Scientia Iranica, Vol. 11, No. 3, pp 185–192

SCIENTIA
IRANICA

Optimal Planning of Equipment Maintenance and Replacement on a Variable Horizon

F. Kianfar[1]

In maintenance and replacement planning of an industry, the purchasing of new equipment on the market with some kind of technological improvement over existing equipment should be taken into account. Because of operating cost, ordinary and preventive maintenance expenses may be lower, production rate may be higher and the quality of output may be better, etc. The industry to be considered in this paper is the mining industry and the replaced equipment is the mine shovel. It is assumed that, at most, one new shovel can be purchased before the mine is exhausted. It is possible that the mine is given up before exhausted, because of the disadvantage of the expense and loss of the salvage value of the shovel compared with the value of the remaining mine. There are four decision variables in this problem, as follows: the maintenance policy in each period, the purchase time of the new shovel, the end of the planning horizon or the time to stop the mining and the value of production during each period. The objective is to determine the values of the decision variables so as to maximize the overall discounted profit of the mine over the planning horizon.

INTRODUCTION

With the increasing demand for products of businesses and other organizations, it is necessary to increase production capacity over time. To increase this capacity, decisions should frequently be made about replacement of existing equipment. The significance of this decision becomes apparent when one notes that businesses spend hundreds of billions of dollars on new plants and equipment and this expenditure is almost 10 percent of the GDP in the United States [1]. Existing equipment is usually replaced by equipment on the market which has some kind of technological improvement. Investment for this replacement may be quickly returned by lower maintenance and operating costs, greater throughput and better quality of output using the new advanced equipment.

The business to be considered in this research is the mining industry and the replaced equipment is the mine shovel. Suppose an existing shovel is working on a mine. If an expensive preventive maintenance policy is applied, the shovel will last longer, operating expenses will be lower due to lower breakdowns, salvage value will be higher and production capacity will reduce more slowly. It is assumed that, at most, one new shovel is required before the mine is exhausted. It is possible that the mine is given up before exhausted because of the disadvantage of expenses and loss of the salvage value of the shovel compared with the value of the remaining mine. There are four decision variables in this problem as follows: The maintenance policy in each period, the purchase time of the new shovel, the end of the planning horizon or the time to stop mining and the value of production during each period. The objective is to determine the values of the decision variables so as to maximize the overall discounted profit of the mine. The mathematical method used to solve this problem is deterministic optimal control [2]. The problem is formulated as a discrete-time dynamic system and, then, the discrete-time maximum principle is used to find the optimal solution [3].

The next sections of this paper are as follows: First, a literature review of papers is presented. Then, problem formulation and solution of the problem by the discrete-time maximum principle are introduced and discussed, respectively. After that, a case study of the paper is presented and, finally, the paper is concluded. All the materials in the third through sixth section of the paper, consisting of the mathematical model, method of solution, applications and conclusions of the problem, are the original contributions of the author.

1. *Department of Industrial Engineering, Sharif University of Technology, Tehran, I.R. Iran.*

LITERATURE REVIEW

Many mathematical models for finding a capital equipment optimal replacement policy have been introduced in recent years. The complication of these models ranges through a wide spectrum from the simple and straightforward to the very sophisticated. Two simple models of this type, taking into account technological improvement with finite and infinite planning horizons, can be found in Jardine [4]. More sophisticated research on this subject has been collected in a recent book by Ben-Daya et al. [5]. In the almost thirty year time interval between the publication of these two references, a lot of papers and books have been published on this subject, some of which are reviewed in this section.

The optimization technique that is used for solving production, maintenance and replacement planning is optimal control. In particular, stochastic optimal control has lately been given much attention by researchers. In [6], Akella and Kumar formulated a one-machine one-part-type production problem as a stochastic optimal control problem, in which part demand is assumed to be a two-state continuous-time Markov chain and the objective function is a discounted inventory/shortage cost over an infinite-time horizon. They showed that optimal control is given by a single threshold inventory level called a hedging point. Then, Bielecki and Kumar treated a long-run average cost in [7], where an optimal hedging point policy was also obtained.

According to this policy, at any point in time, the control guides the production surplus towards a nonnegative level, depending on the capacity state in place. This capacity state specific level is known as the corresponding hedging point. The idea behind this policy is that some nonnegative production surplus should be maintained, at times of excess capacity, to hedge against future capacity shortage [8].

Boukas and Yang [9] extended Akella and Kumar's model to allow the simultaneous planning of production and maintenance in a flexible manufacturing system. Their system is composed of a single machine that produces a given commodity. The machine is subject to some random failures and the probability of machine failure is supposed to be an increasing function of its age. The commodity demand rate is assumed to be a constant and the objective is meeting the demand while minimizing the discounted inventory and maintenance costs. Under some appropriate conditions, they established similar results to the ones given in Akella and Kumar.

