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A Fuzzy Coherent Hierarchical Location-Allocation Model for Congested Systems

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A fuzzy queuing coherent hierarchical location-allocation model is developed for congested systems. The parameters of the model are approximately evaluated and stated as fuzzy-numbers. The coverage of demand nodes is also considered in an approximate manner and is stated by the degree of membership. Using the queuing theory and fuzzy conditions, a coherent hierarchical model is developed for the Maximal Covering Location Problem (MCLP). An example problem is solved and presented, along with results. Conclusions and future extensions are also included.

INTRODUCTION AND LITERATURE REVIEW

There exist many hierarchical structures at service networks, both in the public and private sectors. Here, some examples of hierarchical service networks are elaborated upon. In health care systems, general centers provide low-level services, such as primary health care, and specialized hospitals provide high-level facilities. Schools have a hierarchical structure by nature, because there are primary schools, middle schools and high schools. Numerous other examples of hierarchical structures can be found, such as: Airports, computer service centers, day-care centers, emergency medical centers, regional health facilities, social service centers, police centers, warehouses, distribution systems and so on.

In hierarchical systems, facilities at different levels provide different types of service. However, there is often a linkage between different levels, which makes it impossible to solve the location problem for each level separately. For example, in the area of health care services, when deemed appropriate, the customers of a particular primary health care center can be referred to a hospital that is designated to provide high-level services. Thus, the location problems with a hierarchical structure should be modeled and solved simultaneously for both low- and high-levels, as a unified problem. Due to the nature of relations among the various levels, both on the demand side and the service side, analysis of hierarchical service systems is a challenge to be met.

This research effort is devoted to the development of models for hierarchical Location Set Covering Problems (LSCP) and hierarchical Maximal Covering Location Problems (MCLP) in coherent systems. The LSCP, which was introduced by Toregas et al. [1], attempts to locate the minimum number of servers to cover all the demand nodes within the distance or time standard. Unlike the LSCP, the MCLP, which was developed by Church and ReVelle [2], tries to maximize the covered population within the distance or time standard for a fixed number of servers. Hierarchical service systems can be classified according to their structure as nested and non-nested systems [3].

In a nested system, the high-level servers provide low-level services too, while in non-nested systems, each level offers its own special service. A hierarchical system is labeled as coherent if all customers of a particular low-level server are the customers of a particular high-level server as well. In a referral system, the users can go to a higher-level server only when referred by a low-level server. A non-referral system lacks such restriction.

Church and Eaton [4] and Gerrard and Church [5] provide reviews of early hierarchical models. Serra and ReVelle [6,7] combined hierarchical location and coherent districting in a later effort. Serra, Marianov and

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ReVelle [8] developed a hierarchical maximum capture model for location in a competitive environment. Serra presented his model for the coherent covering location problem in 1996 [9].

The assumption of demand congestion at servers has not been considered in any of the above models. Once the demand rate (for service) exceeds the service rate, congestion occurs and a waiting line emerges. To enhance the quality of rendered service in congested systems, it is obvious that resorting to the queuing theory can be quite helpful. Marianov and Serra [10] published an article on hierarchical location-allocation models for congested systems in 2000. In their article, they developed two hierarchical location models for LSCP and MCLP, based on a queuing theory. The probabilistic nature of their approach makes their models more realistic, even though they adopt the often used crisp conditions. In fact, to make the models more realistic, one can use fuzzy conditions. One good reason for such a notion is that exact evaluation of parameters, such as the rate of demand (for service) at each node and the rate of service etc., is not always easily attainable, while stating these parameters in non-exact forms is readily possible. This paper aims at developing a new location-allocation model for congested systems on the basis of fuzzy and queuing theories. Aside from fuzzifying the parameters, it is intended to consider the variables, which allocate the demand nodes to the servers on the basis of degrees of membership. As such, each node is not required to receive its service from a single server. By considering these conditions, a fuzzy coherent hierarchical queuing model is developed for MCLP, which more closely resembles real world situations. Finally, the performance of this model is compared against the results obtained from one of the existing models and, at the end, proposals are made for future research.

REVIEW OF A COHERENT HIERARCHICAL MAXIMAL COVERING LOCATION PROBLEM (CHIQ-MCLP)

To lay the foundation for presenting the FCHQ-MCLP model, it is appropriate to review the CHQ-MCLP model as presented by Marianov and Serra [10], as follows:

$$\max Z = \sum_i \sum_j \sum_k a_i X_{ijk}, \tag{1}$$

s.t.

$$\sum_{j,k} X_{ijk} \leq 1, \qquad \forall i, \tag{2}$$

$$X_{ijk} \leq Y_{jk}, \qquad \forall i,j,k, \tag{3}$$

$$Y_{jk} \leq Z_k, \qquad \forall j,k, \tag{4}$$

$$Y_{jk} \leq W_j, \qquad \forall j,k, \tag{5}$$

$$\sum_k Y_{jk} \leq 1, \qquad \forall j, \tag{6}$$

$$\sum_j W_j = P_l, \tag{7}$$

$$\sum_k Z_k = P_h, \tag{8}$$

$$P[\text{low-level server } j \text{ has } \leq b \text{ people in queue}] \geq \alpha \quad \forall j, \tag{9}$$

$$P[\text{high-level server } k \text{ has } \leq b \text{ people in queue}] \geq \alpha \quad \forall k,$$

$$X_{ijk}, W_j, Z_k, Y_{jk} = 0, 1, \qquad \forall i,j,k, \tag{10}$$

where:

- X_{ijk} allocation variable that takes value 1, if population at demand node i is allocated to the low-level server j, and to the high-level server k, and zero otherwise;
- Y_{jk} allocation variable that takes value 1, if demand nodes assigned to the low-level server j, are also assigned to the high-level server k. In other words, it relates a low-level server to a high-level server;
- W_j location variable that takes value 1, if a low-level server is located at node j, and zero otherwise;
- Z_k location variable that takes value 1, if a high-level server is located at node k, and zero otherwise;
- a_i the population at demand node i;
- P_l the number of low-level servers to be located;
- P_h the number of high-level servers to be located.

