s)]\}, \tag{33}$$

$$\{\omega^2(J_{bc6y}+A_4\rho_4[(L_z+z_0)^2 z_0^2]/2] \quad K_{bc6y}\}\theta_6$$
$$= \quad \omega^2 A_4\rho_4[(L_z+z_0)^2 \quad z_0^2]/2][y_s X_{11}(h)$$
$$+X_{13}(y_s)+X_5(h)]. \tag{34}$$

EQUATIONS IN *Z*-PINOLE

$$X'_{53}(z)=X_{54}(z), \tag{35a}$$

$$X'_{54}(z)=X_{56}(z) \quad \omega^2\rho_4/kG_4\{X_{53}(z)+(z+z_0)$$
$$\times[X_{41}(x_s)+\theta_4+\theta_5] \quad x_s\theta_2+X_1(h)$$
$$+X_{37}(h)]\}, \tag{35b}$$

$$X'_{55}(z)=X_{56}(z), \tag{35c}$$

$$X'_{56}(z)=1/E_4 I_{4x}\{ \quad \omega^2\rho_4 I_{4x}X_{55}(z)+kG_4A_4$$
$$\times[X_{55}(z) \quad X_{54}(z)]\}, \tag{35d}$$

$$X'_{57}(z)=X_{58}(z), \tag{36a}$$

$$X'_{58}(z)=X_{60}(z)\omega^2\rho_4/kG_4[X_{57}(z)+(z+z_0)\theta_6$$
$$+y_s X_{11}(h)+X_5(h)+X_{13}(y_s)], \tag{36b}$$

$$X'_{59}(z)=X_{60}(z), \tag{36c}$$

$$X'_{60}(z)=1/E_4 I_{4y}\{ \quad \omega^2\rho_4 I_{4y}X_{59}(z)+kG_4A_4$$
$$\times[X_{59}(z) \quad X_{58}(z)]\}. \tag{36d}$$

Now, the corresponding boundary conditions for each member are written.

BOUNDARY CONDITIONS FOR LEFT COLUMN

The left column is considered to be clamped at $\xi=0$. At $\xi=h$, the boundary conditions are determined from Hamilton's principle as follows:

$$X_1(0)=X_3(0)=X_5(0)=X_7(0)=X_9(0)=X_{11}(0)$$
$$=0,$$

$$[2A_2L_y\rho_2+M]X_1(h)+A_3L_x^2\rho_3/2\theta_2 \quad kG_1A_1[X_3(h)$$
$$X_2(h)] \quad A_4\rho_4[(L_z+z_0)^2 \quad z_0^2]/2X_{31}(x_s)$$
$$=0,$$

$$[2A_2L_y\rho_2+M]X_1(h)+A_3L_x^2\rho_3/2\theta_2 \quad kG_1A_1[X_3(h)$$
$$X_2(h)] \quad A_4\rho_4[(L_z+z_0)^2 \quad z_0^2]/2X_{31}(x_s)$$
$$=0,$$

$$[\rho_2A_2L_y^3/3+y_s^2M]X_3(h) \quad E_1I_{1x}X_4(h)[\rho_2A_2L_y^2/2$$
$$+y_sM]X_9(h)+y_sMX_{17}(h)+y_s[x_sM_{bcz}+A_3\rho_3L_x^2/2$$
$$+A_4L_z\rho_4x_s]X_{21}(y_s)+y_s[M_{bcz}+A_4L_z\rho_4]X_{23}(x_s)$$
$$+y_s[x_sM_{bcz}+A_3\rho_3L_x^2/2+A_4L_z\rho_4x_s]X_{31}(x_s)$$
$$=0,$$

$$[\rho_2A_2L_y^2/2+y_s^2M]\omega^2X_5(h)+kG_1A_1[X_7(h)$$
$$X_6(h)] \quad [M \quad M_{bc3}]\omega^2X_{13}(y_s)+[\rho_2A_2L_y^2/2$$
$$+y_s(M \quad M_{bc3})]\times\omega^2X_{11}(h)$$
$$=0,$$

$$\omega^2[\rho_2I_{2y}L_y+J_{bc3y}]X_7(h)+E_1I_{1y}X_7(h)+J_{bc3y}X_{21}(y_s)$$
$$=0,$$