In most of the manufacturing flow-control models considered, it has been assumed that the machine failure rates are independent of production rates and are constant, as long as the system is in one of its discrete capacity states. In reality, however, this assumption is often violated and the failure rate of a machine usually depends on many factors, such as the age of the machine and the instantaneous rate of production. In most cases, it is reasonable to assume that if a machine works at a faster rate, it is more likely to fail. Very few studies have been done for systems with operation dependent failure rates. Boukas and Haurie [10] considered a system that has two machines with age-dependent failure rates and where preventive maintenance is a decision option. They used a numerical method to evaluate the optimal control policy and showed that in their context the optimal hedging surfaces can be defined to represent the optimal production policies.

When the age of the machine is considered, solving optimal control is needed to augment it to the state of the system, which greatly increases the computation burden of the problem. Boukas and Liu avoided using the age of the machine as a state variable in a recent work [11]. They divided the aging of the machine into four segments, associated with the four modes of the machine. The machine is assumed to have three working modes: good, average, bad and a failure mode. In the three working states, the machine can produce parts and some of these parts are rejected at a rate depending on the machine state. In the failure state, no part is produced. The state transition of the machine is governed by a continuous-time Markov process. The jump rates from average and bad states to good state are preventive maintenance rates and the one from failure state to good state is the corrective maintenance rate. By using stochastic dynamic programming, production and maintenance rates are optimized. Some properties of the value function are shown and the optimal control law is characterized.

Rajagopalan [12] attempts to present a unified approach for capacity expansion and equipment replacement. Equipment replacement literature has focused on the replacement issue, usually ignoring aspects such as future demand changes and economies of scale. On the other hand, capacity expansion literature has focused on the expansion of equipment capacity to meet demand growth, considering economies of scale but ignoring the replacement aspect. Rajagopalan formulates and solves a general deterministic model that allows replacement of capacity, as well as expansion and disposal, to adapt to arbitrary demand changes and permits economies of scale in capacity purchases. The model partially captures deterioration and obsolescence effects by permitting operating maintenance costs and salvage values to vary as a function of age and usage. A key contribution of the paper is that it brings together equipment replacement literature and capacity expansion literature. Rajagopalan presents models and solution procedures for the general problem, even in very special cases, which have, so far, been considered difficult.

The application of mathematical models in maintenance, which can be used by maintenance engineers and managers on real problems, is discussed in a paper by Scarf [13]. In this context, developing areas of maintenance modeling are discussed, namely: Inspection maintenance, condition-based maintenance, maintenance for multi-component systems and maintenance management information systems. Some new models relating to capital replacement are also considered. Discussion of maintenance management information systems is included because of their importance in providing data for mathematical modeling and in implementing model-based maintenance policies.

Finally, maintenance optimization models are reviewed in a good survey by Dekker [14], especially from the applications and future prospects point of view.

PROBLEM FORMULATION

Consider buying a new shovel at the beginning of period s and salvaging it at the beginning of period $t > s$. Let J_{st} denote the present value of total profits associated with the new shovel($s > 0$) or present shovel ($s = 0$). To calculate J_{st}, one needs the following notation for $k \in \langle s, t-1\rangle$ where $\langle s, t-1\rangle = \{i | s \leq i \leq t-1\}$:

- x_s^k the resale value of the shovel of vintage s at the beginning of period k. It is assumed that the initial x_s^s is known;
- P_s^k the production capacity during period k. One assumes that P_s^s is known. If the shovel is brand new and purchased at the beginning of period s, P_s^s is known from the manufacturer's specifications for the shovel. For an existing shovel at $s = 0$, P_0^0 could be determined from the knowledge of when it was purchased, its history of preventive maintenance and its production history;
- E_s^k the necessary expense of the ordinary maintenance during period k. This maintenance is for the current period. One assumes that E_s^s is known. Similar remarks, as for P_s^s, apply in connection with how E_s^s and E_0^0 might be determined;
- y_s^k the value of the remaining material in the mine at the beginning of period k that could be mined using the shovel purchased in period s. One lets $y_0^0 = M$, to denote the total material existing in the mine at time zero. The value of y_s^s can, then, be determined as M less the total amount of production in periods $< 0, s-1 >$. In most cases, y_s^k does not depend on s, so y_s^k can be replaced simply by y^k;
- v^k the value of production during period k. This and the past production amounts affect the future values P_s^l, E_s^l and $x_s^l, l \geq k+1$;
- u^k the preventive maintenance expenditure during period k. This and the past preventive maintenance expenditures affect the future values P_s^l, E_s^l and $x_s^l, l \geq k+1$;
- C_s the cost of purchasing a shovel at the beginning of period s. For the existing shovel at time zero, C_0 is the sunk cost and so it may be assumed that $C_0 = 0$, without loss of generality;
- ρ the periodic discount rate, $\rho \geq 0$.