The objective function (Equation 1) attempts to maximize the covered population. The first constraint (Relation 2) means that each demand node can be covered by just one server. The constraints in Relations 3 to 5 assume that allocation variables can take value 1 only when a low-level server and a high-level server have already been located at nodes j and k. The constraint in Relation 6 means a low-level server can relate to just one high-level server. The constraints in Equations 7 and 8 are related to the specific and bounded number of servers. The constraints in Relations 9 and 10 are related to demand congestion at servers or the quality of service, because they try to make sure that the queue length at each server does not exceed b with probability at least α.

To write the constraints in Relations 9 and 10 in mathematical form, Marianov and Serra borrow notions from the queuing theory to arrive at the final form of constraints, as shown below:

$$\sum_{i,k} f_i X_{ijk} \leq \mu_j^l \sqrt[b+2]{1 \quad \alpha} \quad \forall j, \tag{11}$$

$$\sum_{i,j} \beta_j f_i X_{ijk} \leq \mu_k^h \sqrt[b+2]{1 \quad \alpha} \quad \forall k, \tag{12}$$

where:

- f_i rate of appearance of requests for service at node i,
- μ_j^l service rate at low-level server j,
- μ_k^h service rate at high-level server k,
- β_j percentage of the requests referred to the high-level servers by the low-level server j.

Since it is assumed that at each service center there exists just one server and because each node can be served by only one server, the servers are independent and the queuing model at each server is an M/M/1 model.

So, by substituting constraints in Relations 11 and 12 for Relations 9 and 10, one will arrive at the final form of CHQ-MCLP.

FUZZY COHERENT HIERARCHICAL QUEUING MAXIMAL COVERING LOCATION PROBLEM (FCHQ-MCLP)

In this section, the mathematical model for FCHQ-MCLP is developed. For this purpose, first, the parameters, variables and fuzzy sets that are used in the formulation of the model are defined and, then, the formulation process is described.

Parameters, Variables and Fuzzy Sets

The parameters that are used in the model are defined as follows:

- a_i the population of node i; a crisp number,
- $\tilde{b}_l : (b_l^p, b_l^m, b_l^o)$ a triangular fuzzy number, which stands for the maximum number at each low-level service center,
- $\tilde{b}_h : (b_h^p, b_h^m, b_h^o)$ a triangular fuzzy number, which stands for the maximum number at each high-level service center,
- $\tilde{f}_i : (f_i^p, f_i^m, f_i^o)$ a triangular fuzzy number, which stands for the low-level demand rate at node i,
- $\tilde{\mu}_j^l : (\mu_j^{lp}, \mu_j^{lm}, \mu_j^{lo})$ service rate at the low-level server j; a triangular fuzzy number,
- $\tilde{\mu}_k^h : (\mu_k^{hp}, \mu_k^{hm}, \mu_k^{ho})$ service rate at the high-level server k; a triangular fuzzy number,
- $\widetilde{NS}_j^l$ average number of customers at the low-level server j, during the steady state; a triangular fuzzy number,
- $\widetilde{NS}_k^h$ average number of customers at the high-level server k, during the steady state; a triangular fuzzy number,
- $\tilde{\lambda}_j^l : (\lambda_j^{lp}, \lambda_j^{lm}, \lambda_j^{lo})$ arrival rate of demand at the low-level server j; a triangular fuzzy number,
- $\tilde{\lambda}_k^h : (\lambda_k^{hp}, \lambda_k^{hm}, \lambda_k^{ho})$ arrival rate of demand at the high-level server k; a triangular fuzzy number,
- β_i the high-level demand percentage at node i, a crisp number,
- s_{ij}^{dl} degree of membership for the distance between node i and the low-level server j, being almost less than or equal to, the distance standard,
- s_{ik}^{dh} degree of membership for the distance between node i and the high-level server k, being almost less than, or equal to, the distance standard,
- s_{jk}^{lh} degree of membership for the distance between the low-level server j, and the high-level server k, being less than or equal to, the distance standard

The model's decision variables are as follows:

- X_{ij}^l the degree of membership for node i being covered by the low-level server j,
- X_{ik}^h the degree of membership for node i being covered by the high-level server k,
- W_j a zero-one variable, which assumes value one if a low-level server is located at node j; otherwise, it is zero,
- Z_k a zero-one variable, which assumes value one if a high-level server is located at node k; otherwise, it is zero,

Y_{jk} The degree of membership for the low-level server j, to be covered by the high-level server k.

The fuzzy sets used in the model are as follows:

$\tilde{N}_j^{dl}$: This discrete fuzzy set represents the distance of every node from the low-level server j, and is defined as:

$$\tilde{N}_j^{dl} = \left\{ \frac{s_{1j}^{dl}}{1}, \frac{s_{2j}^{dl}}{2}, \cdots, \frac{s_{ij}^{dl}}{i} \right\}, \quad \forall j, \tag{13}$$

where, s_{ij}^{dl} stands for the degree of membership for the distance from node i to the low-level server j, to be approximately smaller than, or equal to, the distance standard and which is calculated as explained below.