$$\omega^2[2\rho_2A_2L_y+\rho_6A_6L_y+M]X_9 \quad (E_1A_1+E_5A_5)X_{10}(h)$$
$$\omega^2M[y_sX_3(h)+X_{17}(y_s)]$$
$$\omega^2(M_{bcz}+A_4L_z\rho_4)[X_{23}(x_s)+x_sX_{21}(y_s)]$$
$$\omega^2lA_4L_z\rho_4lX_1(x_s)=0,$$

$$[3\rho_2A_2L_y^2/2+y_sM]X_5(h)+G_1I_{1z}X_{12}(h)$$
$$[2\rho_2A_2L_y^3/3+y_s^2M]X_{11}(h) \quad y_sMX_{11}(h)=0, \tag{37}$$

where $M = (M_{bc3} + M_{bcz} + A_3 L_x \rho_3 + A_4 L_z \rho_4)$.

BOUNDARY CONDITIONS OF Y-GUIDE WAY

Because the motions of the y-guide way are relative to the left column, it can be assumed that it is clamped at $y = h$ and free at $y = L_y$. Continuity conditions must be satisfied at $y = y_s$.

$$X_{13}(0) = X_{15}(0) = X_{26}(L_y) = X_{17}(0) = 0,$$

$$X_{19}(0) = X_{30}(L_y) = X_{21}(0) = X_{32}(L_y) = 0,$$

$$X_{13}(y_s) \quad X_{23}(y_s) = X_{24}(L_y) \quad X_{25}(L_y) = 0,$$

$$X_{15}(y_s) \quad X_{25}(y_s) = X_{16}(y_s) \quad X_{26}(y_s) = 0,$$

$$X_{17}(y_s) \quad X_{27}(y_s) = X_{28}(L_y) \quad X_{29}(L_y) = 0,$$

$$X_5(h) + y_s X_{11}(h) + X_{13}(y_s) = X_{21}(y_s) \quad X_{31}(y_s) = 0,$$

$$M[y_s X_3(h) \quad X_9(h)] + [M_{bcz} + A_3 \rho_3 L_x^2/2 + A_4 L_z \rho_4] \times X_{21}(y_s) + (M_{bcz} + A_4 L_z \rho_4) X_{33}(x_s) + l A_4 L_z \rho_4 X_{31}(x_s) = 0,$$

$$X_{19}(y_s) \quad X_{29}(y_s) = X_{20}(y_s) \quad X_{30}(y_s) = 0,$$

$$A_4 L_z \rho_4 l x_s X_{31}(x_s) + x_s[M_{bcz} + A_4 L_z \rho_4] \times [X_{23}(x_s) + X_9(h) + x_s X_{17}(y_s) + y_s X_3(h)] = 0. \quad (38)$$

BOUNDARY CONDITIONS OF THE X-GUIDE WAY

The x-guide way is assumed to be a simply supported beam in the xz-plane and to be clamped-free in the xy-plane. It's torsion along the x-axis is assumed to be clamped at both ends. Continuity conditions must be satisfied at $x = x_s$.

$$X_{33}(0) = X_{43}(L_x) = X_{35}(0) = X_{36}(L_x) = X_{37}(0) = 0,$$

$$X_{39}(0) = X_{50}(L_x) = X_{41}(0) = X_{51}(L_x) = 0,$$

$$X_{33}(x_s) \quad X_{43}(x_s) = X_{35}(x_s) \quad X_{45}(x_s) = 0,$$

$$X_{36}(x_s) \quad X_{46}(x_s) = X_{37}(x_s) \quad X_{47}(x_s) = 0,$$

$$l A_4 L_z \rho_4 X_{41}(x_s) + [M_{bcz} + A_4 L_z \rho_4][X_{17}(y_s) + X_{33}(x_s) + y_s \times X_3(h) \quad X_9(h)] = 0,$$

$$X_{49}(L_x) \quad X_{48}(L_x) = X_{39}(x_s) \quad X_{49}(x_s) = 0,$$

$$X_{40}(x_s) \quad X_{60}(x_s) = X_{41}(x_s) \quad X_{51}(x_s) = 0,$$

$$[J_{bc4x} + J_{bc5x} + A_4 \rho_4 (L_z l^2 + [(L_z + z_0)^3 \quad z_0^3]/3)] \times X_{31}(x_s) + A_4 \rho_4 [(L_z + z_0)^2 \quad z_0^2]/2][X_1(h) + X_{37}(x_s)] + A_4 \rho_4 L_z l[\quad X_9(h) + X_{33}(x_s) + y_s X_3(h)] = 0. \quad (39)$$