There is a given budget, U_s^k, for preventive maintenance in period k for the shovel purchased in period s. Hence, it is required that

$$u^k \geq 0, \tag{1}$$

$$U_s^k - u^k \geq 0. \tag{2}$$

Since production cannot exceed the capacity and amount of remaining mine material, it must be required that:

$$v^k \geq 0, \tag{3}$$

$$P_s^k - v^k \geq 0, \tag{4}$$

$$y_s^k - v^k \geq 0. \tag{5}$$

Note that in Equations 1 to 5, U_s^k are given constants and P_s^k and y_s^k are state variables.

J_{st} can be expressed in terms of the variables and functions defined above:

$$J_{st} = \sum_{k=s}^{t-1} \left(v^k - E_s^k - u^k\right)(1+\rho)^{-k} - C_s (1+\rho)^{-s} + x_s^t (1+\rho)^{-t}. \tag{6}$$

In Equation 6, production, v^k, and salvage value, x_s^t, represent revenue terms and expenses E_s^k, u^k; and the purchase cost, C_s, represents expenditure terms.

There must also be functions that will provide the ways in which state variables $(P_s^k, E_s^k, x_s^k, y_s^k)$ evolve over time, given the production amount, v^k , and the amount of preventive maintenance expenditure, u^k. One assumes that the effects of production amount and preventive maintenance are independent. Also, it is assumed that at time s, the only shovels available for purchase are those that are up-to-date with respect to the technology prevailing at s. These functions can, therefore, be subscripted by s to reflect the effect of the shovel's technology. Let $\Pi_s^1(u^k,k), \Psi_s^1(u^k,k)$, and $\Phi_s^1(u^k,k)$ be functions of u^k and k, and let $\Pi_s^2(v^k,k), \Psi_s^2(v^k,k)$ and $\Phi_s^2(v^k,k)$ be functions of v^k and k. With these, one can write the following state equations:

$$\Delta P_s^k = P_s^{k+1} - P_s^k = \Pi_s^1(u^k,k) + \Pi_s^2(v^k,k), \quad P_s^s \text{ is given}, \tag{7}$$

$$\Delta E_s^k = E_s^{k+1} - E_s^k = \Psi_s^1(u^k,k) + \Psi_s^2(v^k,k), \quad E_s^s \text{ is given}, \tag{8}$$

$$\Delta x_s^k = x_s^{k+1} - x_s^k = \Phi_s^1(u^k,k) + \Phi_s^2(v^k,k), x_s^s = (1-\delta)C_s, \tag{9}$$

$$\Delta y_s^k = y_s^{k+1} - y_s^k = -v^k, y_s^s = M - \sum_{i=0}^{s-1} v^i, \tag{10}$$

where δ is the fractional depreciation immediately after the purchase of the shovel at time s.

THE OPTIMAL CONTROL SOLUTION OF THE PROBLEM

The problem is to maximize Equation 6 subject to the state Equations 7 to 10 and Constraints 1 to 5.

The optimal control theory shall be utilized to analyze this problem; (see Chapter 8 in [3]). The Lagrangian function for the problem is:

$$\begin{aligned}L = &\sum_{k=s}^{t-1}(v_s^k - E_s^k - u^k)(1+\rho)^{-k} - C_s(1+\rho)^{-s} \\ &+ x_s^t(1+\rho)^{-t} + \sum_{k=s}^{t-1}\big[\lambda_1^{k+1}(\Pi_s^1 + \Pi_s^2 - P_s^{k+1} + P_s^k) \\ &+ \lambda_2^{k+1}(\Psi_s^1 + \Psi_s^2 - E_s^{k+1} + E_s^k) \\ &+ \lambda_3^{k+1}(\Phi_s^1 + \Phi_s^2 - x_s^{k+1} + x_s^k) \\ &+ \lambda_4^{k+1}(-v^k - y_s^{k+1} + y_s^k)\big] \\ &+ \sum_{k=s}^{t-1}\big[\mu_1^k u^k + \mu_2^k(U_s^k - u^k) + \mu_3^k v^k \\ &+ \mu_4^k(P_s^k - v^k) + \mu_5^k(y_s^k - v^k)\big].\end{aligned} \tag{11}$$