Let d_{ij} represent the distance between node i and the low-level server j. Also, let S_{dl} denote the distance standard for low-level services. Now, the statement that "the node i's distance from node j is approximately less than, or equal to, the distance standard", can be represented by the following fuzzy notation:

$$d_{ij} \overset{\sim}{\leq} S_{dl}. \tag{14}$$

Such a definition allows us to include any node i in the set $\tilde{N}_j^{dl}$ for the low-level server j. As Relation 14 shows, each node arrives with a degree of membership in the set $\tilde{N}_j^{dl}$. Regardless of the fact that the distance of node i from node j is within the distance standard or not, the degree of membership for node i, i.e., s_{ij}^{dl}, can be calculated as follows:

$$s_{ij}^{dl} = \begin{cases} 0, & d_{ij} > u_{dl} \\ \frac{u_{dl} \quad d_{ij}}{u_{dl} \quad s_{dl}}, & s_{dl} \leq d_{ij} < u_{dl} \\ 1, & d_{ij} \leq s_{dl} \end{cases}, \tag{15}$$

where, u_{dl} stands for the acceptable upper bound for the distance standard. The relation in Equation 15 is obtained on the basis of Figure 1;

$\tilde{N}_k^{dh}$: This discrete fuzzy set stands for the distance from all nodes to the high-level server k, and is defined as:

$$\tilde{N}_k^{dh} = \left\{ \frac{s_{1k}^{dh}}{1}, \frac{s_{2k}^{dh}}{2}, \cdots, \frac{s_{ik}^{dh}}{i} \right\}, \quad \forall k.$$

$\tilde{N}_k^{lh}$: This discrete fuzzy set represents the distance from low-level servers to the high-level server k, and is defined as:

$$\tilde{N}_k^{lh} = \left\{ \frac{s_{1k}^{lh}}{1}, \frac{s_{2k}^{lh}}{2}, \cdots, \frac{s_{jk}^{lh}}{j} \right\}, \quad \forall k.$$

The technicalities of evaluating s_{ik}^{dh} and s_{jk}^{lh} are similar to those when evaluating s_{ij}^{dl}.

$\tilde{N}c_{ij}^{l}$: This fuzzy set includes the nodes which are approximately covered by the low-level server j, i.e.:

$$\tilde{N}c_{ij}^{l} = \left\{ \frac{X_{1j}^{l}}{1}, \frac{X_{2j}^{l}}{2}, \cdots, \frac{X_{ij}^{l}}{i} \right\}, \quad \forall j.$$

$\tilde{N}c_{ik}^{h}$: This fuzzy set includes the nodes which are approximately covered by the high-level server k, i.e.:

$$\tilde{N}c_{ij}^{h} = \left\{ \frac{X_{1k}^{h}}{1}, \frac{X_{2k}^{h}}{2}, \cdots, \frac{X_{ik}^{h}}{i} \right\}, \quad \forall k.$$

Nc_{jk}^{lh} : This fuzzy set includes the low-level servers which are approximately covered by the high-level server k, i.e.:

$$\tilde{N}c_{jk}^{lh} = \left\{ \frac{Y_{1k}}{1}, \frac{Y_{2k}}{2}, \cdots, \frac{Y_{ik}}{i} \right\}, \quad \forall k.$$

In this work, it is intended to develop models which cover the demand nodes that are within the distance standard. Thus, in the case of coverage by low-level servers, one has to find the intersection of the fuzzy sets, $\tilde{N}c_{ij}^{l}$ and $\tilde{N}_j^{dl}$, to determine the issue of coverage for low-level services, with respect to the distance standard. So, the degree of membership for coverage of the demand nodes by the low-level server j, within the distance standard, will be the minimum of the degrees of membership across the fuzzy sets, $\tilde{N}c_{ij}^{l}$ and $\tilde{N}_j^{dl}$, i.e.:

$$V_{ij} = \min \left\{ X_{ij}^{l}, s_{ij}^{dl} \right\}. \tag{16}$$

In case conditions are provided such that X_{ij}^{l}always stays less than, or equal to s_{ij}^{dl}, then, it is always true that $V_{ij} = X_{ij}^{l}$, which eliminates the need to define V_{ij}.

Thus, in order to drop the need for consideration of V_{ij}, the following constraint is enforced:

$$X_{ij}^{l} \leq S_{ij}^{dl}. \tag{17}$$

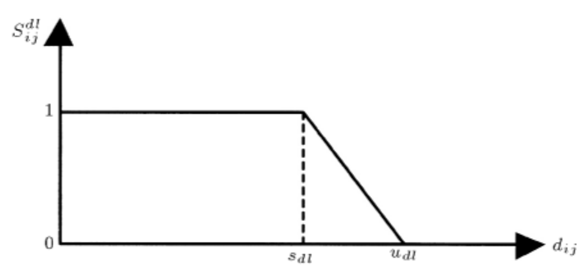

Figure 1. Membership function of the distance standard for low-level services.

With the same consideration for $(\widetilde{N}c_{ik}^{h}, \widetilde{N}_{k}^{dh})$ and $(\widetilde{N}c_{jk}^{lh}, \widetilde{N}_{k}^{lh})$, one also has the following constraints:

$$X_{ik}^{h} \leq S_{ik}^{dh}, \tag{18}$$

$$Y_{jk} \leq S_{jk}^{lh}. \tag{19}$$

$\widetilde{D}c_{j}^{l}$: This is the set of demands which are approximately covered by the low-level server j, i.e.:

$$\widetilde{D}c_{j}^{l} = \left\{ \frac{X_{1j}^{l}}{\widetilde{f}_{1}}, \frac{X_{2j}^{l}}{\widetilde{f}_{2}}, \cdots, \frac{X_{ij}^{l}}{\widetilde{f}_{i}} \right\}, \quad \forall j. \tag{20}$$

Each element of this set is a triangular fuzzy number. In the following section, this set is employed to determine the arrival rates of the service demand for low-level servers.

