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Designing an Infinite Channel Server

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In this paper, the designing of a parallel channel queuing system is considered in which it is desirable to serve the customers immediately upon their arrival. To design such a service facility, the first and the second moments of the number of customers are obtained in an infinite channel server under the assumption that the arrival process has a general probability structure and the service times of customers are independent and identically distributed random variables. With the knowledge about the above two moments, one can obtain the approximate number of channels such that the queue length becomes zero.

INTRODUCTION AND LITERATURE REVIEW

There are many situations in the production of goods and services in industries in which the cost of waiting time for customers is very large or where it is desirable to serve the customers immediately upon their arrival at the service facility. Freeway toll crossings, hospitals services, telephone switch boards, etc. are some examples of such situations. A proper design of such systems leads to a considerable saving of time and money.

One way of fulfilling this desire is to provide a large number of identical channels in parallel. However, to design such service facilities, it is important to determine the number of parallel channels such that there is a negligible probability for there to be a queue of customers waiting to enter the service. Knowing the first and the second moments of the number of customers in an infinite channel queue, one might set the number of parallel channels equal to the mean plus a few standard deviations.

In the literature of the queuing theory and stochastic processes, the approximate value of the first moment of the queue length in a multi-channel queuing system, with a Poisson arrival process and a general service time distribution, has been determined. When the service time distribution is gamma, this approximation is very close to the true mean and, in the case of exponential distribution, it is actually equal to the true mean [1].

Determining the distribution of the number of customers in a service facility with an infinite number of servers, for the case where the arrival process is Poisson and the service times are independent and identically distributed (i.i.d.) random variables with a general distribution function G, has been investigated [2-5]. In the case of transient behavior, it has been shown that the number of customers in the system at time t and, also, the number of departures from the system up to time t, have a Poisson distribution [4,5]. Also, it is well known that for the steady-state case, the variance of the number of customers, N, for the $M/G/\infty$ queue is equal to the traffic intensity, ρ (the ratio of the arrival rate to the service rate). This follows from the fact that the distribution of N for an $M/G/\infty$ queue is Poisson with parameter ρ [5,6].

In this paper, no assumptions are made about the arrival process, i.e., the arrival process can posses any structure. Formulas are derived for the first and second moment of the number of customers in the system for this arrival process and for the case where the service times are i.i.d. random variables.

In the following sections, first, the notations and the assumptions needed to obtain the first and the second moments of the number of customers in the system are defined. Then, the formula for the first moment in the steady-state situation is obtained, and a formula for the second moment of the number of customers in the system is derived. After that, the formula derived for the first and the second moments of N is specialized into two cases. In case one, the

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arrival process is Poisson and in the second case it is a stationary process with an orderliness property. Finally, the conclusions are presented.

NOTATIONS AND ASSUMPTIONS

In order to evaluate the first and the second moment of the number of customers in the system at time t, the time axis is divided into small intervals of width δ. For any fixed t, the intervals will be measured and numbered backward from t with the jth interval from $t - j\delta$ to $t - (j-1)\delta$. The notations and assumptions are defined as follows:

- $N(t)$: the number of customers in the system at time t,
- $E[N(t)]$: the expected value of $N(t)$,
- $\mathrm{Var}[N(t)]$: the variance of $N(t)$,
- $\sigma(t)$: the standard deviation of $N(t)$,
- $N_j(t)$: the number of customers who arrive in the jth interval and are still in the system at time t, $j = 1, 2, \cdots$,
- $A_j(t)$: the number of customers who arrive during the jth interval, $j = 1, 2, \cdots$,
- $m(v, u)$: the expected value of the number of customers who arrive in the interval (u, v), assuming a customer has arrived at time u.

To simplify the notations, $A_j(t)$ is indicated with A_j and $N_j(t)$ with N_j, keeping in mind that A_j and N_j depend on t.