In Equation 11, $\lambda_1^k, \lambda_2^k, \lambda_3^k, \lambda_4^k$ are known as adjoint variables and $\mu_1^k, \mu_2^k, \mu_3^k, \mu_4^k, \mu_5^k$ are known as Lagrange multipliers. The Hamiltonian function is defined as:

$$\begin{aligned}&H_s^k\left(P_s^k, E_s^k, x_s^k, y_s^k, u^k, v^k, k\right) \\ &= (v^k - E_s^k - u^k)(1+\rho)^{-k} \\ &+ \left[\lambda_1^{k+1}\left(\Pi_s^1 + \Pi_s^2\right) + \lambda_2^{k+1}\left(\Psi_s^1 + \Psi_s^2\right)\right. \\ &\left.+ \lambda_3^{k+1}\left(\Phi_s^1 + \Phi_s^2\right) + \lambda_4^{k+1}\left(-v^k\right)\right], \\ &k \in \langle s, t-1\rangle .\end{aligned} \tag{12}$$

Using Equation 12 in Equation 11, the Lagrangian function can be written in terms of the Hamiltonian as:

$$\begin{aligned}L = &x_s^t(1+\rho)^{-t} - C_s(1+\rho)^{-s} + \sum_{k=s}^{t-1}\big[H_s^k \\ &- \lambda_1^{k+1}\left(P_s^{k+1} - P_s^k\right) - \lambda_2^{k+1}\left(E_s^{k+1} - E_s^k\right) \\ &- \lambda_3^{k+1}\left(x_s^{k+1} - x_s^k\right) - \lambda_4^{k+1}\left(y_s^{k+1} - y_s^k\right)\big] \\ &+ \sum_{k=s}^{t-1}\big[\mu_1^k u^k + \mu_2^k\left(U_s^k - u^k\right) + \mu_3^k v^k \\ &+ \mu_4^k\left(P_s^k - v^k\right) + \mu_5^k\left(y_s^k - v^k\right)\big].\end{aligned} \tag{13}$$

Necessary conditions for an optimal solution are given by the Kuhn-Tucker conditions for the problem. These conditions yield the adjoint equations and their terminal conditions when the derivatives of the Lagrangian function, with respect to the state variables, are set equal to zero. That is:

$$\lambda_1^{k+1} - \lambda_1^k = -\frac{\partial H_s^k}{\partial P_s^k} - \mu_4^k = -\mu_4^k, \quad \lambda_1^t = 0, \tag{14}$$

$$\lambda_2^{k+1} - \lambda_2^k = -\frac{\partial H_s^k}{\partial E_s^k} = (1+\rho)^{-k}, \quad \lambda_2^t = 0, \tag{15}$$

$$\lambda_3^{k+1} - \lambda_3^k = -\frac{\partial H_s^k}{\partial x_s^k} = 0, \quad \lambda_3^t = (1+\rho)^{-t}, \tag{16}$$

$$\lambda_4^{k+1} - \lambda_4^k = -\frac{\partial H_s^k}{\partial y_s^k} - \mu_5^k = -\mu_5^k, \quad \lambda_4^t = 0. \tag{17}$$

Furthermore, if one assumes $\Pi_s^1, \Pi_s^2, \Phi_s^1$ and Φ_s^2 to be strictly concave and Ψ_s^1 and Ψ_s^2 to be strictly convex, then, one can set the derivatives of the Lagrangian function, with respect to u^k and v^k, equal to zero. These relations, along with the usual complimentary slackness conditions, provide the following conditions on the Lagrange multipliers $\mu_1^k, \mu_2^k, \mu_3^k, \mu_4^k, \mu_5^k$:

$$-(1+\rho)^{-k} + \lambda_1^{k+1}\frac{\partial \Pi_s^1}{\partial u^k} + \lambda_2^{k+1}\frac{\partial \Psi_s^1}{\partial u_k} + \lambda_3^{k+1}\frac{\partial \Phi_s^1}{\partial u^k} + \mu_1^k - \mu_2^k = 0, \quad (18)$$

$$\mu_1^k \geq 0, \qquad \mu_1^k u^k = 0, \quad (19)$$

$$\mu_2^k \geq 0, \qquad \mu_2^k \left(U_s^k - u^k\right) = 0, \quad (20)$$

$$(1+\rho)^{-k} + \lambda_1^{k+1}\frac{\partial \Pi_s^2}{\partial v_k} + \lambda_2^{k+1}\frac{\partial \Psi_s^2}{\partial v_k} + \lambda_3^{k+1}\frac{\partial \Phi_s^2}{\partial v_k} + \mu_3^k - \mu_4^k - \mu_5^k = 0, \quad (21)$$

$$\mu_3^k \geq 0, \qquad \mu_3^k v^k \geq 0, \quad (22)$$

$$\mu_4^k \geq 0, \qquad \mu_4^k \left(P_s^k - v^k\right) = 0, \quad (23)$$

$$\mu_5^k \geq 0, \qquad \mu_5^k \left(y_s^k - v^k\right) = 0. \quad (24)$$

Of course, if the Π and Ψ functions are linear, then, the control u^k will be bang-bang. Likewise, if the Φ functions are linear, v^k will be bang-bang.