$\widetilde{D}c_{k}^{h}$: This is the set of demand calls which are approximately covered by the high-level server k, and is formed as the sum of two fuzzy sets. The first set, designated by $\widetilde{D}c_{k}^{dh}$, is the set of demands covered by high-level servers and the next, designated by $\widetilde{D}c_{k}^{lh}$, is the fuzzy set of the low-level demands which come from low-level to high-level servers.

$$\widetilde{D}c_{k}^{dh} = \left\{ \frac{X_{1k}^{h}}{\beta_{1}\widetilde{f}_{1}}, \frac{X_{2k}^{h}}{\beta_{2}\widetilde{f}_{2}}, \cdots, \frac{X_{ik}^{h}}{\beta_{i}\widetilde{f}_{i}} \right\} \quad \forall k, \tag{21}$$

$$\widetilde{D}c_{k}^{lh} = \left\{ \frac{Y_{1k}}{\widetilde{\mu}_{1}^{l}}, \frac{Y_{2k}}{\widetilde{\mu}_{2}^{l}}, \cdots, \frac{Y_{ik}}{\widetilde{\mu}_{i}^{l}} \right\}, \quad \forall k, \tag{22}$$

$$\widetilde{D}c_{k}^{h} = \left\{ \frac{\min(X_{ik}^{h}, Y_{ik})}{\beta_{i}\widetilde{f}_{i} + \widetilde{\mu}_{i}^{l}} \right\}, \quad \forall k. \tag{23}$$

Now, a new variable, Z_{ik}, is defined that satisfies the following constraints:

$$Z_{ik} = \min(X_{ik}^{h}, Y_{ik}), \tag{24}$$

$$Z_{ik} \leq X_{ik}^{h}, \tag{25}$$

$$Z_{ik} \leq Y_{jk}. \tag{26}$$

Next, the set $\widetilde{D}c_{k}^{h}$ is converted to the following form and, along with constraints in Relations 25 and 26, are added to the model.

$$\widetilde{D}c_{k}^{h} = \left\{ \frac{Z_{ik}}{\beta_{i}\widetilde{f}_{i} + \widetilde{\mu}_{i}^{l}} \right\}, \quad \forall k. \tag{27}$$

In the following section, this fuzzy set is employed to determine the arrival rates of service demand for the high-level servers.

$\widetilde{P}c_{j}^{l}$: This fuzzy set of populations are approximately covered by the low-level servers. In other words:

$$\widetilde{P}c_{j}^{l} = \left\{ \frac{X_{1j}^{l}}{a_{1}}, \frac{X_{2j}^{l}}{a_{2}}, \cdots, \frac{X_{ij}^{l}}{a_{i}} \right\}. \tag{28}$$

Since the objective of this model is maximizing the covered populations within the distance standard, one can define the objective function for the case of low-level servers as follows:

$$\max Z^{l} = \sum_{i,j} a_{i} X_{ij}^{l}. \tag{29}$$

Reasoning in a similar manner, the second part of the objective function, which is related to the populations which are approximately covered by high-level servers, becomes as follows:

$\widetilde{P}c_{k}^{h}$: This fuzzy set of populations are approximately covered by high-level servers. In other words:

$$\widetilde{P}c_{k}^{h} = \left\{ \frac{X_{1k}^{h}}{a_{1}}, \frac{X_{2k}^{h}}{a_{2}}, \cdots, \frac{X_{ik}^{h}}{a_{i}} \right\}. \tag{30}$$

Since the second part of the objective function is defined as:

$$\max Z^{h} = \sum_{i,k} a_{i} X_{ik}^{h}, \tag{31}$$

the objective function can be written as follows:

$$\max Z = \max Z^{l} + \max Z^{h},$$

or:

$$\max Z = \sum_{i,j} a_{i} X_{ij}^{l} + \sum_{i,k} a_{i} X_{ik}^{h}. \tag{32}$$

Mathematical Model for FCHQ-MCLP

The FCHQ-MCLP mathematical model, which is a mixed integer programming model, is as follows:

$$\max Z = \sum_{i,j} a_{i} X_{ij}^{l} + \sum_{i,k} a_{i} X_{ik}^{h}, \tag{33}$$

s.t.:

$$X_{ij}^{l} \leq W_{j}, \quad \forall i, j, \tag{34}$$

$$X_{ij}^{h} \leq Z_{k}, \quad \forall i, k, \tag{35}$$

$$Y_{jk} \leq W_{j}, \quad \forall j, k, \tag{36}$$

$$Y_{jk} \leq Z_k, \quad \forall j, k,$$

$$X_{ij}^l \leq S_{ij}^{dl}, \quad \forall i, j,$$

$$X_{ij}^h \leq S_{ik}^{dh}, \quad \forall i, k,$$

$$Y_{jk} \leq S_{jk}^{lh}, \quad \forall j, k,$$

$$Z_{ik} \leq X_{ik}^h, \quad \forall i, k,$$

$$Z_{ik} \leq Y_{jk}, \quad \forall (i = j), k, \tag{37}$$

$$\widetilde{N}S_j^l \leq \widetilde{b}_l, \quad \forall j, \tag{38}$$

$$\widetilde{N}S_k^h \leq \widetilde{b}_h, \quad \forall k, \tag{39}$$

$$\sum_j W_j = P_l, \tag{40}$$

$$\sum_k Z_k = P_h, \quad 0 \leq X_{ij}^l \leq 1, \quad 0 \leq X_{ik}^h \leq 1,$$

$$0 \leq Y_{jk} \leq 1, \quad 0 \leq Z_{ik} \leq 1, \quad W_j = 0, 1, \quad Z_k = 0, 1. \tag{41}$$

This model tries to maximize the approximate covered population within the distance standard. The first four constraints are meant to guarantee that the demand nodes can be covered by a specific node's server only when a server is located at that node. The constraints in Relations 38 and 39 are incorporated in the model to check the quality of rendering service by both types of servers. The criterion for such quality is defined as the condition where the average number of low- and high-level customers calling upon any server must not exceed a given value, $\widetilde{b}_l$ or $\widetilde{b}_h$, accordingly. The constraints in Relations 40 and 41 are included to enforce a limitation on the number of low- and high-level servers. The rest of the constraints in the model have already been elaborated on in the body of the paper.