Given the above notations, the number of customers in the system at time t is:

$$N(t) = \sum_{j=1}^{\infty} N_j. \tag{1}$$

It is assumed that the service times, (S)'s, are i.i.d. random variables with a distribution function denoted by:

$$G(x) = P(S \leq x), \tag{2}$$

$$\overline{G}(x) = 1 - G(x). \tag{3}$$

FIRST MOMENT DERIVATION

To evaluate $E[N(t)]$, it is first noted that from Equation 1, one can write:

$$E[N(t)] = E\left[\sum_{j=1}^{\infty} N_j\right] = \sum_{j=1}^{\infty} E[N_j].$$

$E[N_j]$ is obtained by conditioning on A_j, thus;

$$E[N(t)] = \sum_{j=1}^{\infty} E\left[E[N_j/A_j]\right]. \tag{4}$$

If δ is sufficiently small, then the random variable, N_j/A_j, has approximately a binomial distribution with the number of trials equal to A_j, the number of customers arrived in the interval $t - j\delta$ to $t - (j-1)\delta$, and the probability of success equal to $\overline{G}(j\delta)$. Thus;

$$E[N_j/A_j] = A_j\overline{G}(j\delta), \tag{5}$$

$$\mathrm{Var}[N_j/A_j] = A_j\overline{G}(j\delta)G(j\delta). \tag{6}$$

If one substitutes Equation 5 into Equation 4, one has:

$$E[N(t)] = E\left[\sum_{j=1}^{\infty} \overline{G}(j\delta)A_j\right] = \sum_{j=1}^{\infty} E[\overline{G}(j\delta)A_j],$$

or:

$$E[N(t)] = \sum_{j=1}^{\infty} \overline{G}(j\delta)E[A_j]. \tag{7}$$

Now, in the limit, when $\delta \to 0$, one can write Equation 7 as:

$$E[N(t)] = \int_{-\infty}^{t} \overline{G}(t - u)dE[A(u)]. \tag{8}$$

Equation 8 gives the first moment of the number of the customers in an infinite channel queuing system in a steady-state situation, in which the arrival process has a general probability structure and the service times are i.i.d. random variables.

SECOND MOMENT DERIVATION

To evaluate $E[N^2(t)]$, using Equation 1, one can write:

$$E[N^2(t)] = E\left[\left(\sum_{j=1}^{\infty} N_j\right)^2\right]$$

$$= E\left[\sum_{j=1}^{\infty} N_j^2 + 2\sum_{k>j}\sum N_jN_k\right]$$

$$= \sum_{j=1}^{\infty} E[N_j^2] + 2\sum_{k>j}\sum E[N_jN_k]. \tag{9}$$

$E[N_j^2]$ and $E[N_jN_k]$ are obtained by conditioning on A_j and A_jA_k, respectively. That is, Equation 9 can be written as:

$$E[N^2(t)] = \sum_{j=1}^{\infty} E\left[E[N_j^2/A_j]\right] + 2\sum_{k>j}\sum E\left[E[N_jN_k/A_j, A_k]\right]. \tag{10}$$

Using the fact that for any random variable, X, the equation $E(X^2) = \text{Var}(X) + [E(X)]^2$ holds, from Equations 5 and 6, one has:

$$E[N_j^2/A_j] = \overline{G}(j\delta)G(j\delta)A_j + \overline{G}^2(j\delta)A_j^2. \tag{11}$$

Furthermore, because of the independence of service times, one has:

$$E[N_jN_k/A_j, A_k] = E[N_j/A_j, A_k]E[N_k/A_j, A_k] = E[N_j/A_j]E[N_k/A_k]. \tag{12}$$

Thus, from Equation 5, one can write Equation 12 as:

$$E[N_jN_k/A_j, A_k] = \overline{G}(j\delta)\overline{G}(k\delta)A_jA_k. \tag{13}$$

Now, from Equations 11 and 13, Equation 10 can be rewritten as:

$$E[N^2(t)] = \sum_{j=1}^{\infty}\overline{G}(j\delta)G(j\delta)E[A_j] + \sum_{j=1}^{\infty}\overline{G}^2(j\delta)\left\{\text{Var}(A_j) + [E(A_j)]^2\right\} + 2\sum_{k>j}\sum\overline{G}(j\delta)\overline{G}(k\delta)\Big[\text{Cov}(A_j, A_k) + E(A_j)E(A_k)\Big]. \tag{14}$$