The Kuhn-Tucker conditions (Equations 14 to 24), which are the same as the maximum principle conditions of optimal control, are necessary for optimality. Because appropriate concavity and convexity conditions have been assumed, these conditions are also sufficient for optimality in this problem. In the next section, these conditions are analyzed further.

A CASE STUDY

It is reasonable to assume the following properties of the technology functions: Π_s^1, and Π_s^2 are negative, Π_s^1 is increasing and concave in u^k, and Π_s^2 is decreasing and concave in v^k. This recognizes that the production capacity decreases over time and its decrease is smaller at higher preventive maintenance levels. Moreover, the value of the preventive maintenance is marginally diminishing. On the other hand, decrease in production capacity is larger at larger production levels, with the effect of production level on production capacity to be marginally diminishing. The influences of preventive maintenance and production level on the salvage value are similar. Finally, the necessary expense rate increases over time and its increase is smaller at higher preventive maintenance levels, with this effect marginally diminishing. With respect to production level, the increase in the necessary expense rate is larger at larger production levels and with marginally diminishing effects. These properties are sketched in Figures 1 to 4 and are expressed mathematically as follows:

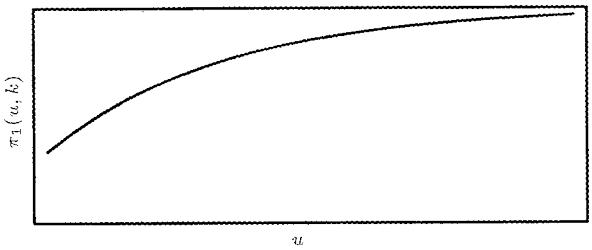

Figure 1. Graph of typical $\pi(u,k)$ and $\phi(u,k)$ with respect to u.

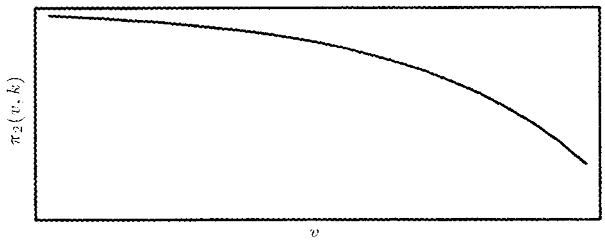

Figure 2. Graph of typical $\pi_2(v,k)$ and $\phi_2(v,k)$ with respect to v.

i) $\Pi_s^1(u^k,k) \leq 0$, $\frac{\partial \Pi_s^1(u^k,k)}{\partial u^k} \geq 0$, $\frac{\partial^2 \Pi_s^1(u^k,k)}{\partial u_k^2} \leq 0$,

ii) $\Pi_s^2(v^k,k) \leq 0$, $\frac{\partial \Pi_s^2(v^k,k)}{\partial v^k} \leq 0$, $\frac{\partial^2 \Pi_s^2(v^k,k)}{\partial^2 v_k^2} \geq 0$,

iii) $\Psi_s^1(u^k,k) \geq 0$, $\frac{\partial \Psi_s^1(u^k,k)}{\partial u^k} \leq 0$, $\frac{\partial^2 \Psi_s^1(u^k,k)}{\partial u_k^2} \geq 0$,

iv) $\Psi_s^2(v^k,k) \geq 0$, $\frac{\partial \Psi_s^2(v^k,k)}{\partial v^k} \geq 0$, $\frac{\partial^2 \Psi_s^2(v^k,k)}{\partial v_k^2} \leq 0$,

v) $\Phi_s^1(u^k,k) \leq 0$, $\frac{\partial \Phi_s^1(u^k,k)}{\partial u^k} \geq 0$, $\frac{\partial^2 \Phi_s^1(u^k,k)}{\partial u_k^2} \leq 0$,

vi) $\Phi_s^2(v^k,k) \leq 0$, $\frac{\partial \Phi_s^2(v^k,k)}{\partial v^k} \leq 0$, $\frac{\partial^2 \Phi_s^2(v^k,k)}{\partial v_k^2} \geq 0$.

Equations 14 to 17 can be solved, as follows:

$$\lambda_1^k = \sum_{i=k}^{t-1} \mu_4^i \quad k \in \langle s,t \rangle, \quad (25)$$

$$\lambda_2^k = -\sum_{i=k}^{t-1} (1+\rho)^{-i}, \quad k \in \langle s,t \rangle, \quad (26)$$

$$\lambda_3^k = (1+\rho)^{-t}, \quad k \in \langle s,t \rangle, \quad (27)$$

$$\lambda_4^k = \sum_{i=k}^{t-1} \mu_5^k, \quad k \in \langle s,t \rangle. \quad (28)$$