Applying the Quality Constraints and Converting the Model to a Mixed Integer Programming

This section attempts to explain how fuzzy constraints in Relations 38 and 39 can be transformed to classical forms. For this reason, one starts with the method proposed by Dubois and Prade [11], which deals with calculating the correctness of fuzzy inequalities like Relations 38 and 39 to hold true. According to this method, for any two fuzzy numbers, $\widetilde{I}$ and $\widetilde{J}$, the correctness of $\widetilde{I} \leq \widetilde{J}$ holding true is calculated as:

$$T(\widetilde{I} \leq \widetilde{J}) = \mathrm{Sup}_{x \leq y} \left\{ \min \left\{ \mu_{\widetilde{I}}(x), \mu_{\widetilde{J}}(y) \right\} \right\}, \tag{42}$$

where $\mu_{\widetilde{I}}(x)$ and $\mu_{\widetilde{J}}(y)$ represent the membership functions for x belonging to $\widetilde{I}$ and y belonging to $\widetilde{J}$. Using this method, constraints in Relations 38 and 39 are converted to:

$$T(\widetilde{N}S_j^l \leq \widetilde{b}_l) \geq 1 \quad \alpha, \tag{43}$$

and:

$$T(\widetilde{N}S_k^h \leq \widetilde{b}_h) \geq 1 \quad \beta. \tag{44}$$

Next, it is demonstrated how $\widetilde{N}S_j^l$ (and $\widetilde{N}S_k^h$) can be calculated. To calculate $\widetilde{N}S_j^l$, one first finds the arrival rate of service demand to the low-level servers $\widetilde{D}c_j^l$, which was defined in Equation 20 as:

$$\widetilde{D}c_j^l = \left\{ \frac{X_{1j}^l}{\widetilde{f}_1}, \frac{X_{2j}^l}{\widetilde{f}_2}, \cdots, \frac{X_{ij}^l}{\widetilde{f}_i} \right\} \quad \forall j,$$

Since $\widetilde{D}c_j^l$ is a convex fuzzy set and $\widetilde{f}_i$ is covered by the low-level server j, according to a degree of membership equal to 1, then, $\widetilde{D}c_j^l$ is a discrete fuzzy number. Furthermore, the centroid method can be applied ([12,13]), which is intended for transforming a fuzzy number to a crisp number. Since the elements of $\widetilde{D}c_j^l$, i.e., $\widetilde{f}_i$'s, are all triangular fuzzy numbers, the centroid method transforms $\widetilde{D}c_j^l$ to a triangular fuzzy number. To demonstrate the way that the centroid method works, let one assume that:

$$\widetilde{Z} = \left\{ \frac{\mu_{\widetilde{c}}(z_1)}{z_1}, \frac{\mu_{\widetilde{c}}(z_2)}{z_2}, \cdots, \frac{\mu_{\widetilde{c}}(z_i)}{z_i} \right\},$$

is a discrete fuzzy number, where z_i's are crisp numbers in $\widetilde{Z}$ and $\mu_{\widetilde{c}}(z_i)$ stands for z_i's degree of membership in $\widetilde{Z}$. According to the centroid method, the fuzzy number, $\widetilde{Z}$, is transformed to the crisp number, Z^*, as:

$$Z^* = \frac{\sum_i \mu_{\widetilde{c}}(z_i) z_i}{\sum_i \mu_{\widetilde{c}}(z_i)}. \tag{45}$$

Applying Relations 41 transforms the fuzzy number, $\widetilde{D}c_j^l$, to the triangular fuzzy number:

$$\widetilde{\lambda}_j^l = \frac{\sum_i \widetilde{f}_i X_{ij}^l}{\sum_i X_{ij}^l}, \quad \forall j, \tag{46}$$

which is actually a triangular fuzzy number, like:

$$\widetilde{\lambda}_j^l = (\lambda_j^{lp}, \lambda_j^{lm}, \lambda_j^{lo}), \quad \forall j, \tag{47}$$

where:

$$\lambda_j^{lp} = \frac{\sum\limits_i f_i^p X_{ij}^l}{\sum\limits_i X_{ij}^l}, \quad \lambda_j^{lm} = \frac{\sum\limits_i f_i^m X_{ij}^l}{\sum\limits_i X_{ij}^l}, \quad \lambda_j^{lo} = \frac{\sum\limits_i f_i^o X_{ij}^l}{\sum\limits_i X_{ij}^l}.$$

Assuming that each node's demand for service behaves as a Poisson process, then, the arrival rate of each call at a server j, also follows a Poisson process. Furthermore, assuming that the time to serve a demand by server j is approximately distributed, one will have a fuzzy Markovian (FM/FM/1) queuing model at each server. Thus, the fuzzy Little relations, as proposed by Jo et al. [14], can be employed to evaluate $\widetilde{NS}_j^l$, i.e.:

$$\widetilde{NS}_j^l = \frac{\overset{\sim}{\lambda}{}_j^l}{\overset{\sim}{\mu}{}_j^l \quad \overset{\sim}{\lambda}{}_j^l}. \tag{48}$$

Since $\overset{\sim}{\lambda}{}_j^l$ and $\overset{\sim}{\mu}{}_j^l$ are triangular fuzzy numbers, so is $\widetilde{NS}_j^l$, that is:

$$\widetilde{NS}_j^l = (NS_j^{lp}, NS_j^{lm}, NS_j^{lo}), \tag{49}$$

where:

$$NS_j^{lp} = \frac{\sum_i f_i^p X_{ij}^l}{\mu_j^{lp} \sum_i X_{ij}^l \quad \sum_i f_i^p X_{ij}^l},$$

$$NS_j^{lm} = \frac{\sum_i f_i^m X_{ij}^l}{\mu_j^{lm} \sum_i X_{ij}^l \quad \sum_i f_i^m X_{ij}^l},$$

$$NS_j^{lo} = \frac{\sum_i f_i^o X_{ij}^l}{\mu_j^{lo} \sum_i X_{ij}^l \quad \sum_i f_i^o X_{ij}^l}.$$