From Equation 7, one can write:

$$E^2[N(t)] = \left(\sum_{j=1}^{\infty}\overline{G}(j\delta)E[A_j]\right)^2 = \sum_{j=1}^{\infty}\overline{G}^2(j\delta)E^2[A_j] + 2\sum_{k>j}\sum\overline{G}(j\delta)\overline{G}(k\delta)E[A_j]E[A_k].$$

Using Equations 3, 7, 14 and the above relation, the standard deviation of the number of customers in the system, $\sigma(t)$, can be obtained by:

$$\sigma^2(t) = E[N^2(t)] \quad \{E[N(t)]\}^2 = E[N(t)] + \sum_{j=1}^{\infty}\overline{G}^2(j\delta)\left[\text{Var}(A_j) \quad E(A_j)\right] + 2\sum_{k>j}\sum\overline{G}(j\delta)\overline{G}(k\delta)\text{Cov}(A_j, A_k). \tag{15}$$

In the limit as $\delta \to 0$, Equation 15 can be written as:

$$\sigma^2(t) = E[N(t)] + \int_{\infty}^{t}\overline{G}^2(t \quad u)[\text{Var}(dA(u)) \quad E(dA(u))] + \int_{\infty}^{t}\int_{u}^{t}\overline{G}(t \quad u)\overline{G}(t \quad v)\text{Cov}[dA(u), dA(v)]. \tag{16}$$

Hence, Equations 8 and 16 give the mean and the standard deviation of the number of customers in an infinite channel server with independent and identically distributed service times without any assumptions on the arrival processes. Knowing the mean and the standard deviation of $N(t)$, one can easily find the approximate number of servers such that the queue length is negligible.

In what follows, Equations 8 and 16 are employed in two scenarios of arrival processes, the Poisson and the stationary arrival processes.

Poisson Arrival Process

Equations 8 and 16 give the steady-state mean and standard deviation of the number of customers in an infinite channel queuing system for a general case where the service times are i.i.d. random variables and no assumption is made on the probability structure of the arrival process. However, if the arrival process is Poisson with rate λ and the service times are i.i.d. with a general distribution with mean $\frac{1}{\mu}$, then;

$$\text{Var}(A_j) = E(A_j), \quad \text{Cov}(A_j, A_h) = 0,$$

$$\text{for } j \neq k \quad \text{and} \quad E[A_j] = \lambda\delta.$$

Thus, from Equation 10 one has:

$$E[N(t)] = \lambda E[S] = \frac{\lambda}{\mu} = \rho. \tag{17}$$

Also, from Equations 8 and 15 or 16, one can write:

$$\sigma^2(t) = E[N(t)] = \rho. \tag{18}$$

That is, in the long run, the standard deviation of the number of the customers in the system is equal to traffic intensity. This result is the same as that obtained in [2,3,6].

Stationary Arrival Process

Consider an infinite channel queue in which the service times of the customers are i.i.d. random variables. The arrival process is stationary and has an orderliness property. For an orderliness property, the probability of occurrence of two or more arrivals in a sufficiently small interval of time is negligible when compared with the probability of one (or no) arrival. In mathematical notation, if $P_i(h)$ denotes the probability of precisely i arrivals in the interval with length h, the latter assumption may be defined by the following equation [3]:

$$\sum_{i=2}^{\infty} P_i(h) = 1 \quad P_0(h) \quad P_1(h) = o(h) \quad \text{as } h \to 0. \tag{19}$$

Note that a function, f, is said to be $o(h)$, if [3]:

$$\lim_{h \to 0} \frac{f(h)}{h} = 0,$$

For the stationary arrival process, one has [3,6]:

$$\lim_{\delta \to 0} \frac{1 \quad P_0(\delta)}{\delta} = \omega,$$

in which, ω is a constant number. In other words, as $\delta \to 0$, one has:

$$1 \quad P_0(\delta) = \omega\delta + o(\delta). \tag{20}$$

It is also noted that for a stationary process with rate λ, except for the case $P_0(x) = 1$, the expected number of arrivals in the interval $(0, x)$ is [3,6]:

$$E[A(x)] = \lambda x. \tag{21}$$

In addition, if the arrival process has the orderliness property, then [3]:

$$\omega = \lambda. \tag{22}$$

Thus, when the arrival process is stationary with rate λ, then, the expected number of customers who arrive in an interval with length δ from Equation 21 is $E[A_j] = \lambda\delta$ and Equation 7 can be written as:

$$E[N(t)] = \lambda \sum_{j=1}^{\infty} \overline{G}(j\delta)\delta. \tag{23}$$

When $\delta \to 0$, from Equation 8, one has:

$$E[N(t)] = \int_{\infty}^{t} \overline{G}(t \quad u)\lambda du = \lambda \int_{0}^{\infty} \overline{G}(u) du, \tag{24}$$

or:

$$E[N(t)] = \lambda E[S] = \rho.$$

As seen, this result is a special case of Little's formula "$L = \lambda W$" [7-10].

To obtain the standard deviation of the number of customers in the system for a stationary arrival process with an orderliness property, the general Equation 16 is used. To do this, it is first noted that both the first and the second moment of an indicator random variable are equal to the probability of its being equal to one. In addition, according to the orderliness property, i.e. Equation 19, $dA(t)$ is either zero or one. Hence:

$$\begin{aligned} E\left\{[dA(t)]^2\right\} &= E[dA(t)] \\ &= P\{\text{One arrival in } (t, t+dt)\}. \end{aligned} \tag{25}$$

Furthermore, from Equations 19, 20 and 22, one obtains:

$$E[dA(t)] = \lambda dt. \tag{26}$$

Also, note that from Equation 26, one has:

$$E^2[dA(t)] = (\lambda dt)^2 = o(dt). \tag{27}$$

From Equations 25 and 26, one can write:

$$E\{[dA(t)]^2\} = \lambda dt. \tag{28}$$

Hence, Equations 27 and 28 give the variance of $dA(t)$ as:

$$\text{Var}[dA(t)] = \lambda dt + o(dt). \tag{29}$$

Note that for stationary arrival processes $m(v, u) = m(v \quad u)$. Then, by conditioning on $dA(u)$ and using Equations 25 and 26, one can write:

$$E[dA(u)dA(v)] = d[m(v \quad u)]\lambda du. \tag{30}$$

Hence, using Equations 26, 29 and 30 in Equation 16, one will have:

$$\sigma^2(t) = \rho + 2\lambda \int_{\infty}^{t} \int_{u}^{t} \overline{G}(t \quad u)\overline{G}(t \quad v) d[m(v \quad u) \quad \lambda v] du. \tag{31}$$

Let $y = t \quad u$ and $x = v \quad u$, then, Equation 31 can be written as:

$$\sigma^2(t) = \rho + 2\lambda \int_{0}^{\infty} \int_{0}^{y} \overline{G}(y)\overline{G}(y \quad x) d[m(x) \quad \lambda x] dy. \tag{32}$$

When the interarrival times of customers are i.i.d random variables, $m(x)$ is the renewal function.

Thus, knowing that the arrival process is stationary and has an orderliness property, the standard deviation of the number of the customers in the system can be obtained by Equation 32. Clearly, for the Poisson arrival process $m(x) = \lambda x$ and from Equation 32, once again, Equation 18 is obtained for $M/G/\infty$.

Having information on the first moment (ρ) and the second moment (Equation 32) of the number of customers in an infinite channel queue, one might choose the number of parallel channels equal to the mean plus a few standard deviations.

CONCLUSIONS

In this paper, an infinite channel queuing system is considered in which the arrival process has a general probability structure and the service times are i.i.d. random variables. Two general formulae were derived, one for the first moment and the other for the second moment on the number of the customers in the system in a steady-state situation. Then, the formulae were employed in two special cases of the arrival processes; the first arrival process being Poisson and the second one a stationary process and the mean and the standard deviation of the number of the customers in the system were obtained. Given the mean and the standard deviation of the number of the customers, one can easily obtain the approximate number of parallel servers in an infinite channel queuing system such that there is a negligible probability for formation of a queue of customers waiting to enter the service.

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