To simplify the analysis, let one assume that the changes in production capacity, necessary expense and salvage value of the shovel are not affected by the rate of production, i.e., $\Pi_s^2(v_k,k) = \Psi_s^2(v_k,k) = \Phi_s^2(v_k,k) = 0$. In this case, Equation 21 is replaced by:

$$v^k = \text{bang}\left\{\min\left(P_s^k, y_s^k\right), 0; (1+\rho)^{-k} + \mu_3^k - \mu_4^k - \mu_5^k\right\}. \quad (29)$$

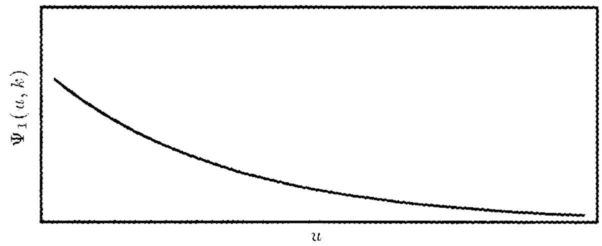

Figure 3. Graph of typical $\Psi_1(u,k)$ with respect to u.

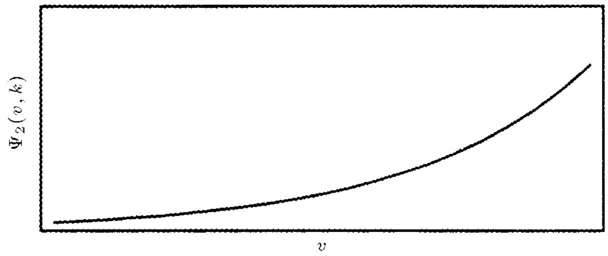

Figure 4. Graph of typical $\Psi_2(v,k)$ with respect to v.

This means that as long as there is a shovel, one should produce at the maximum possible production level in period k, which is min (P_s^k, y_s^k), if $(1+\rho)^{-k} + \mu_3^k - \mu_4^k - \mu_5^k > 0$, and one should not produce if otherwise. It is easy to show that the case of singular control, i.e., $(1+\rho)^{-k} + \mu_3^k - \mu_4^k - \mu_5^k = 0$, does not arise in this problem. It should also be obvious that if $P_s^l > y_s^l$ in a period l, then the entire remaining material would be mined in period l and the shovel would be sold at the beginning of period $l+1$, if $t = l+1$. If $t > l+1$, then clearly $v_i = 0$, for $i \in \langle l+1, t-1\rangle$. Note that, in this case, $J_{st} < J_{s,l+1}$ and, since one will be obtaining J_{st} successively for $t = s+1, s+2, ...$, one does not need to compute J_{st} for $t > l+1$. Note, further, that this does not affect cases when the shovel is optimally sold before the mine is exhausted.

With the above observations, one can set:

$$\mu_3^k = 0, \quad \forall\ k \in \langle s, t-1\rangle. \tag{30}$$

As for obtaining μ_5^k, one proceeds as follows.

Since P_s^k and y_s^k both decline over time from their respective initial values P_s^s and y_s^s, one can define the first period $\tau \in \langle s, t-1\rangle$, if it exists, in which $P_s^\tau > y_s^\tau$. It should be obvious that $\mu_5^k = 0$ for all $k < \tau$. In period τ, one should produce $v^\tau = y_s^\tau$. This means that $\mu_5^\tau > 0$. Moreover, the shovel will be sold in period $\tau+1$, as argued earlier, since the last of the remaining mine will be extracted in period τ. Therefore, in this case, one has $t = \tau+1$ and, from Equations 25 and 28, one has:

$$\lambda_4^k = \begin{cases} 0, & k = t \\ \mu_5^{t-1}, & k < t, \end{cases} \tag{31}$$

$$\lambda_1^k = \sum_{i=k}^{t-2} \mu_4^i, \quad k \in \langle s, t\rangle. \tag{32}$$

If τ, as defined above, does not exist, then, $\mu_5^k = 0$ for all $k \in \langle s, t-1\rangle$ and $\lambda_4^k = 0$ for all $k \in \langle s, t\rangle$. In this case, since $v^k = P_s^k, \lambda_1^k$ is as given in Equation 25.