BOUNDARY CONDITIONS OF THE Z-PINOLE

The motion of the z-pinole in the zx- and zy-planes is assumed to be clamped-free.

$$X_{53}(0) = X_{55}(0) = X_{56}(L_z) = X_{57}(0) = X_{59}(0) = X_{60}(L_z) = 0,$$

$$X_{54}(L_z) \quad X_{55}(L_z) = X_{58}(L_z) \quad X_{59}(L_z) = 0. \quad (40)$$

METHOD OF SOLUTION

In order to obtain the natural frequencies of the system, initially, the rotations θ_2, θ_4, θ_5 and θ_6 will be found from Equations 27 and 32 to 34 and substituted in the model equations. Note that the derived equation can be listed into two categories. The first category contains equations corresponding to the motions of the left column and may be written in the following form:

$$\{X'(\xi)\} = [D]\{X(\xi)\}, \quad (41)$$

where $\{X\} = \{X_i, X_j, \ldots, X_n\}^t$ represents the vector of corresponding state variables. The sign ($'$) denotes derivation, with respect to ξ. Furthermore, the matrix, $[D]$ is in the form:

$$[D] = [K] \quad \omega^2[M], \quad (42)$$

where $[K]$ and $[M]$ can be found readily by arranging the equations in terms of ω^2. The solution of Equation 41 can be written as:

$$\{X'\} = [U]^t[\Lambda][U]\{C\}, \quad (43)$$

where $[U]$ is the matrix of the eigenvectors of $[D]$ and $\{C\}$ is a column vector of constants. The diagonal matrix, $[\Lambda]$, is in the form:

$$[\Lambda] = \begin{bmatrix} \ddots & & 0 \\ & e^{\lambda_i \xi} & \\ 0 & & \ddots \end{bmatrix}, \tag{44}$$

where λ_i's are eigenvalues of the dynamic matrix $[D]$ depending on ω^2. Taking $\xi = h$, the value of $X_i(h)$ in terms of C_i's can be found and will be used in other formulas.

The second category of equations contains equations corresponding to the motion of members in the y-guide way, x-guide way and z-pinole. These equations are in the form:

$$\{X'\} = [D]\{X\} + [F], \tag{45}$$

where, in the above equation, the vector, $\{X\}$, may be a function of x, y or z. In order to solve the equation, initially, the matrix of the eigenvectors of $[D]$, i.e. $[U]$, should be found. Using the following transformation:

$$\{Y\} = [U]\{X\}, \tag{46}$$

it can be obtained:

$$\{Y'\} = [U]^t[D][U]\{Y\} + [U]^t[F], \tag{47}$$

or:

$$\{Y'\} = [\Lambda]\{Y\} + [U]^t[F]. \tag{48}$$

Because the matrix, $[\Lambda]$, is diagonal, an ordinary differential equation for each Y_i is obtained and can be solved readily. Note that for each Y_i, there is a constant of integration, C_i. Next, using Equation 46, each X_i can be found. Taking $y = y_s$ and $x = x_s$, the values of $X_i(x_s)$ or $X_i(y_s)$ can be found in terms of C_i's and implemented in other formulas.

So far, X_i's are obtained in terms of C_i's and λ_i's. Now, substituting X_i's into boundary conditions and rearranging in terms of C_i's, the following equation may be obtained:

$$[B]\{C\} = \{0\}. \tag{49}$$

In order to have a nontrivial solution, the determinant of coefficient matrix $[B]$ must be zero, i.e.:

$$\text{Det}[B] = 0. \tag{50}$$

This yields an equation in terms of ω^2. Because the presented model is based on continuous beam modeling, the obtained equation is transcendental and has an infinite number of solutions. For each value of x_s and y_s, it can solve for ω^2.