Reasoning in a similar manner for high-level servers, by using the centroid method (Equation 45), $\widetilde{Dc}_k^h$ is transformed to a triangular fuzzy number, $(\overset{\sim}{\lambda}{}_k^h)$, as:

$$\overset{\sim}{\lambda}{}_k^h = \frac{\sum_{i,(i=j)} (\beta_i \tilde{f}_i + \overset{\sim}{\mu}{}_j^l) Z_{ik}}{\sum_i Z_{ik}}, \quad \forall k. \tag{50}$$

Since $\tilde{f}_i$ and $\overset{\sim}{\mu}{}_j^l$ are triangular fuzzy numbers, then, $(\beta_i \tilde{f}_i + \overset{\sim}{\mu}{}_j^l)$, which is called $\overset{\sim}{\theta}{}_i^h$, is also a triangular fuzzy number and, with respect to the arithmetic operations on triangular fuzzy numbers, one has:

$$\overset{\sim}{\theta}{}_i^h = (\beta_i \tilde{f}_i + \overset{\sim}{\mu}{}_j^l), \tag{51}$$

$$\overset{\sim}{\theta}{}_i^h = (\theta_i^{hp}, \theta_i^{hm}, \theta_i^{ho}), \tag{52}$$

where:

$$\theta_i^{hp} = \beta_i f_i^p + \mu_j^{lp},$$

$$\theta_i^{hm} = \beta_i f_i^m + \mu_j^{lm},$$

$$\theta_i^{ho} = \beta_i f_i^o + \mu_j^{lo}.$$

Then, $\overset{\sim}{\lambda}{}_k^h$, according to Equations 50, 51 and 52 is presented as:

$$\overset{\sim}{\lambda}{}_k^h = (\lambda_k^{hp}, \lambda_k^{hm}, \lambda_k^{ho}), \quad \forall k, \tag{53}$$

where:

$$\lambda_k^{hp} = \frac{\sum_i \theta_i^{hp} Z_{ik}}{\sum_i Z_{ik}},$$

$$\lambda_k^{hm} = \frac{\sum_i \theta_i^{hm} Z_{ik}}{\sum_i Z_{ik}},$$

$$\lambda_k^{ho} = \frac{\sum_i \theta_i^{ho} Z_{ik}}{\sum_i Z_{ik}}.$$

So, in a similar manner as before and with respect to the fuzzy Little law, one has:

$$\widetilde{NS}_k^h = (NS_k^{hp}, NS_k^{hm}, NS_k^{ho}), \tag{54}$$

where:

$$NS_k^{hp} = \frac{\sum_i \theta_i^{hp} Z_{ik}}{\mu_k^{hp} \sum_i Z_{ik} \quad \sum_i \theta_i^{hp} Z_{ik}},$$

$$NS_k^{hm} = \frac{\sum_i \theta_i^{hm} Z_{ik}}{\mu_k^{hm} \sum_i Z_{ik} \quad \sum_i \theta_i^{hm} Z_{ik}},$$

$$NS_k^{ho} = \frac{\sum_i \theta_i^{ho} Z_{ik}}{\mu_k^{ho} \sum_i Z_{ik} \quad \sum_i \theta_i^{ho} Z_{ik}}.$$

Now, the following lemma, proved in [15], is used to transform constraints in Relations 55 and 56 to the linear form.

Lemma

Given two triangular fuzzy numbers $\tilde{I} = (I^p, I^m, I^o)$ and $\tilde{J} = (J^p, J^m, J^o)$, one has:

a) $T(\tilde{I} \leq \tilde{J}) = 1 \Leftrightarrow I^m \leq J^m,$ (55)

b) $T(\tilde{I} \leq \tilde{J}) \geq 1 \quad \alpha \Leftrightarrow I^m \leq J^o \quad (1 \quad \alpha)(J^o \quad J^m).$ (56)

Using Equation 45, one can transform the constraint in Equation 32 to the linear form, as:

$$T\left(\widetilde{NS}_j^l \leq \tilde{b}_l\right) \geq 1 \quad \alpha \equiv NS_j^{lm} \leq b_l^o \quad (1 \quad \alpha)\,(b_l^o \quad b_l^m). \tag{57}$$

By substituting the equivalent of NS_j^{lm} from Equation 49 and doing appropriate mathematical manipulations, one will arrive at the linear form:

$$\sum_{i=1}^{n}(\beta_i^l \quad \gamma_j^l)X_{ij}^l \leq 0, \quad \forall j, \tag{58}$$

where:

$$\beta_i^l = f_i^m + b_l^o f_i^m \quad (1 \quad \alpha)(b_l^o \quad b_l^m) f_i^m, \quad \forall i, \tag{59}$$

$$\gamma_j^l = b_l^o \mu_j^{lm} \quad (1 \quad \alpha)(b_l^o \quad b_l^m)\mu_j^{lm}, \quad \forall j. \tag{60}$$

The constraint in Equation 33 will be transformed to the following form:

$$\sum_{i=1}^{n}(\beta_i^h \quad \gamma_k^h)Z_{ik} \leq 0, \quad \forall k, \tag{61}$$

where:

$$\beta_i^h = \theta_i^{hm} + b_h^o \theta_i^{hm} \quad (1 \quad \beta)\,(b_h^o \quad b_h^m)\,\theta_i^{hm}, \quad \forall i, \tag{62}$$

$$\gamma_k^h = b_h^o \mu_k^{hm} \quad (1 \quad \beta)(b_h^o \quad b_h^m)\mu_k^{hm}, \quad \forall k. \tag{63}$$