Now, suppose the conditions leading to Equation 29 are satisfied and this equation, together with its following analysis, is valid. Let a numerical example be mentioned, to justify the validity of the model. Despite performing activities to collect real world data from appropriate mines, this kind of data has not been accessible. So, simulated type data were constructed according to the reasonable range of the model's variables and parameter changes. The data on state variables x_s^k, P_s^k and y_s^k are summarized in Tables 1 to 3, respectively. All the numbers in these three tables and any succeeding table are in million of dollars. The time period of the problem is considered to be a month. The data on the ordinary maintenance expense, E_s^k, for $k = 1$, starts from 1 and 0.5 million dollars for $s = 0$ and $s = 1$, respectively and, then, it is increased by 0.1 million dollars for each increment in k and in s, separately. The data on parameter U_s^k starts from 1 million dollars for $k = 1$ and $s = 1$, then, it is

Table 1. The resale value of the mine shovel of vintage s at the beginning of period $k(x_s^k)$.

s \ k	1	2	3	4	5	6	7	8	9	10
1	10									
2	9	11								
3	8	9.9	12							
4	7	8.8	10.8	13						
5	6	7.7	9.6	11.7	14					
6	5	6.6	8.4	10.4	12.6	15				
7	4	5.5	7.2	9.1	11.2	13.5	16.5			
8	3	4.4	6.0	7.8	9.8	12	14.85	18		
9	2	3.3	4.8	6.5	8.4	10.5	13.2	16.2	19.5	
10	1	2.2	3.6	5.2	7.0	9.0	11.55	14.4	17.55	21
11	0	1.1	2.4	3.9	5.6	7.5	9.9	12.6	15.6	18.9
12		0	1.2	2.6	4.2	6.0	8.25	10.8	13.65	16.8
13			0	1.3	2.8	4.5	6.6	9	11.7	14.7
14				0	1.4	3.0	4.95	7.2	9.75	12.6
15					0	1.5	3.3	5.4	7.8	10.5
16						0	1.65	3.6	5.85	8.4
17							0	1.8	3.9	6.3
18								0	1.95	4.2
19									0	2.1
20										0

Table 2. The production capacity of the mine shovel of vintage s at the beginning of period $k(P_s^k)$.

$s \backslash k$	0	1	2	3	4	5	6	7	8	9	10
1	20	25									
2	19.5	24	28								
3	19	23	27	31							
4	18.5	22	26	30	34						
5	18	21	25	29	33	37					
6	17.5	20	24	28	32	36	40				
7	17	19	23	27	31	35	39	43			
8	16.5	18	22	26	30	34	38	42	46		
9	16	17	21	25	29	33	37	41	45	49	
10	15.5	16	20	24	28	32	36	40	44	48	52
11	15	15	19	23	27	31	35	39	43	47	51
12	14.5	14	18	22	26	30	34	38	42	46	50
13	14	13	17	21	25	29	33	37	41	45	49
14	13.5	12	16		24	28	32	36	40	44	48
15	13	11	15			27	31	35	39	43	47
16	12.5	10					30	34	38	42	46
17	12	9						33	37	41	45
18	11.5	8							36	40	44
19	11	7								39	43
20	10.5	6									42

Table 3. The value of the remaining material in the mine at the beginning of period k that could be mined using the shovel purchased in period $s(y_s^k)$.

$s \backslash k$	0	1	2	3	4	5	6	7	8	9	10
1	300	300									
2	280	275	280								
3	260.5	251	252	260.5							
4	241.5	228	225	229.5	241.5						
5	223	206	199	199.5	207.5	223					
6	205	185	174	170.5	174.5	186	205				
7	187.5	165	150	142.5	142.5	150	165	187.5			
8	170.5	146	127	115.5	111.5	115	126	144.5	170.5		
9	154	128	105	89.5	81.5	81	88	102.5	124.5	154	
10	138	111	84	64.5	52.5	48	51	61.5	79.5	105	138
11	122.5	95	64	40.5	24.5	16	15	21.5	35.5	57	86
12	107.5	80	45	17.5	0	0	0	0	0	10	35
13	93	66	27	0						0	0
14	79	53	10								
15	65.5	41	0								
16	52.5	30									
17	40	20									
18	28	11									
19	16.5	3									
20	5.5	0									

increased by 0.2 million dollars for each increment in k and in s, separately. The values of C_s, for $0 \leq s \leq 10$ are as follows: $C_0 = 0$ and considering $\delta = 0$, one will have $C_s = x_s^s$, for $1 \leq s \leq 10$, where the values of x_s^s can be extracted from Table 1. Finally, the monthly discount rate, ρ, is assumed to be 1%.

As mentioned above, the values of the decision variables, i.e., production value v^k, preventive maintenance expenditure u^k, the end of the planning horizon t, and the present value of profit J_{st}, are computed according to the analysis following Equation 29, for different values of s and all are summarized in Table 4. As seen in this table, the maximum value of J_{st} is 207.22 million dollars and belongs to the case $s = 2$, i.e., the optimal plan of the problem is to replace the present shovel by a new one at the beginning of period two and continue mining the remaining material in the mine up to the end of period fourteen.

CONCLUSIONS

It was shown in this paper that any company can increase its profits by equipment maintenance and replacement planning. Existing equipment is usually replaced with equipment on the market with some kind of technological improvement. Investment for this replacement may be quickly returned by lower maintenance and operating costs, greater output and better quality output using the new advanced equipment. The question is how to determine when to take advantage of the technologically improved equipment?