In order to illustrate an example of modeling, a CMM with the following specifications is modeled.

$\rho_i = 7850$ kg/m^3,	$E_i = 200$ Gpa,
$G_i = 70$ Gpa,	$h = 0.825$ m,
$L_y = 0.6$ m,	$L_x = 1.2$ m,
$L_z = 0.75$ m,	$l = 0.16$ m,
$z_0 = 0.125$ m,	$K_{\text{bearings}} = 5e7$ N.m/rad,
$A_1 = 7.6e\quad 3$ m^2,	$A_2 = 4.6e\quad 3$ m^2,
$A_3 = 7.6e\quad 3$ m^2,	$A_4 = 1.0e\quad 3$ m^2,
$A_5 = 4.6e\quad 3$ m^2,	$A_6 = 4.6e\quad 3$ m^2,
$I_{1x} = 7.86e\quad 5$ m^4,	$I_{1y} = 1.3e\quad 5$ m^4,
$I_{1z} = 3.58e\quad 5$ m^4,	$I_{2x} = 1.35e\quad 5$ m^4,
$I_{2y} = 1.38e\quad 6$ m^4,	$I_{2z} = 2.04e\quad 5$ m^4,
$I_{3x} = 5.94e\quad 5$ m^4,	$I_{3y} = 2.82e\quad 5$ m^4,
$I_{3z} = 6.35e\quad 5$ m^4,	$I_{4x} = 4.21e\quad 7$ m^4,
$I_{4y} = 4.21e\quad 7$ m^4,	$I_{5x} = 1.35e\quad 5$ m^4,
$I_{6x} = 1.35e\quad 5$ m^4,	$I_{6y} = 2.04e\quad 5$ m^4,
$I_{6z} = 6.95e\quad 5$ m^4,	$J_{bc3z} = 0.18$ kgm^4,
$J_{bc3z} = 0.3$ kgm^4,	$J_{bc4x} = 2.7$ kgm^4,
$J_{bc5x} = 0.05$ kgm^4,	$J_{bc6y} = 0.05$ kgm^4,
$M_{bc3} = 10$ kg,	$M_{bcz} = 35$ kg.

The natural frequencies of the system depend on the location of the probe. The variation of the 1st and 2nd natural frequencies versus the $\{XY\}$ position of the probe is illustrated in the 3-D plots of Figures 2 and 3, when the tip of the probe is at its maximum distance from the x-guideway ($z = L_z$). It is clear that increasing the values of x_s and y_s reduces the natural frequencies of the system. Note that the driving systems of the x- and y-motions are located at the right side of the CMM, consequently, the x-guide way slides on the left y-guide way, and the minimum value of the natural frequency occurs when the probe is located at the end of the x-guide way. The variation of the 1st to 4th natural frequencies of the system at different $\{XY\}$ positions of the probe ($z = L_z$), obtained from the

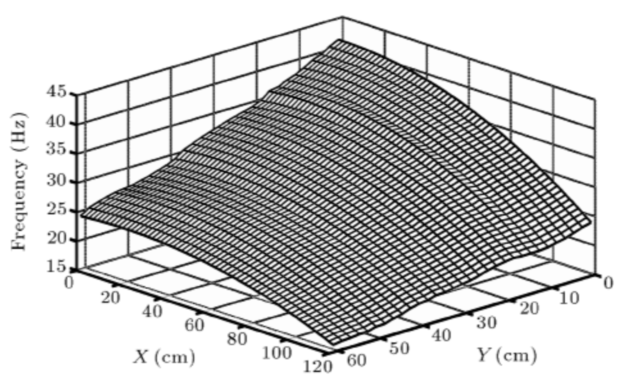

Figure 2. First natural frequency ($z = L_z$).