Therefore, the final FCHQ-MCLP model can be written as follows:

$$\max Z = \sum_{i,j} a_i X_{ij}^l + \sum_{i,k} a_i X_{ik}^h,$$

s.t.:

$$X_{ij}^l \leq W_j, \quad \forall i,j, \quad X_{ij}^h \leq Z_k, \quad \forall i,k,$$

$$Y_{jk} \leq W_j, \quad \forall j,k, \quad Y_{jk} \leq Z_k, \quad \forall j,k,$$

$$X_{ij}^l \leq S_{ij}^{dl}, \quad \forall i,j, \quad X_{ij}^h \leq S_{ik}^{dh}, \quad \forall i,k,$$

$$Y_{jk} \leq S_{jk}^{lh}, \quad \forall j,k, \quad Z_{ik} \leq X_{ik}^h, \quad \forall i,k,$$

$$Z_{ik} \leq Y_{jk}, \quad \forall (i=j),k,$$

$$\sum_{j=1}^{n} W_j = P_l, \qquad \sum_{k=1}^{n} Z_k = P_h,$$

$$\sum_{i=1}^{n}(\beta_i^l \quad \gamma_j^l)X_{ij}^l \leq 0, \quad \forall j,$$

$$\sum_{i=1}^{n}(\beta_i^h \quad \gamma_k^h)Z_{ik} \leq 0, \quad \forall k,$$

$$0 \leq X_{ij}^l \leq 1, \quad 0 \leq X_{ik}^h \leq 1, \quad 0 \leq Y_{jk} \leq 1,$$

$$0 \leq Z_{ik} \leq 1, \quad W_j = 0,1, \quad Z_k = 0,1.$$

AN EXAMPLE

In this section, the results obtained from solving a typical problem are presented. The typical problem is solved for CHQ-MCLP and FCHQ-MCLP and, then, the results are compared. To solve the problem, the branch and bound method and IBM OSL v3, on a Pentium 2, 333 MHZ were used. Table 1 illustrates the parameter values for the problem and Tables 2 and 3 display the results of solving the CHQ-MCLP and FCHQ-MCLP, respectively.

Suppose this example relates to health care services, where the low-level servers provide primary health care and the high-level servers provide high-level health care services. In this problem, there is a network with 15 nodes that represent different regions, each region has a population, (a_i), and estimation of the approximate demand rate for low-level services is given by $\widetilde{f}_i = (f_i^p, f_i^m, f_i^o)$. The number of low-level servers to be located, P_l, is 3, and the number of high-level centers to be located, P_h, is 2. The distance between two nodes is measured and treated in terms of the distance standard for low-level and high-level services and, on the basis of such treatment, the membership degrees are determined. In this problem, it is assumed that the distance standards are the same for low-level and high-level services, so that the membership degrees for the distance between nodes are identical for both low-level and high-level services, i.e., $s_{ij}^{dl} = s_{jk}^{lh} = s_{ik}^{dh}$. The maximum allowable number of customers is determined approximately for both levels and is assumed to be $\widetilde{b}_l = (2,3,4)$ and $\widetilde{b}_h = (1,2,3)$. The service rate at each level of servers is determined by $\widetilde{\mu}_l = (30,40,50)$ and $\widetilde{\mu}_h = (10,20,30)$. The percentage of low-level service demands that are referred to high-level centers is $\beta_i = 0.2$, for all nodes. So, under these circumstances, one seeks to locate the servers and allocate the demand nodes to the servers in such a way that the population covered approximately around the distance standards is maximized. To achieve this purpose, the branch and bound method is used to solve the following small-scaled typical problem. The optimal solutions obtained for the probabilistic model, proposed by Marianov and Serra [10], as well as the fuzzy model, are compared.

On the basis of these results, a comparison between the probabilistic CHQ-MCLP and FCHQ-MCLP models is appropriate. Table 2 shows the optimal answer for the probabilistic CHQ-MCLP. In this problem, the low-level servers are located at nodes 1, 2 and 5 and the high-level servers are located at nodes 8 and 10. In the probabilistic version, node i can be covered by the low-level server j, and the high-level server k, only when its distance to these servers is less than, or equal to, the distance standard. So,

Table 1. Parameter values for the example.

Number of Nodes (n) = 15															
n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
ai	937	503	524	654	585	597	580	679	782	914	582	628	854	695	912
f^p	2	3	7	6	5	2	4	1	1	7	9	8	4	3	5
f^m	4	5	9	8	7	4	6	3	3	9	11	10	6	5	7
f^o	5	8	11	12	9	6	8	5	5	13	13	12	9	8	10
C_j	100	120	110	980	850	760	950	115	125	102	130	90	80	92	105
K_k	250	220	185	159	145	220	200	215	198	212	211	196	168	175	185
$s_{ij}^{dl} = s_{jk}^{lh} = s_{ik}^{dh}$															
1	1	1	0.2	0.5	1	0	0.6	1	0	0.9	0.7	0.14	0.51	0.3	0
2		1	0.2	0	0.5	0.14	0.32	1	0.25	0.64	0.9	0.15	0.62	0	0.7
3			1	0	0.3	0.5	0.18	0.51	0.61	0.71	0.2	0.02	0	1	0.9
4				1	0.21	0.51	0.54	0.61	0.12	0.15	0	1	0.29	0.84	0.17
5					1	1	0.9	0.8	0.14	0.21	0.51	0.3	0	1	0.24
6						1	0.2	1	0	0.3	0.6	0.9	0.4	0.7	0.6
7							1	0.2	0	0.9	0.4	0.61	0.72	0.1	0.2
8								1	0.6	0	0.9	0.8	0.4	0.7	0.61
9									1	1	0.3	0.8	0.47	0.16	0.92
10										1	0.2	0.7	0.8	0.14	0.61
11											1	0.9	0.2	0.4	0.31
12												1	0.2	0.1	0.09
13													1	0.8	0.12
14														1	0
15															1
$\widetilde{b}_l = (2,3,4)$				$\widetilde{\mu}_l = (30,40,50)$				$\alpha = 0.05$				$p_l = 3$			
$\widetilde{b}_h = (1,2,3)$				$\widetilde{\mu}_h = (10,20,30)$				$\beta_i = 0.2$				$p_h = 2$			