This question and three other related questions, regarding the equipment maintenance and replacement planning problem, were answered in this paper. The other three questions are: The maintenance policy in each period, the end of the planning horizon or the time to stop the mining and the value of production during each period. In this paper, the mining industry is the business considered and the replaced equipment is the mine shovel.

A powerful technique for the mathematical modeling of the equipment maintenance and replacement planning problem is optimal control. The problem in this paper is formulated as a discrete-time dynamic system, on which the discrete-time maximum principle is used to find the optimal control solution. The type of optimal control solution is presented in the fourth and fifth sections of the paper.

Table 4. The production value (v^k), the preventive maintenance expenditure (u^k), the end of the planning horizon (t) and the profit present value (J_{st}) of the example.

	$s=0$		$s=1$		$s=2$		$s=3$		$s=4$		$s=5$	
k	v^k	u^k	v^k	u^k	v^k	u^k	v^k	u^k	v^k	u^k	v^k	u^k
1	20	1.5	25	1								
2	19.5	1.7	24	1.2	28	1.2						
3	19	1.9	23	1.4	27	1.4	31	1.4				
4	18.5	2.1	22	1.6	26	1.6	30	1.6	34	1.6		
5	18	2.3	21	1.8	25	1.8	29	1.8	33	1.8	37	1.8
6	17.5	2.5	20	2	24	2	28	2	32	2	36	2
7	17	2.7	19	2.2	23	2.2	27	2.2	31	2.2	35	2.2
8	16.5	2.9	18	2.4	22	2.4	26	2.4	30	2.4	34	2.4
9	16	3.1	17	2.6	21	2.6	25	2.6	29	2.6	33	2.6
10	15.5	3.3	16	2.8	20	2.8	24	2.8	28	2.8	32	2.8
11	15	3.5	15	3	19	3	23	3	24.5	3	16	3
12	14.5	3.7	14	3.2	18	3.2	17.5	3.2	0	3.2	0	3.2
13	14	3.9	13	3.4	17	3.4	0	3.4				
14	13.5	4.1	12	3.6	10	3.6						
15	13	4.3	11	3.8	0	3.8						
16	12.5	4.5	10	4								
17	12	4.7	9	4.2								
18	11.5	4.9	8	4.4								
19	11	5.1	3	4.6								
20	5.5	5.3	0	4.8								
21	0	5.5										
t	21		20		15		13		12		12	
J_{st}	179.13		196.74		207.22		199.36		188.97		173.85	

REFERENCES

1. *Annual Survey of Manufacturers, Bureau of the Census*, U.S. Department of Commerce (1995).
2. Disuit, A.K., *Optimization in Economic Theory*, New York: Oxford University Press (1990).
3. Sethi, S.P. and Thompson, G.L., *Optimal Control Theory: Applications to Management Science*, Amsterdam, Martinus Nijhoff (2000).
4. Jardine, A.K.S., *Maintenance, Replacement and Reliability*, Wiley, New York, USA (1973).
5. Ben-Daya, M., Raouf, A. and Duffuaa, S., *Maintenance, Modelling and Optimization*, Kluwer Academic Publishers, Boston, USA (2000).
6. Akella, R. and Kumar, P.R. "Optimal control of production rate in a failure-prone manufacturing system", *IEEE Trans. Automatic Control*, **AC-31**, pp 116-126 (1986).
7. Bielecki, T. and Kumar, P.R. "Optimality of zero-inventory policies for unreliable manufacturing systems", *Operations Research*, **36**, pp 532-541 (1988).
8. Kimemia, J.G. and Gershwin, S.B. "An algorithm for the computer control production in flexible manufacturing systems", *IIE Transactions*, **15**, pp 353-362 (1983).
9. Boukas, E.K. and Yang, H. "Optimal control of manufacturing flow and preventive maintenance", *IEEE Transactions on Automatic Control*, **41**, pp 881-885 (1996).
10. Boukas, E.K. and Haurie, A. "Manufacturing flow control and preventive maintenance: A stochastic control approach", *IEEE Trans. Automat. Control*, **35**, pp 1024-1031 (1990).
11. Boukas, E.K. and Liu, Z.K. "Production and maintenance control for manufacturing systems", *IEEE Transactions on Automatic Control*, **46**, pp 1455-1460 (2001).
12. Rajagopalan, S. "Capacity expansion and equipment replacement: A unified approach", *Operations Research*, **46**, pp 846-857 (1998).
13. Scarf, P.A. "On the application of mathematical models in maintenance", *European Journal of Operational Research*, **99**, pp 493-506 (1997).
14. Dekker, R. "Application of maintenance optimization models: A review and analysis", *Reliability Engineering and System Safety*, **51**, pp 229-240 (1996).