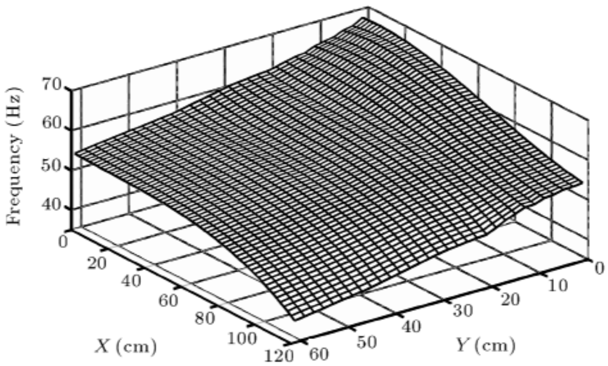

Figure 3. Second natural frequency ($z = L_z$).

analytical and finite elements method, are presented in Figures 4 to 7. Findings from the analytical and finite elements method indicate high accuracy and good agreement between the results.

CONCLUSION

The modeling of a gantry type CMM, based on Timoshenko beam theory using Hamilton's principle, was developed and the natural frequencies of the system were calculated at various positions of the probe in the system. The parameters of motion are defined so that the kinetic energy of each element is influenced by the motion of adjacent connected elements, while the strain energy is independent of the position and velocity of these elements. Results indicate that in the first four natural frequencies, for any y-position of the probe, the system has the lowest natural frequency when the probe moves toward the end of the x-guide way. Moreover, the values of natural frequencies again decrease when the system moves toward the end of the y-guide way. However, when the system is at the end of both x- and y-guide ways, the minimum value of the

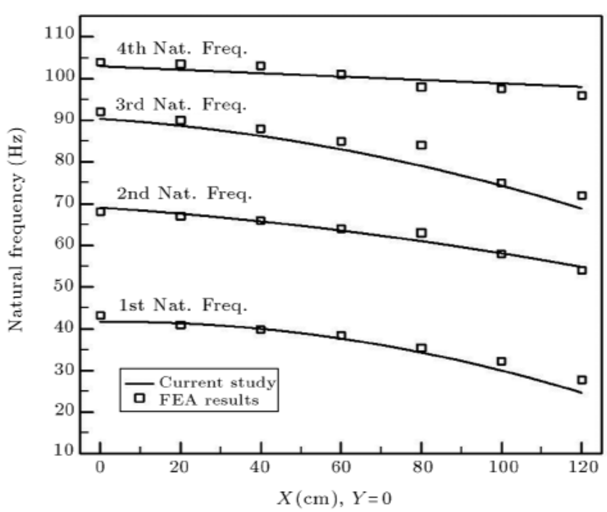

Figure 4. Natural frequencies in $Y = 0$ for $X = 0$ to $X = 120$ cm.

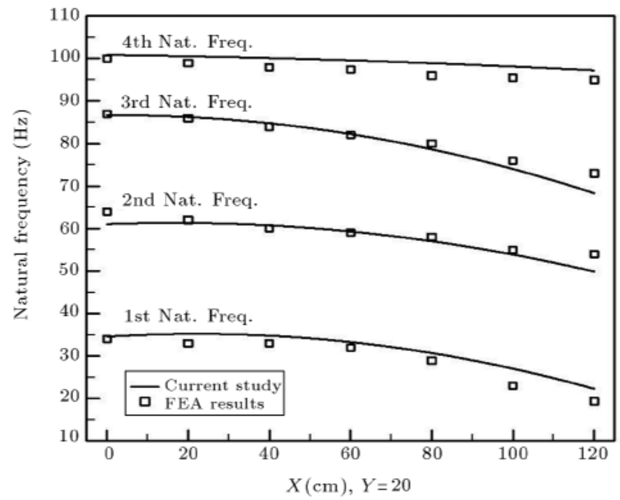

Figure 5. Natural frequencies in $Y = 20$ for $X = 0$ to $X = 120$ cm.