the nodes 3, 4, 9, 12, 13 and 15 are not covered by any low-level or high-level servers, because the distance from these nodes to the servers exceeds the distance standard. Among the covered nodes, for instance, node 6 is covered by the low-level server 5 and the high-level server 10. Therefore, in the probabilistic CHQ-MCLP, each demand node can be covered by just one server and no demand node can select a server from the available ones, rather, it must ask for service at the specified server determined for it.

In reality, it does not sound acceptable to restrict each node to receive service from just one server. Besides, it does not seem very real to deprive a demand node from receiving service on the basis that its distance from a server is somewhat larger than the distance standard. The FCHQ-MCLP model, on the other hand, is equipped to consider priorities on the basis of which to ask for and to render service. In fact, in this model, each server provides service on the basis of its own priorities, in the same way that each demand node chooses to receive service from servers according to its own priorities. When the conditions of rendering service are identical for all servers, distance becomes the measure on the basis of which demand nodes assign priorities to servers. In this way, each demand node prefers to go to its nearest server and, if this server is occupied, to go to the next nearest server, and so on.

In the FCHQ-MCLP model, each demand node assigns a priority to each low-level and high-level server on the basis of the degree of membership for its own distance from each one of them, $(s_{ij}^{dl}, s_{jk}^{lh}, s_{ik}^{dh})$. As Table 3 indicates, low-level servers for FCHQ-MCLP are located at nodes 1, 8 and 10 and high-level servers are located at nodes 1 and 10. All of the demand nodes are covered by servers according to the degrees of membership. For example, the demand node 5 is

Table 2. The optimum solution for the CHQ-MCLP

Covering the Nodes by the Low-Level and High-Level Servers (X_{ijk})															
Low-Level Server	**Nodes**														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0
5	0	0	0	0	1	1	1	0	0	0	0	0	0	1	0
High-Level Server															
8	1	1	0	0	0	0	0	1	0	1	1	0	0	0	0
10	0	0	0	0	1	1	1	0	0	0	0	0	0	1	0

Relation Between the Low-Level and High-Level Servers (Y_{jk})		
Low-Level Server	**High-Level Server**	
	8	10
1	1	0
2	1	0
5	0	1
Optimal Objective Function Value : 6072		

Low-level servers' locations: 1, 2, 5; High-level servers' locations: 8, 10.

Table 3. The optimum solution for the FCHQ-MCLP.

Nodes Covered by the Low-Level Servers (X^l_{ij})															
Low-Level Server	**Nodes**														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	0	0.5	1	0	0.6	1	0	0.9	0.7	0	0.5	0.3	0
8	1	1	0.5	0.4	0.8	1	0	1	0.6	0	0.9	0.8	0	0.7	0.6
10	0.9	0	0.7	0	0	0.3	0.9	0	1	1	0	0.7	0.8	0	0.6
High-Level Server	**Nodes Covered by the High-Level Servers (X^h_{ik})**														
1	1	1	0.2	0.5	1	0	0.6	1	0	0.9	0.7	0	0.5	0.3	0
10	0.9	0	0.6	0.1	0.2	0.3	0.9	0	1	1	0	0.7	0.8	0	0.6

Relation Between the Low-Level and High-Level Servers (Y_{jk})		
Low-Level Server	**High-Level Server**	
	1	10
1	1	0.9
8	1	0
10	0.9	1
Optimal Objective Function Value : 27569.8		

Low-level servers' locations: 1, 8, 10; High-level servers' locations: 1, 10.

covered by the low-level servers 1 and 8 with the degrees of membership 1 and 0.8 and, for high-level services, it is covered by the high-level servers 1 and 10 with the degrees of membership 1 and 0.2. This means that node 5 gives the highest priority to the high-level server 1 and the least priority to the high-level server 10. The FCHQ-MCLP model makes it possible for the servers to assign their own priorities to the demand nodes as well. This is accomplished by X^l_{ij} and X^h_{ik}, which stand for the degrees of membership for covering nodes. For instance, for the low-level server $j = 1$, nodes 1, 2, 5 and 8 have the highest priority for receiving service, node 10 has the second highest priority and so on. As can be seen in Table 3, each demand node may be covered by various servers and there is a possibility that none of the nodes will be deprived of receiving service. This, obviously, is the advantage of a fuzzy treatment of the problem.

CONCLUSIONS AND FUTURE EXTENSIONS

A new mathematical coherent hierarchical location-allocation model for MCLP with a fuzzified queuing structure is developed for congested systems, where each demand node submits its service call to any low-level, as well as high-level, server according to degree of membership. The final model is transformed to a mixed integer programming model. Since the previous model (MCLP) is NP-Hard [2] and the 0-1 integer programming model derived in this paper can be reduced to the MCLP model in polynomial time, so, it is NP-Hard as well and one can attempt to develop a heuristic method, such as a genetic algorithm, a tabu search and/or an ant algorithm, etc., for its solution. Other extensions include developing the models of coherent fuzzy hierarchical queuing systems for a maximal availability location problem (MALP) and fuzzy queuing hierarchical location models with up to two service levels. Considering non triangular fuzzy numbers is another avenue to be explored.

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