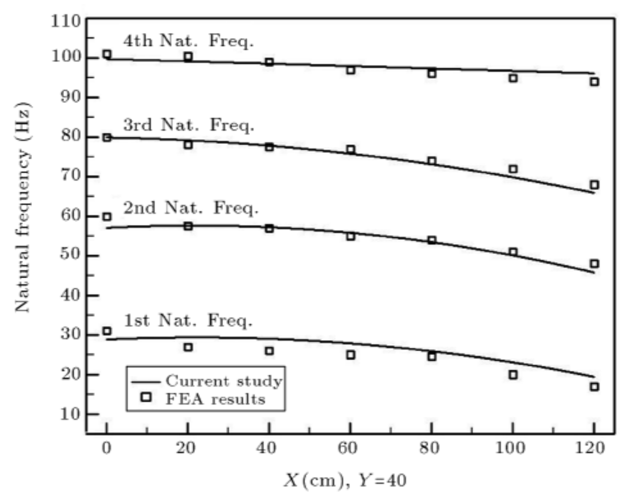

Figure 6. Natural frequencies in $Y = 40$ for $X = 0$ to $X = 120$ cm.

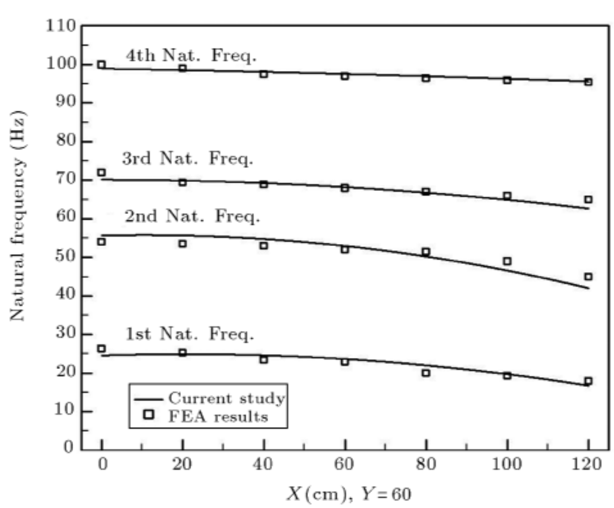

Figure 7. Natural frequencies in $Y = 50$ for $X = 0$ to $X = 120$ cm ($Z = L_z$).

natural frequencies occurs. This is due to the minimum stiffness of the system in that position.

These frequencies help us to present any resonant when the system is under operation. Any interference between the systems natural frequencies and the frequencies of the driving system should be prevented.

A good agreement and high accuracy in comparison with the results found by the finite element analysis can be observed.

REFERENCES

1. Weekers, W.G. "Compensation for dynamic errors of coordinate measuring machines", PhD Thesis, Eindhoven University of Technology (1996).
2. Weekers, W.G. and Schellekens, P.H.J. "Compensation for dynamic errors of coordinate measuring machines", *Measurement*, **20**, pp 197-209 (1997).
3. Barakat, N.A., Elbestawi, M.A. and Spence, A.D. "Kinematic and geometric error compensation of a coordinate measuring machine", *Machine Tools & Manufacture*, **40**, pp 833-850 (2000).
4. De Nijs, J.F.C., Schellekens, M.G.M. and Van der Wolf, A.C.H. "Modeling of a coordinate measuring machine for analysis of its dynamic behavior", *Annals of the CIRP*, **37**, pp 507-510 (1988).
5. Vermeulen, M. "High precision 3D-coordinate measuring machine, design and prototype development", PhD Thesis, Eindhoven University of Technology (1995).
6. Teewwsen, J.W.M., Soons, J.A. and Schelekens, P.H. "A general method for error description of CMMs using polynomial fitting procedure", *Annals of the CIRP*, **38**, pp 505-510 (1989).
7. Zhang, G. "Error compensation of coordinate measuring machines", *Annals of the CIRP*, **34**, pp 445-448 (1985).
8. Lee, H.P. "Transverse vibration of a Timoshenko beam acted on by an accelerating mass", *Applied Acoustics*, **47**, pp 319-330 (1996).
9. Esmailzadeh, E. and Ghorashi, M. "Vibration analysis of a Timoshenko beam subjected to a traveling mass", *Journal of Sound and Vibration*, **199**, pp 615-628 (1997).
10. Reddy, J.N. "Energy and variational methods in applied mechanics", John Wiley & Sons (1984).
11. Ramezani, S. "Vibration analysis of gantry type CMMs and optimization using genetic algorithm", M.Sc. Thesis, Sharif University of Technology, Tehran, Iran (2003).